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On finite symmetries of simply connected four-manifolds

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Abstract. For most positive integer pairs (a, b), the topological space $\#a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$ is shown to admit infinitely many inequivalent smooth structures which dissolve upon performing a single connected sum with $S^2 \times S^2$. This is then used to construct infinitely many nonequivalent smooth free actions of suitable finite groups on the connected sum $\#a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$. We then investigate the behavior of the sign of the Yamabe invariant for the resulting finite covers, and observe that these constructions provide many new counter-examples to the 4-dimensional Rosenberg Conjecture.

Contents

1. Introduction

The geometry of manifolds in dimension 4 is remarkably intricate, and in key respects differs markedly from the corresponding story in any other dimension. For example, many compact topological 4-manifolds can be shown to support infinitely many inequivalent smooth structures. Indeed, the remarkable results of Freedman [Fre82] and Donaldson [Don83] show that oriented simply-connected compact 4-manifolds without boundary are classified, up to homeomorphism, by just three invariants: the Euler characteristic χ, the signature τ, and the parity of the intersection form on $H^2(\mathbb{Z})$; consequently, any simply connected nonspin closed 4-manifold is homeomorphic to a connected sum $\#a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$, where $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ respectively

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denote the complex projective plane with its standard and nonstandard orientations. By contrast, however, gauge theory gives rise to diffeomorphism invariants, such as Seiberg–Witten basic classes, that can often be used to show that pairs of 4-manifolds are nondiffeomorphic, even though the above-mentioned results imply that they are actually homeomorphic. If a 4-manifold M is orientedly diffeomorphic to $\#a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$, equipped with the familiar smooth structure arising from the standard smooth structure on the complex projective plane via the connect-sum construction, we will therefore find it useful to say that M carries the *conventional* smooth structure; on the other hand, if M is homeomorphic but not diffeomorphic to the connected sum $\#a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$, we will then say that its differential structure is *exotic*.

We first discuss the existence of exotic structures and the region in which they can be exhibited, and emphasize a special property of these structures, namely their solubility. We say that a nonspin 4-manifold is $(S^2 \times S^2)$-*soluble* or $\mathbb{CP}^2$-*soluble,* if after taking the connected sum with one copy of $S^2 \times S^2$ or with one copy of $\mathbb{CP}^2$, respectively, its differential structure is the conventional structure. The $(S^2 \times S^2)$-solubility is more adequate if one studies the general class of 4-manifolds, while the $\mathbb{CP}^2$-solubility is more suitable to the study of complex surfaces or symplectic 4-manifolds. The $\mathbb{CP}^2$-solubility property is equivalent to the almost complete decomposability in the sense of Mandelbaum [Man80].

We study the geography of exotic structures on simply connected nonspin $(S^2 \times S^2)$-soluble manifolds. This generalizes the results on the geography of symplectic manifolds of [BrKo05].

Theorem A. *For any $\epsilon > 0$ there is a constant $N_\epsilon > 0$ such that given any integer pair (a, b) in the first quadrant satisfying either one of the two conditions*

$$(1) \qquad b \geq \left(\frac{1}{2} + \epsilon\right) a + N_\epsilon \qquad \textit{and} \quad a \not\equiv 0 \mod 8,$$

$$(2) \qquad b \leq \frac{2}{1+2\epsilon}(a - N_\epsilon) \qquad \textit{and} \quad b \not\equiv 0 \mod 8,$$

the topological space $M = \#a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$ admits infinitely many pairwise nondiffeomorphic smooth structures, which are all $(S^2 \times S^2)$-soluble.

This result is used to exhibit an infinite number of inequivalent smooth free actions on the *conventional* nonspin smooth structure on 4-manifolds:

Theorem B. *Let $d \geq 2$ be an integer and Γ any group of order d which acts freely on the sphere S^3. For any $\epsilon > 0$ there exists a constant $N'_\epsilon > 0$ such that for any point (a, b) in the region R_ϵ satisfying the divisibility conditions* (D_1) *and* (D_2)*, below, the manifold $M = \#a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$ has the following properties:*

(B_1) *M admits infinitely many smooth orientation preserving free actions of the group Γ, which we denote by $\Gamma_i, i \in \mathbb{N}$.*

(B_2) *the actions Γ_i are conjugate by homemorphisms, but are not conjugate by the diffeomorphisms of M.*

The region R_ϵ is defined as:

$$R_{\epsilon,1} = \left\{ (a,b) \in \mathbb{N} \times \mathbb{N} \;\middle|\; b \geq \left(\frac{1}{2} + \epsilon\right) a + N'_\epsilon \right\}$$

$$R_{\epsilon,2} = \left\{ (a,b) \in \mathbb{N} \times \mathbb{N} \;\middle|\; b \leq \frac{2}{1+2\epsilon}(a - N'_\epsilon) \right\}$$

$$R_\epsilon = R_{\epsilon,1} \cup R_{\epsilon,2}$$

while the divisibility conditions are:

(D_1) $$a + 1 \equiv 0 \mod d \text{ and } b + 1 \equiv 0 \mod d,$$

(D_2) $$\begin{cases} \text{if } (a,b) \in R_{\epsilon,1} \text{ then } \frac{a+1}{d} \not\equiv 1 \mod 8, \text{ or} \\ \text{if } (a,b) \in R_{\epsilon,2} \text{ then } \frac{b+1}{d} \not\equiv 1 \mod 8. \end{cases}$$

For any integer d there exists at least one group of order d which acts freely on the 3-sphere, for example the finite cyclic group

$$\mathbb{Z}_d \cong \{\rho \in \mathbb{C} |\ \rho^d = 1\}$$

acting on $S^3 \subset \mathbb{C}^2$ by multiplication. The divisibility condition (D_1) is necessary in order to assure that the quotient manifolds M/Γ_i have the Betti numbers $b_2^+ = \frac{1+a}{d} - 1$ and $b_2^- = \frac{1+b}{d} - 1$ integer valued. The constant ϵ can be chosen to be arbitrary small. Unfortunately, due to the nature of the constructions involved we are unable to compute the value of N'_ϵ. The constant depends on the constant N_ϵ from Theorem A and increases linearly in d, for the exact formula see Equation (11) in Section §2. Nevertheless, the region R_ϵ covers all the integer lattice points in the first quadrant with the exception of finitely many.

This theorem generalizes earlier work of Ue [Ue96, Main Theorem], LeBrun [LeB03, Theorem 2] and Hanke–Kotschick–Wehrheim [HKW03, Theorem 12], putting an emphasis on the geography of the 4-manifolds. Torres [Tor15, Theorems 1.6 and 1.7] also discusses similar results but with more restrictive divisibility conditions and for a much smaller region. Using different constructions, Akhmedov–Ishida–Park [AIP15] exhibited a similar behaviour for manifolds with zero signature. One of the aims of these papers is to provide examples for which the Rosenberg Conjecture [Ros86, Conjecture 1.2] fails, meaning that there are manifolds with finite fundamental group of odd order which do not admit metrics of positive scalar curvature, while their universal cover does. Such a phenomenon can be detected by a differential invariant of the manifold arising from Riemannian geometry, the Yamabe invariant, see Equation (12) in Section §4. As $\mathbb{CP}^2$ admits a metric of positive curvature, for example the Fubini–Study metric, then a result of Gromov–Lawson [GrLa80] tells us that this property is preserved under the connected sum operation, and hence so does $M = \#a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$. The Yamabe invariant of M is, then, positive. The existence of a symmetry group on M has an immediate impact on the Riemannian properties of the manifold. If we consider the manifold M endowed with one of the actions Γ_i, we show that:

Proposition 3. *For any integer d and any point (a, b) in the region R_ϵ satisfying the divisibility conditions (D_1) and (D_2) let g be a Γ_i-invariant metric on the manifold $M = \#a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$, for any of the Γ_i actions defined in Theorem B. Then the Yamabe invariant of the conformal class of g is negative.*

Remark 4. In some cases it can be shown that these actions also give obstructions to the existence of invariant Einstein metrics, see Şuvaina [Su08].

For the examples in Proposition 3 the Yamabe invariant of the conformal class of a Γ_i-invariant metric, or equivalently of any conformal class on M/Γ_i, is negative. Thus our constructions provide a large class of counterexamples to the Rosenberg Conjecture. Moreover, the Yamabe invariant of the conformal class is bounded away from zero. This is not necessary true for an arbitrary action of Γ and Γ-invariant metrics.

Proposition 5. *For any integer $d \geq 2$ and any integer pair (a, b) in the region:*

$$R_0 = \left\{ (a,b) \in \mathbb{N}\times\mathbb{N} \;\middle|\; b \geq 5a + 12d + 4,\ \textit{and}\ \frac{1+a}{d} \not\equiv 1 \mod 8 \right\}$$
$$\bigcup \left\{ (a,b) \in \mathbb{N}\times\mathbb{N} \;\middle|\; b \leq \frac{1}{5}(a - 12d - 4),\ \textit{and}\ \frac{1+b}{d} \not\equiv 1 \mod 8 \right\}$$

on the manifold $M = \#a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$ there exist infinitely many conjugate homeomorphic conjugate nondiffeomoprhic actions of the group Γ, which we denote by $\Gamma'_i, i \in \mathbb{N}$, such that the Yamabe invariant of the conformal class of a Γ'_i-invariant metric g is nonpositive and for different choices of Γ'_i-invariant metrics g the invariant can be made arbitrary close to zero.

As an immediate consequence of Propositions 3 and 5 we have:

Corollary 6. *On any manifold $M = \#a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$ for (a, b) in the region $R_0 \cap R_\epsilon$ there are infinitely many, free actions of the group Γ, denoted by $\Gamma_i, i \in \mathbb{N}$, such that the Yamabe invariant of the manifold M/Γ_i is negative, and there are also infinitely many free actions of the group Γ, denoted by $\Gamma'_i, i \in \mathbb{N}$, such that the Yamabe invariant of the manifold M/Γ'_i is zero.*

2. Exotic smooth structures

In this section we discuss the existence of exotic smooth structures and prove Theorem A.

Proof of Theorem A. It is enough to prove the statement for the points (a, b) satisfying the inequality (1). The second inequality is immediately implied by the first if one considers a change in orientation. The solubility property is preserved under this operation as $S^2 \times S^2$ admits a diffeomorphism which reverses its orientation, for example one can consider the antipodal map on one of the factors and identity on the other.

The construction relies on a result of Braungardt–Kotschick, which proves the following theorem about the geography of $\mathbb{CP}^2$-soluble symplectic manifolds:

Theorem 7 ([BrKo05, Theorem 4])**.** *For every $\epsilon > 0$ there is a constant $N_\epsilon > 0$ such that every lattice point (a, b) satisfying the conditions*

$$a \equiv 1 \mod 2 \tag{8}$$

$$b \leq 4 + 5a \tag{9}$$

$$b \geq \left(\frac{1}{2} + \epsilon\right)a + N_\epsilon \tag{10}$$

is realized by the Betti two invariants $(a, b) = (b_2^+, b_2^-)$ of infinitely many pairwise nondiffeomorphic simply connected minimal symplectic manifolds $M_{(a,b,i)}$, all of which are $\mathbb{CP}^2$-soluble.

As the manifolds considered are simply connected and symplectic then $a = b_2^+$ must be odd, namely condition (8), and furthermore condition (10) implies that a is a large integer. For minimal symplectic manifolds with $b^+ > 1$, Taubes [Tau96] showed that $c_1^2 = 4 + 5b_2^+ - b_2^- \geq 0$, which is equivalent to inequality (9). In the original paper, the geography statement is given in terms of the Chern number c_1^2, and the Todd number χ_h. These can be computed as $c_1^2 = 2\chi + 3\tau = 2 + 5b_2^+ - b_2^-$ and $\chi_h = \frac{1}{4}(\chi + \tau) = \frac{1}{2}(1 + b_2^+)$. After relabeling the constants ϵ and N_ϵ, the Kotschick-Braungardt inequality in [BrKo05] can be formulated as inequality (10). The infinitely many smooth structures in the theorem are constructed by doing logarithmic transforms of different multiplicities along a symplectic 2-torus of self-intersection zero. To prove that this construction generates infinitely many smooth structures, one considers a smooth invariant called the bandwidth $\mathcal{BW}$. This is defined to be the highest divisibility of the difference between two distinct Seiberg–Witten basic classes, see Ishida–LeBrun [IsLe02, Definition 2]. As the number of basic classes of a manifold is finite while $\sup_{i \in \mathbb{N}} \mathcal{B}W(M_{(a,b,i)}) = +\infty$, we conclude that infinitely many diffeotypes are represented.

The blow-up of the manifold $M_{(a,b,i)}$ at p points is a symplectic manifold diffeomorphic to $M_{(a,b,i)} \# p\overline{\mathbb{CP}^2}$. Moreover, it is simply connected and $\mathbb{CP}^2$-soluble. A trick of Wall [Wa64] tells us that if X is a nonspin manifold then $X \# (S^2 \times S^2)$ is diffeomorphic to $X \# (\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})$. Hence, $M_{(a,b,i)} \# p\overline{\mathbb{CP}^2}, p \geq 0$, is also $(S^2 \times S^2)$-soluble.

We denote by E_j the exceptional 2-spheres of self-intersection (-1) introduced by the blow-ups, and by $e_j = c_1(E_j)$ the Poincare dual of the homology class of E_j. If s is a basic class of $M_{(a,b,i)}$, then $\pm s + \sum_{j=1}^{p} \pm e_j$ are basic classes of $M_{(a,b,i)} \# p\overline{\mathbb{CP}^2}$. Hence $\mathcal{B}W(M_{(a,b,i)} \# p\overline{\mathbb{CP}^2}) \geq \mathcal{B}W(M_{(a,b,i)})$, and the bandwidth argument implies that the family $M_{(a,b,i)} \# p\overline{\mathbb{CP}^2}$ contains infinitely many diffeotypes. In particular, we showed that for each integer pair (a, b) satisfying condition (8) and condition (10) there are infinitely many pairwise nondiffeomorphic simply connected $S^2 \times S^2$-soluble 4-manifolds with $(b_2^+, b_2^-) = (a, b)$.

In order to generalize condition (8) in Theorem 7 we use the stable cohomotopy Seiberg–Witten invariant, due to Bauer–Furuta [BaFu04, Bau04]. In [Bau04, Proposition 4.5], Bauer considers manifolds obtained by taking the connected sum of $k = 2, 3$ or 4 manifolds $X = X_1 \# \cdots \# X_k$ satisfying $b_1(X_j) = 0, b_2^+(X_j) \equiv 3 \mod 4$, and if $k = 4$, $\sum_1^4 b_2^+(X_j) \equiv 4 \mod 8$. If the Seiberg–Witten invariant of a basic class s_j of X_j is odd, then X has a nonvanishing Bauer–Furuta invariant for basic classes of the form

$$\pm s_1 \pm \cdots \pm s_k, \quad \text{for } k = 2, 3 \text{ or } 4.$$

Let $X_{1,i} = M_{(a,b,i)} \# p\overline{\mathbb{CP}^2}$ and $X_2 = X_3 = X_4 = K3$, the simply connected complex surface with trivial canonical line bundle. The $K3$ surface can be realized, for example, as a hypersurface of degree 4 in $\mathbb{CP}^3$, and has $b_2^+ = 3$ and $b_2^- = 19$. Consider the following three families:

$$X_{1,i}, \quad X_{2,i} = X_{1,i} \# X_2,$$

$$X_{3,i} = X_{1,i} \# X_2 \# X_3 \# X_4, \quad \text{only when } b_2^+(X_{1,i}) + 3 \cdot 3 \equiv 4 \mod 8.$$

As $X_{1,i}, X_2, X_3, X_4$ are symplectic manifolds with $b_2^+ > 1$, Taubes [Tau94] showed that the Seiberg–Witten invariant associated to a canonical almost complex structure is ± 1. Hence we can apply Bauer's theorem and the bandwidth argument to conclude that infinitely many smooth structures are represented. These three families cover all the points in the region defined by inequality (1), up to a change of the constant N_ϵ.

The $K3$ manifold is $\mathbb{CP}^2$-soluble [Man80]. As $X_{1,i}$ is $(S^2 \times S^2)$-soluble, for $k = 2$, we can now conclude that:

$$\begin{aligned} X_{2,i} \# (S^2 \times S^2) &\cong X_{1,i} \# (S^2 \times S^2) \# K3 \cong (a+1)\mathbb{CP}^2 \# (b+p+1)\overline{\mathbb{CP}^2} \# K3 \\ &\cong (a+1+3)\mathbb{CP}^2 \# (b+p+1+19)\overline{\mathbb{CP}^2}. \end{aligned}$$

A similar computation for $k = 4$ shows that $X_{3,i}$ is $S^2 \times S^2$-soluble. This concludes the proof of Theorem A. □

3. Infinitely many free actions

In this section we use the examples in Theorem A to construct infinitely many actions of finite groups on manifolds with the conventional smooth structure and prove Theorem B.

Proof of Theorem B. We only need to prove that the statement is true for the points in the first subset, as considering the opposite orientation will imply the result for the second subset.

For any finite group Γ acting freely on S^3 consider the orientable rational homology sphere S_Γ with fundamental group $\pi_1(S_\Gamma) = \Gamma$ and universal cover $\#(d-1)(S^2 \times S^2)$, constructed by Ue [Ue96, Proposition 1-3]. Corollary 8 in [IsLe03] shows that the Bauer–Furuta invariant of $X_{k,i}\#S_\Gamma$ is nontrivial for monopole classes of the form $s_i \in H^2(X_{k,i}, \mathbb{Z}) \hookrightarrow H^2(X_{k,i}\#S_\Gamma, \mathbb{Z})/torsion$, where s_i is a basic class for the Bauer–Furuta invariant on $X_{k,i}$, for $k = 1, 2$ or 3. The bandwidth argument can be used again to argue that we constructed infinitely many diffeotypes.

Moreover, the universal cover of $X_{k,i}\#S_\Gamma$ is diffeomorphic to

$$\begin{aligned}&\#dX_{k,i}\#(d-1)(S^2 \times S^2)\\ &\cong [d \cdot b_2^+(X_{k,i}) + d - 1]\mathbb{CP}^2\#[d \cdot b_2^-(X_{k,i}) + d - 1]\overline{\mathbb{CP}^2}.\end{aligned}$$

Hence, on the conventional smooth structure of

$$\#a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2} = [d \cdot b_2^+(X_{k,i}) + d - 1]\mathbb{CP}^2\#[d \cdot b_2^-(X_{k,i}) + d - 1]\overline{\mathbb{CP}^2}$$

we constructed infinitely many free actions of the group Γ such that the quotient spaces are homeomorphic but pairwise nondiffeomorphic.

It is easy to check that these constructions cover all the points in the region R_ϵ satisfying conditions (D_1) and (D_2) for

$$N'_\epsilon = dN_\epsilon + \left(\frac{1}{2} - \epsilon\right)(d-1), \tag{11}$$

as we consider all the manifolds $X_{k,i}$ used in the proof of Theorem A. □

4. The sign of the Yamabe invariant

We consider next an invariant associated to Riemannian metrics. On a closed 4-manifold M, given a Riemannian metric g, one defines the Yamabe invariant of the conformal class $[g]$ of g as:

$$Y(M, [g]) = \inf_{\widetilde{g} \in [g]} \frac{\int_M s_{\widetilde{g}} d\mu_{\widetilde{g}}}{Vol(\widetilde{g})^{\frac{1}{2}}},$$

where $[g] = \{\widetilde{g} = e^f g |\ f : M \to \mathbb{R} \text{ smooth}\}$, see for example [Sch84, LePa87].

The Yamabe invariant is a diffeomorphism invariant of the manifold and is defined [Kob87, Sch87] as:

$$Y(M) = \sup_{[g]} Y(M, [g]). \tag{12}$$

Proof of Proposition 3. Given one of the Γ_i actions on M constructed in Theorem B the quotient M/Γ_i is diffeomorphic to $X_{k,i}\#S_\Gamma$, with the previous notations. As the manifold $X_{k,i}\#S_\Gamma$ has a nontrivial Bauer–Furuta invariant, then $Y(X_{k,i}\#S_\Gamma) \leq 0$ [IsLe03]. Moreover, by [IsLe03, Proposition 12 and 14] the Yamabe invariant satisfies:

$$Y(X_{k,i}\#S_\Gamma) \leq -4\pi\sqrt{2c_1^2(M_{(a,b,i)}) + 2\sum_{j=2}^{k} c_1^2(X_j)} = -4\pi\sqrt{2c_1^2(M_{(a,b,i)})},$$

where $M_{(a,b,i)}$ denotes the minimal symplectic manifold used in the proof of Theorem A. Moreover, we can assume that $M_{(a,b,i)}$ is a symplectic manifold of general type, meaning that $c_1^2(M_{(a,b,i)}) > 0$. This implies that $Y(X_{k,i}\#S_\Gamma) \leq -4\pi\sqrt{2}$ and, in particular, that there is a metric of negative constant scalar curvature in the conformal class of $\pi_*(g)$, where π_* denotes the push forward of the Γ_i-invariant metric g. Then the conformal class $[g]$ contains a negative constant scalar curvature metric (which is Γ_i-invariant) and hence, its Yamabe invariant is negative [Sch84]. This concludes the proof of the proposition. □

In order to prove Proposition 5 we need a different construction of group actions.

Proof of Proposition 5. The main blocks in the construction are the elliptic surfaces. As they are well understood from the algebraic geometry, differential topology, or Seiberg–Witten theory point of view, we give the reader a textbook reference [GoSt99, Chapter 3], where all these aspects are presented and complete references are included. We denote by $E(n)$ the simply connected elliptic surface with Euler characteristic $\chi(E(n)) = 12n$ and signature $\tau(E(n)) = -8n$ which admits a section. Let $E(n)_{p,q}$ the complex surface obtained by doing two logarithmic transforms of multiplicities p and q on two generic fibers and require that $(p, q) = 1$, in order for the manifold to be simply connected. It is well known that $E(n)_{p,q}$ is diffeomorphic to $E(n)_{p',q'}$ if and only if $\{p, q\} = \{p', q'\}$ for $n \geq 2$, [GoSt99, Corollary 3.3.7] and when $n = 1$ if $\{p, q\} = \{p', q'\}$ or if $1 \in \{p, q\} \cap \{p', q'\}$ [GoSt99, Theorem 3.3.8]. Moreover, $E(n)_{p,q}$ is spin if and only if n is even and pq is odd [GoSt99, Lemma 3.3.4]. In particular, if n is odd or if n is even and pq is even then the manifolds $E(n)_{p,q}$ are all nonspin and homeomorphic, for a fixed n. Moreover, in this case $E(n)_{p,q}$ is $\mathbb{CP}^2$-soluble [Man80, Theorem 2.15]. Using Wall's trick [Wa64], then this is also $S^2 \times S^2$-soluble.

As before, Γ is a finite group of order d acting freely on S^3 and S_Γ is Ue's rational homology sphere with fundamental group Γ. We consider the following three families of manifolds:

$$\mathcal{F}_1(n) = \{E(n)_{p,q}\#k\overline{\mathbb{CP}^2}\#S_\Gamma \mid k \in \mathbb{N}, (p,q) = 1, pq \text{ even} \\ \text{and if } n = 1 \text{ then } p, q \geq 2\}$$

$$\mathcal{F}_2(n) = \{E(n)_{p,q}\#k\overline{\mathbb{CP}^2}\#S_\Gamma\#E(2) \mid k \in \mathbb{N}, (p,q) = 1, pq \text{ even} \\ \text{and } n \text{ even}\}$$

$$\mathcal{F}_3(n) = \{E(n)_{p,q}\#k\overline{\mathbb{CP}^2}\#S_\Gamma\#3E(2) \mid k \in \mathbb{N}, (p,q) = 1, pq \text{ even} \\ \text{and } n \equiv 2 \mod 4\}.$$

For fixed n and k the manifolds in each family are homeomorphic, moreover by using the bandwidth arguments due to [IsLe02] we conclude that infinitely many diffeotypes are constructed. Their universal cover is the conventional nonspin manifold. The topological invariants of the first family are of the form

$$(a, b) = (b_2^+, b_2^-) = (2n - 1, 10n - 1 + k), \quad n \geq 1, k \geq 0,$$

covering the region

$$b \geq 5a + 4 \quad \text{for } a \text{ odd}.$$

Hence, their universal coverings cover all the integer lattice points in the region:

$$\left(\frac{1+b}{d} - 1\right) \geq 5\left(\frac{1+a}{d} - 1\right) + 4 \quad \text{or} \quad b \geq 5a + 4 \quad \text{for } \frac{1+a}{d} \text{ even.}$$

A similar computation for the second and third families shows that we are able to cover all the integer lattice points on the regions:

$$b \geq 5a + 4 + 4d \qquad \text{when} \quad n = \frac{1}{2}\left(\frac{1+a}{d} - 3\right) \text{ even,}$$
$$\text{or equivalently } \frac{1+a}{d} \equiv 3 \mod 4,$$
$$b \geq 5a + 4 + 12d \qquad \text{when} \quad n = \frac{1}{2}\left(\frac{1+a}{d} - 9\right) \equiv 2 \mod 4,$$
$$\text{or } \frac{1+a}{d} \equiv 5 \mod 8.$$

The union of these sets covers the needed region.

The Yamabe invariant of the manifolds in the families $\mathcal{F}_{1,2,3}$ is zero, by [IsLe03, Theorem A]. Hence, we can always choose a family of negative constant scalar curvature metrics for which the scalar curvature converges to zero. These lift to metrics on the universal covering such that the Yamabe invariants of their conformal classes are negative and converge to zero. □

Proof Corollary 6. Propositions 3 and 5 provide the constructions for the infinitely many actions for which the quotients have the Yamabe invariant negative or zero, respectively. The quotient manifolds are all homeomorphic by the Donaldson and Freedman's classification [Don83, Fre82], as they are all of the form $M_*\#S_\Gamma$, with the manifolds M_* being simply connected nonspin with the same topological invariants b_2^+ and b_2^-. They are pairwise nondiffeomorphic, as the quotients in different families have different Yamabe invariants while in the same family they are nondiffeomorphic by construction. □

Remark 13. If the Yamabe invariant of S_Γ is positive, then we can also construct an action on M such that M/Γ is diffeomorphic to $\#m\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}\#S_\Gamma$ and has positive Yamabe invariant. This is true, if for example $\Gamma = \mathbb{Z}_2$ and $S_{\mathbb{Z}_2}$ is the quotient of $S^2 \times S^2$ by the diagonal antipodal map on each factor. It is difficult to exhibit two homeomorphic, nondiffeomorphic structures with positive Yamabe invariants as there are no known invariants to distinguish such smooth structures.

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