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On smoothly superslice knots

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Abstract. We find smoothly slice (in fact doubly slice) knots in the 3-sphere with trivial Alexander polynomial that are not superslice, answering a question posed by Livingston and Meier.

1. Introduction

A recent paper of Livingston and Meier raises an interesting question about *superslice knots*. Recall [3] that a knot K in S^3 is said to be superslice if there is a slice disk D for K such that the double of D along K is the unknotted 2-sphere S in S^4. We will refer to such a disk as a *superslicing disk*. In particular, a superslice knot is slice and also doubly slice, that is, a slice of an unknotted 2-sphere in S^4. Livingston and Meier ask about the converse in the smooth category.

Problem 4.6 (Livingston–Meier [10])**.** *Find a smoothly slice knot K with $\Delta_K(t) = 1$ that is not smoothly superslice.*

The corresponding question in the topological (locally flat) category is completely understood [10, 12], for a knot K with $\Delta_K(t) = 1$ is topologically superslice.

In this note we give a simple solution to Problem 4.6, making use of Taubes' proof [16] that Donaldson's diagonalization theorem [5] holds for certain noncompact manifolds. For K a knot in S^3, we write $\Sigma_k(K)$ for a k-fold cyclic branched cover of S^3 branched along K. The same notation will be used for the corresponding branched cover along an embedded disk in B^4 or sphere in S^4.

Theorem 1.1. *Suppose that J is a knot with Alexander polynomial* 1*, so that $\Sigma_k(J) = \partial W$, where W is simply connected and the intersection form on W is definite and not diagonalizable. Then the knot $K = J \# -J$ is smoothly doubly slice, but is not smoothly superslice.*

An unpublished argument of Akbulut says that the positive Whitehead double of the trefoil is a knot J satisfying the hypotheses of the theorem, with $k = 2$. The construction is given as [1, Exercise 11.4] and is also

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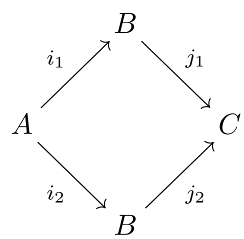
documented, along with some generalizations, in the paper [4]. Hence J gives an answer to Problem 4.6. We remark that for the purposes of the argument, it doesn't matter if W is positive or negative definite, as one could replace J by $-J$ and change all the signs.

We need a simple and presumably well-known algebraic lemma.

Lemma 1.2. *Suppose that*



is a pushout of groups, and that $i_1 = i_2$. *Then* C *surjects onto* B.

Proof. This follows from the universal property of pushouts; the identity map id_B satisfies $\mathrm{id}_B \circ i_1 = \mathrm{id}_B \circ i_2$, and hence defines a homomorphism $C \to B$ with the same image as id_B. □

Applying Lemma 1.2 to the decomposition of the complement of the unknot in S^4 into two disk complements, we obtain the following useful facts. (The first of these was presumably known to Kirby and Melvin; compare [8, Addendum, p. 58], and the second is due to Gordon and Sumners [6].)

Corollary 1.3. *If* K *is superslice and* D *is a superslicing disk, then*

$$\pi_1(B^4 - D) \cong \mathbb{Z} \quad \text{and} \quad \Delta_K(t) = 1.$$

Proof. The lemma says that there is a surjection

$$\mathbb{Z} \cong \pi_1(S^4 - S) \to \pi_1(B^4 - D).$$

Hence $\pi_1(B^4 - D)$ is abelian and so must be isomorphic to $\mathbb{Z}$. This condition implies, using Milnor duality [13] in the infinite cyclic covering, that the homology of the infinite cyclic covering of $S^3 - K$ vanishes, which is equivalent to saying that $\Delta_K(t) = 1$. □

Proof of Theorem 1.1. It is standard [15] that any knot of the form $J \# -J$ is doubly slice. In fact, it is a slice of the 1-twist spin of J, which was shown by Zeeman [17] to be unknotted.

Suppose that K is superslice and let D be a superslicing disk, so $D \cup_K D = S$, an unknotted sphere. Then $S^4 = \Sigma_k(S) = V \cup_Y V$, where we have written $Y = \Sigma_k(K)$ and $V = \Sigma_k(D)$. By Corollary 1.3, the k-fold cover of $B^4 - D$ has $\pi_1 \cong \mathbb{Z}$, so the branched cover V is simply connected.

Note that $\Sigma_k(K) = \Sigma_k(J) \# -\Sigma_k(J)$. Since $\Delta_J(t) = 1$, the same is true for $\Delta_K(t)$, moreover this implies that both $\Sigma_k(J)$ and $\Sigma_k(K)$ are homology spheres. An easy Mayer-Vietoris argument says that $V = \Sigma_k(D)$ is a homology ball; in fact Claim 1.3 implies that it is contractible. Adding a

3-handle to V, we obtain a simply-connected homology cobordism V' from $\Sigma_k(J)$ to itself. By hypothesis, there is a manifold W with boundary $\Sigma_k(J)$ and nondiagonalizable intersection form. Stack up infinitely many copies of V', and glue them to W to make a definite periodic-end manifold M, in the sense of Taubes [16]. Since $\pi_1(V)$ is trivial, M is *admissible* (see [16, Definition 1.3]), and Taubes shows that its intersection form (which is the same as that of W) is diagonalizable. This contradiction proves the theorem. □

The fact that $\pi_1(B^4 - D) \cong \mathbb{Z}$ for a superslicing disk leads to a second obstruction to supersliceness, based on the Ozsváth–Szabó d-invariant [14]. Recall from [11] (for degree 2 covers) and [7] in general that for a knot K and prime p, that one denotes by $\delta_{p^n}(K)$ the d-invariant of a particular spin structure $\mathfrak{s}$ on $\Sigma_{p^n}(K)$ pulled back from the 3-sphere. The fact that a p^n fold branched cover of a slicing disk is a rational homology ball implies that if K slice then $\delta_{p^n}(K) = 0$. For a non-prime-power degree k, the invariant $\delta_k(K)$ might not be defined, because $\Sigma_k(K)$ is not a rational homology sphere. (One might define such an invariant using Floer homology with twisted coefficients as in [2, 9], but there's no good reason that it would be a concordance invariant.)

Theorem 1.4. *If K is superslice, then for any k, the d-invariant*

$$d(\Sigma_k(K), \mathfrak{s}_0)$$

is defined and vanishes.

Proof. Since by Corollary 1.3 the Alexander polynomial is trivial, so $\Sigma_k(K)$ is a homology sphere, and hence $d(\Sigma_k(K), \mathfrak{s}_0)$ is defined. (There is only the one spin structure.) As in the proof of Theorem 1.1, the branched cover $\Sigma_k(D)$ is contractible, and hence [14, Theorem 1.12], $d(\Sigma_k(K), \mathfrak{s}_0) = 0$. □

Sadly, we do not know any examples of a slice knot where Theorem 1.4 provides an obstruction to it being superslice. For such a knot would not be ribbon, so we would also have a counterexample to the slice-ribbon conjecture!

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