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Subdivision rule constructions on critically preperiodic quadratic matings

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Abstract. 'Mating' describes a collection of operations that combine two complex polynomials to obtain a new dynamical system on a quotient topological 2-sphere. The dynamics of the mating are then dependent on the two polynomials and the manner in which the quotient space was defined, which can be difficult to visualize. In this article, we use Hubbard trees and finite subdivision rules as tools to examine quadratic matings with preperiodic critical points. In many cases, discrete parameter information on such quadratic pairs can be translated into topological information on the dynamics of their mating. The central theorems in this work provide methods for explicitly constructing subdivision rules that model nonhyperbolic matings. We follow with several examples and connections to the current literature.

Contents

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1. Introduction

Even the simplest of rational maps can have surprisingly complicated dynamics. Many rational maps may exhibit behavior resembling polynomials though, which are better understood. In the early 1980's, Douady described *polynomial mating*—a way to combine two polynomials in order to obtain a new map with dynamics inherited from both original maps [4]. Sometimes, this mating is dynamically similar to a rational map. In such a case, we can examine the dynamics of the constituent polynomials in the mating to better understand the rational map.

There are many kinds of polynomial matings, but their constructions typically begin in a similar manner. We consider the compactification $\widetilde{\mathbb{C}}$ of $\mathbb{C}$ given by adding in the circle at infinity, $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty \cdot e^{2\pi i\theta} | \theta \in \mathbb{R}/\mathbb{Z}\}$. Then, we take two monic polynomials of the same degree with locally connected and connected filled Julia sets acting on two disjoint copies of $\widetilde{\mathbb{C}}$. If we use these domains to form a quotient space in an appropriate manner, our polynomial pair will determine a map that descends to this new space. The map on the quotient space then exhibits combined dynamical behavior from the two polynomials. Both *topological matings* and *formal matings* are quotient maps formed in this manner, differing only in the equivalence relation which identifes points on our copies of $\widetilde{\mathbb{C}}$: In a formal mating, the opposing circles at infinity are identified so that the quotient space is a topological 2-sphere. In a topological mating, the domain of the map is given by a quotient space which identifies two filled Julia sets along their boundaries. (While we provide more details later, an excellent overview of some fundamental mating constructions is given in [10].)

Since the equivalence relation used in the topological mating is typically more complicated than that for the formal mating, one might naively expect that this difference in complexity is reflected in the associated quotient spaces. The domain of the topological mating can sometimes be surprising: by results of Lei, Rees, and Shishikura, it is possible to develop an equivalence relation on two connected filled Julia sets—including ones with no interior—such that the associated quotient space is a topological 2-sphere [8], [12], [13]. In [1], a general method is presented for developing the mating resulting from a given polynomial pairing—but this method is best suited for the hyperbolic case. As visualization of how the boundary identifications develop can be useful, this paper presents a construction to model the case involving two critically preperiodic polynomials. We expand here upon preliminary results given in [14].

In §2, we detail the prerequisites needed to define and construct polynomial matings. We also describe finite subdivision rules and Hubbard trees, and why their use is relevant here.

In §3 we introduce the essential construction for obtaining finite subdivision rules from matings, and demonstrate using several examples. We then close with connections to the current literature and future avenues for exploration in §4.

2. Prerequisites

2.1. Fundamentals. Let $\mathbb{S}^2$ denote a topological 2-sphere and $\widehat{\mathbb{C}}$ denote the Riemann sphere. In this paper, we will make reference to many self-maps of these spaces, and discuss instances in which these maps may 'behave similarly.' To clarify statements of this sort, one way to ensure similar behavior is the existence of a topological conjugacy. If f, g, and h are two maps such that $h \circ f \circ h^{-1} = g$, we may note that compositions of f and g are also topologically conjugate. We may then think of conjugation as a means of providing a coordinate change between the dynamical systems given by iterating f and g.

We may obtain a similar, but weaker statement on the similarity of behavior of two maps in the event that they are *Thurston equivalent* to each other:

Definition 2.1. Let $f, g : \mathbb{S}^2 \to \mathbb{S}^2$ be two branched mappings with respective postcritical sets P_f and P_g. The mappings f and g are said to be *Thurston equivalent* if and only if there exist homeomorphisms

$$h, h' : (\mathbb{S}^2, P_f) \to (\mathbb{S}^2, P_g)$$

such that the diagram

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{h'} & (\mathbb{S}^2, P_g) \\ \big\downarrow f & & \big\downarrow g \\ (\mathbb{S}^2, P_f) & \xrightarrow{h} & (\mathbb{S}^2, P_g) \end{array}$$

commutes, and h is isotopic to h' relative to P_f [5].

This suggests that when f and g are Thurston equivalent, f acts on a sphere containing its postcritical set much in the same manner that g acts on a sphere containing its postcritical set. These similarities in behavior may not necessarily pass through to iteration as with topological conjugacy, but we may be able to find maps h'', h''', etc. to extend the commutative diagram. It should also be noted that in the definition above, we may allow $\widehat{\mathbb{C}}$ to stand in for $\mathbb{S}^2$ if f or g necessitate such, since the pairing of a metric with $\mathbb{S}^2$ is not relevant to the definition.

In studying the behavior of an individual map and its iterates, a typical point of investigation is how the map affects the space that it acts on. Two of the most fundamental dynamical structures associated with a map are its Fatou and Julia sets, defined as follows:

Definition 2.2. A sequence of maps $f_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is said to *converge locally uniformly* to the limit $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ if for every compact set $K \subset \widehat{\mathbb{C}}$ the sequence $\{f_n|_K\}$ converges uniformly to $g|_K$ in the spherical metric on $\widehat{\mathbb{C}}$. We say that a collection $\mathcal{F}$ of holomorphic self-maps of $\widehat{\mathbb{C}}$ is *normal* if every infinite sequence of maps from $\mathcal{F}$ contains a subsequence which converges locally uniformly.

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function, regarded as an holomorphic self-map of the Riemann sphere. The *Fatou set* of f, denoted by $\mathcal{F}_f$ or $\mathcal{F}$, consists of all points on $\widehat{\mathbb{C}}$ with an open neighborhood U such that the restrictions of the iterates of f to U form a normal family of analytic functions on W. The *Julia set* of $f(z)$, denoted by either $\mathcal{J}_f$ or $\mathcal{J}$, is the set $\widehat{\mathbb{C}} \backslash \mathcal{F}$. [11]

In a sense, the Fatou set for f is the portion of the domain on which the iterates of f behave somewhat predictably based on the surrounding local behavior. The Julia set is the portion of the domain on which the iterates of f respond with much higher sensitivity to initial conditions. A rudimentary example is given by the map $z \mapsto z^2$: Off the unit circle in $\widehat{\mathbb{C}}$, iterates of this map converge to either 0 or ∞. On the unit circle, the map doubles arguments of complex points—which means that upon iteration, points on the unit circle may eventually have wildly differing itineraries from even their closest neighbors. Almost all points on the unit circle do not converge to any limit under these iterations. The unit circle here is the Julia set of $z \mapsto z^2$, while its complement is the Fatou set.

It should be noted that both the Fatou set and Julia set of $f(z)$ are sets that are invariant under f. Further, there are other equivalent means of defining the Julia set for certain maps: when f is a polynomial, the collection of points in $\widehat{\mathbb{C}}$ that stay bounded away from ∞ under iteration by f form a set called the *filled Julia set* for f, denoted either $\mathcal{K}_f$ or $\mathcal{K}$. For such f we may also obtain the Julia set by taking $\mathcal{J} = \partial\mathcal{K}$.

Later we will examine matings of quadratic polynomials. This necessitates an understanding of the Julia sets of the following special family of quadratic functions:

Definition 2.3. Let $c \in \mathbb{C}$ and $f_c(z) = z^2 + c$. The Mandelbrot set, M, is the set of all values of c such that the forward orbit of 0 under f_c is bounded. Equivalently, the Mandelbrot set may also be defined as the set of all values of c for which the Julia set of f_c is connected.

In this paper, we will emphasize quadratic polynomials which are *postcritically finite*—i.e., functions f for which the forward orbit of the critical point(s) yields a finite set of points, P_f. Then, any f_c which is postcritically finite must clearly be associated with some parameter c which is contained in M.

2.2. Parameter space. Suppose that c is contained in the Mandelbrot set, and that $\mathcal{K}_c$ is the filled Julia set of the map $f_c(z) = z^2 + c$. Since this implies that $\mathcal{K}_c$ is connected, $\widehat{\mathbb{C}} \backslash \mathcal{K}_c$ is conformally isomorphic to the complement of the closed unit disk via some holomorphic map $\phi : \widehat{\mathbb{C}} \backslash \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \backslash \mathcal{K}_c$. The map ϕ can be chosen to conjugate $z \mapsto z^2$ on $\widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ to f_c on $\hat{\mathbb{C}} \backslash \mathcal{K}_c$ so that $\phi(z^2) = f_c(\phi(z))$, in which case ϕ is a unique map.

Taking the image of rays of the form $\{re^{2\pi it} : r \in (1, \infty)\}$ under ϕ for fixed $t \in \mathbb{R}/\mathbb{Z}$ then yields the *external ray* of angle t, $R_c(t)$. (See Figure 1.) If $\mathcal{K}_c$ is locally connected, the map ϕ extends continuously to a map from the unit circle to the Julia set $\mathcal{J}_c$ and external rays of angle t are said to *land* at the point $\gamma(t) = \lim_{r \to 1^+} \phi(re^{2\pi it})$. The map $\gamma : \mathbb{R}/\mathbb{Z} \to \mathcal{J}_c$ is called the *Carathéodory semiconjugacy*, with the associated identity

$$\gamma(2 \cdot t) = f_c(\gamma(t))$$

in the degree 2 case. This identity allows us to easily track forward iteration of external rays and their landing points in $\mathcal{J}_c$ by doubling the angle of their associated external rays modulo 1.

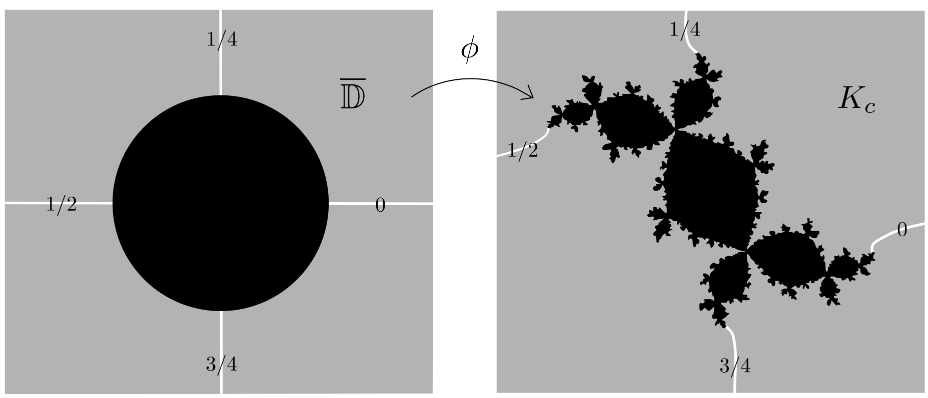

FIGURE 1. The conformal isomorphism ϕ and selected external rays for the rabbit polynomial.

The work in this paper will be restricted to the use of polynomials whose parameters are obtained from *Thurston–Misiurewicz* points—values of c at which the critical point of f_c is strictly preperiodic. Such values of c are always contained in the boundary of the Mandelbrot set. Critically preperiodic polynomials are typically parameterized by the angle θ of some external ray landing at the critical value rather than by the critical value c. (In the event that the critical value is accessible by multiple external rays, it is possible for multiple parameters to refer to the same polynomial.) We will follow this convention from this point on, using f_θ in lieu of f_c. These critically preperiodic polynomials have filled Julia sets that are *dendrites*: locally connected continuua that contain no simple closed curves. In other words, the filled Julia set of such a polynomial possesses a possibly infinite tree-like structure and has no interior. Further, since these f_θ have Julia sets that are locally connected, recall that external rays land on $\mathcal{J}_\theta$. This means that the conformal isomorphism ϕ and Carathéodory semiconjugacy γ can be used to recover the mapping behavior of f_θ on its Julia set. As a brief example, consider Figure 2: we could obtain that the critical orbit is preperiodic and follows the pattern $c_0 \mapsto c_1 \mapsto c_2 \mapsto c_3 \mapsto c_2$ by evaluation in $f_{1/6}$, or we could double the angles of external rays landing at these points to obtain the same pattern.

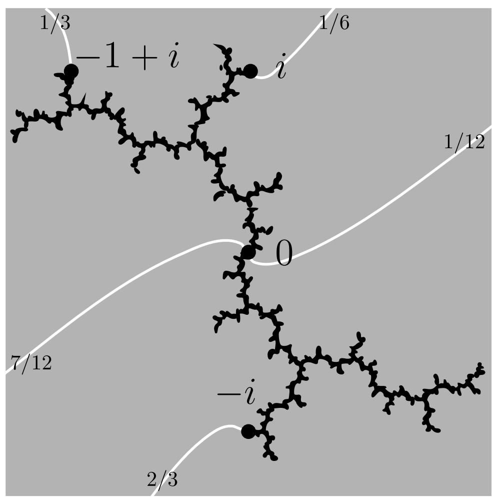

FIGURE 2. External rays landing on the critical orbit of $f_{1/6}(z) = z^2 + i$.

2.3. Matings. The operation of 'mating' may refer to one of several ways to combine two polynomials to form a new map. We will focus on three types of mating operations in this paper: formal, topological, and essential. In general, the $\perp\!\!\!\perp$ symbol denotes that a mating operation is being performed between two polynomials, and each kind of mating operation is defined using a prescribed equivalence relation on the polynomial domains. We will utilize $\perp\!\!\!\perp_f$, $\perp\!\!\!\perp_t$, and $\perp\!\!\!\perp_e$ respectively to reflect when we are discussing the formal, topological, or essential mating; and will use subscripts in a similar manner to identify the associated equivalence relations.

We now define the formal mating. Let $f_\alpha : \widetilde{\mathbb{C}}_\alpha \to \widetilde{\mathbb{C}}_\alpha$ and $f_\beta : \widetilde{\mathbb{C}}_\beta \to \widetilde{\mathbb{C}}_\beta$ be postcritically finite monic quadratic polynomials taken on two disjoint copies of $\widetilde{\mathbb{C}}$. Form the topological 2-sphere $\mathbb{S}^2$ by taking $\mathbb{S}^2 = \widetilde{\mathbb{C}}_\alpha \bigsqcup \widetilde{\mathbb{C}}_\beta / \sim_f$, where $\sim_f$ identifies $\infty \cdot e^{2\pi i t}$ on $\widetilde{\mathbb{C}}_\alpha$ with $\infty \cdot e^{-2\pi i t}$ on $\widetilde{\mathbb{C}}_\beta$. This yields a topological 2-sphere by gluing two copies of $\tilde{\mathbb{C}}$ together along their circles at infinity with opposing angle identifications. (See Figure 3.) This quotient space serves as the domain of the *formal mating* $f_\alpha \perp\!\!\!\perp_f f_\beta$, which is the map that applies f_α and f_β on their respective hemispheres of $\mathbb{S}^2$. The Carathéodory semiconjugacy guarantees that $f_\alpha \perp\!\!\!\perp_f f_\beta$ is well-defined on the equator and provides a continuous branched covering of $\mathbb{S}^2$ to itself. We will use $F = f_\alpha \perp\!\!\!\perp_f f_\beta$ to denote the formal mating whenever it is unambiguous to do so.

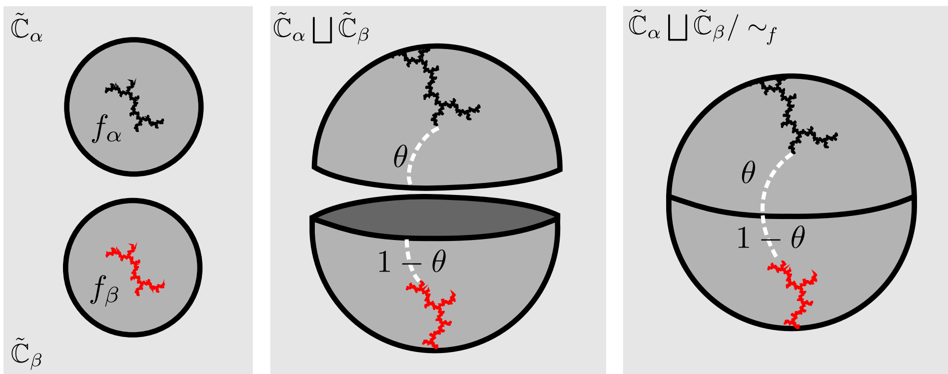

FIGURE 3. Steps in the formation of the formal mating.

The *topological mating* $f_\alpha \perp\!\!\!\perp_t f_\beta$, on the other hand, is formed by using the quotient space $\mathcal{K}_\alpha \bigsqcup \mathcal{K}_\beta / \sim_t$, where $\sim_t$ identifies the landing point of $R_\alpha(t)$ on $\mathcal{J}_\alpha$ with the landing point of $R_\beta(-t)$ on $\mathcal{J}_\beta$. This glues the Julia sets of f_α and f_β together at opposing external angles. Similar to the formal mating, we obtain the map $f_\alpha \perp\!\!\!\perp_t f_\beta$ by applying f_α and f_β on their respective filled Julia sets. The Carathéodory semiconjugacy similarly guarantees that the resulting map is well-defined and continuous, but it is possible that the quotient space is no longer a topological 2-sphere, even though there is an induced map.

The quotient space obtained in developing the topological mating sometimes is a 2-sphere, however—and further, $f_\alpha \perp\!\!\!\perp_t f_\beta$ may be topologically conjugate to a rational map on the Riemann sphere. Such a rational map is called a *geometric mating* of f_α and f_β. The following elegant result highlights a case that we will consider in this paper:

Theorem 2.4 (Lei, Rees, Shishikura)**.** *The topological mating of the post-critically finite maps $z \mapsto z^2 + c$ and $z \mapsto z^2 + c'$ is Thurston equivalent to a rational map on $\hat{\mathbb{C}}$ if and only if c and c' do not lie in complex conjugate limbs of the Mandelbrot set* [8], [12], [13].

This is useful since we can determine if the mating exists merely from the values of the parameters c and c'. Given that this mating acts on a 2-sphere obtained by identifying the boundaries of two Julia sets, and that one or both of these Julia sets may be dendrites though, this result may appear somewhat counterintuitive.

To help understand the boundary identifications in the quotient space of the topological mating, we will examine the *essential* mating, $f_\alpha \amalg_e f_\beta$ in the setting of Theorem 2.4. (Similar to our convention for the formal mating, we will use $E = f_\alpha \amalg_e f_\beta$ to denote the essential mating whenever it is unambiguous to do so.) Starting with the 2-sphere $\mathbb{S}^2$ developed as the domain of the formal mating F, the essential mating is constructed as detailed below and in [8].

Definition 2.5. Let $\{l_1, \ldots, l_n\}$ be the set of connected graphs of external rays on $\mathbb{S}^2$ containing at least two points of the postcritical set P_F, and let $\{\tau_1, \ldots, \tau_m\}$ be the set of maximal connected graphs of external rays in $\bigcup_{k \in \mathbb{N}} \bigcup_{i=1}^{n} F^{-k}(l_i)$ containing at least one point on the critical orbit of F. Take each of the $\{\tau_1, \ldots, \tau_m\}$ to be an equivalence class of the equivalence relation $\sim_e$. (See the white dashed lines on the top left of Figure 4.) Note that in the setting of Theorem 2.4, $\mathbb{S}'^2 = \mathbb{S}^2 / \sim_e$ is homeomorphic to a sphere since these equivalence classes will contain no loops. Further, F maps equivalence classes to equivalence classes, so letting $\pi : \mathbb{S}^2 \to \mathbb{S}'^2$ denote the natural projection yields that $\pi \circ F \circ \pi^{-1}$ is well-defined and preserves the mapping order of the equivalence classes $\{\tau_1, \ldots, \tau_m\}$. (See the mapping behavior demonstrated by Case A in Figure 4.)

This composition is not necessarily a branched covering if $\{\tau_1, \ldots, \tau_m\}$ is nonemtpy, though. (In this case, $F^{-1}(\{\tau_1, \ldots, \tau_m\})$ will contain a component which is not an element of $\{\tau_1, \ldots, \tau_m\}$—and this component will be a collection of arcs mapping to a point under $\pi \circ F \circ \pi^{-1}$, as in Case B of Figure 4.) To rectify this, set V_j to be an open neighborhood of τ_j such that $V_j \cap (P_F \cup \Omega_F) = \tau_j \cap (P_f \cup \Omega_F)$ for each j, and such that distinct V_j are nonintersecting. (See the dark grey region on the top left of Figure 4.) For each j, denote by $\{U_{ij}\}$ the set of connected components of $F^{-1}(V_j)$ for which $U_{ij} \cap \bigcup_{p=1}^{m} \tau_p = \varnothing$.

Finally, we set $E : \mathbb{S}'^2 \to \mathbb{S}'^2$ to be equivalent to $\pi \circ F \circ \pi^{-1}$ off of the set $\bigcup_{i,j} \pi(U_{ij})$, and for each i, j set $E : \pi(U_{ij}) \to \pi(V_j)$ to be a homeomorphism that extends continuously to the boundary of each $\pi(U_{ij})$. E is the *essential mating* of f_α and f_β.

This is a dense definition, so we will unpack it a bit: the essential mating resembles the formal mating, except we have 'collapsed' the domain of the formal mating along certain external ray pair groupings, and tweaked the resulting quotient map to ensure that it is a branched covering of a 2-sphere. The ray pair groupings τ_j that we collapse to form the essential mating are those which connect two or more postcritical points of F, and preimages of

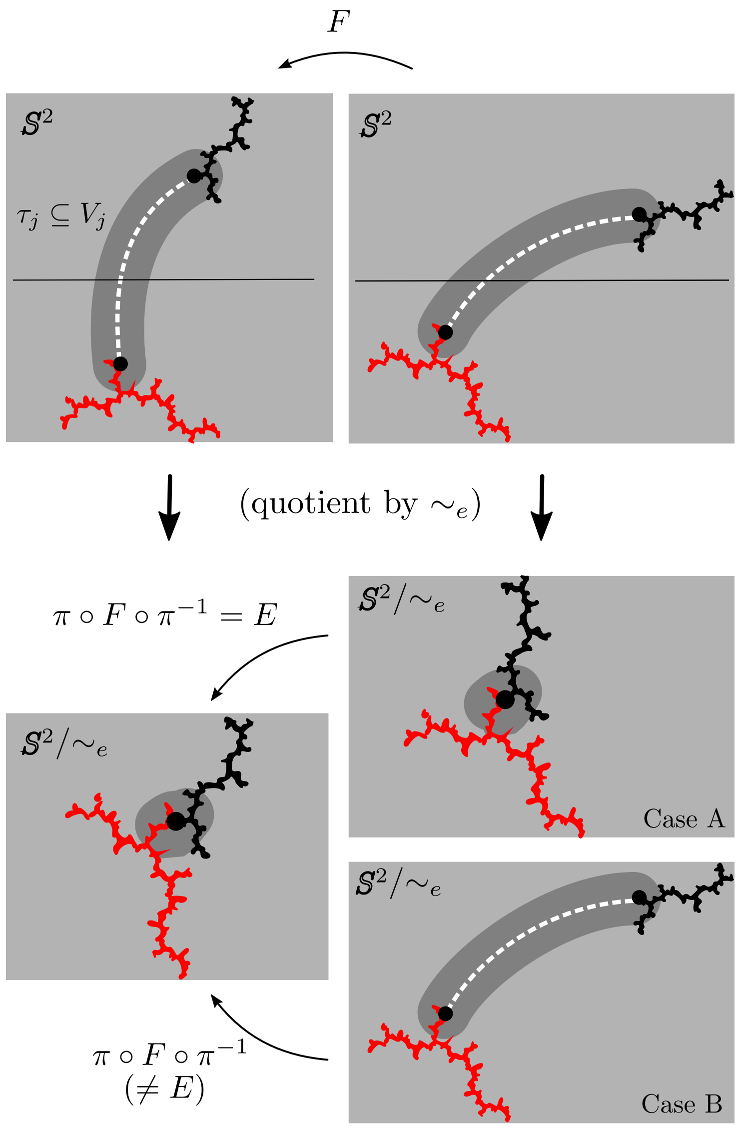

FIGURE 4. Above, the action of F on an external ray pair grouping which maps to some τ_j. Below, note that on much of the 2-sphere, the essential mating is equivalent to the map $\pi \circ F \circ \pi^{-1}$. This would be locally true for Case A where we have behavior that locally resembles a branched cover, but not true for Case B where we have an arc which maps to a point.

these ray pair groupings containing a point on the critical orbit of F, such as the white dashed lines demonstrate at the top of Figure 4.

Note that F maps each of the τ_j forward to some element in $\{\tau_1, \ldots, \tau_m\}$. This is not necessarily true for inverse images under F, though. On the bottom of Figure 4, we demonstrate two possible cases. In Case A, the inverse image of τ_j under F has a component which *is* an element of $\{\tau_1, \ldots, \tau_m\}$; in Case B there is a component of the preimage of τ_j which is *not*. In most locations on $\mathbb{S}'^2$ (including those like the one pictured in Case A), the essential mating E is defined as $\pi \circ F \circ \pi^{-1}$. Situations like Case B on the other hand force $\pi \circ F \circ \pi^{-1}$ to not be a branched covering—so we instead define E as a homeomorphism on open neighborhoods of problematic ray pair groupings such as this.

Despite the appearance of E being defined rather arbitrarily in the last step, the essential mating is uniquely determined up to Thurston equivalence, and is in fact a degree 2 branched covering map which is Thurston equivalent to the associated topological mating [8]. In a sense, the essential mating captures the "essential" identifications—i.e., mostly ones on the critical orbit—that are made in forming the topological mating. We may see Figure 5 for an example highlighting the key identifications in the mating of $f_{1/6}$ with itself—the formal self-mating is a map with six postcritical points, while the essential and topological self-matings are both maps with only four.

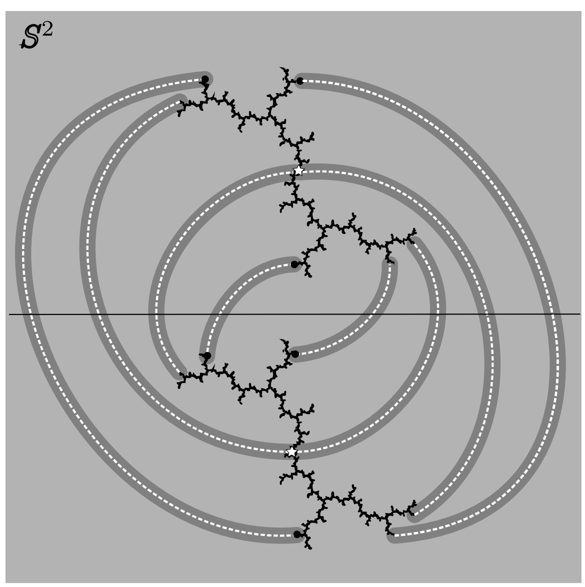

FIGURE 5. Depicted is the domain $\mathbb{S}^2$ for the formal self-mating of $f_{1/6}$. The white dashed lines represent τ_j and the open grey regions surrounding these are the corresponding V_j.

The essential mating mostly behaves like the map F with the fundamental difference being that the domain and range of E are a quotient space where important identifications on the critical orbit of F are collapsed together. It should thus be noted that if no postcritical points of F can be connected by a graph of external rays on $\mathbb{S}^2$, then the collection of τ_j in Definition 2.5 is actually the null set and $\sim_e$ is the equality equivalence relation. In this case, the natural projection $\pi : \mathbb{S}^2 \to \mathbb{S}'^2$ behaves much like the identity map on $\mathbb{S}^2$. Further, no τ_j means no neighborhoods V_j, and thus no sets U_{ij}. Since off of the U_{ij} (i.e. on the whole sphere) we have $E = \pi \circ F \circ \pi^{-1}$, this implies that we could take the essential and formal mating to be the same map in any case where there are no postcritical identifications under $\sim_t$ or $\sim_e$. This would, for example, be the case when mating two polynomials with periodic critical points.

2.4. Finite subdivision rules. Our ultimate motivation in examining the essential mating is to develop a tiling construction that highlights the identifications formed in the topological mating. We will develop this construction using *finite subdivision rules.*

Definition 2.6. A *finite subdivision rule* $\mathcal{R}$ consists of the following three components:

1. A tiling. Formally, this is a finite 2-dimensional CW complex $S_{\mathcal{R}}$, called the subdivision complex, with a fixed cell structure such that $S_{\mathcal{R}}$ is the union of its closed 2-cells. We assume that for each closed 2-cell $\tilde{s}$ of $S_{\mathcal{R}}$ there is a CW structure s on a closed 2-disk such that s has ≥ 3 vertices, the vertices and edges of s are contained in ∂s, and the characteristic map $\psi_s : s \to S_{\mathcal{R}}$ which maps onto $\tilde{s}$ restricts to a homeomorphism on each open cell.
2. A subdivided tiling. Formally, this is a finite 2-dimensional CW complex $\mathcal{R}(S_{\mathcal{R}})$ which is a subdivision of the above CW complex $S_{\mathcal{R}}$.
3. A continuous cellular map $g_{\mathcal{R}} : \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$, called the subdivision map, whose restriction to any open cell is a homeomorphism. [3]

In essence, a finite subdivision rule is a finite combinatorial rule for subdividing tilings on some 2-complex. We restrict, however, to tilings formed by "filling in" connected finite planar graphs on a 2-sphere with open tiles that are topological polygons. None of these tiles are allowed to be monogons or digons, and further, each edge of the tiling must be a boundary edge to some tile. These tiles may be nonconvex, though—to the potential extreme of allowing both sides of a single edge to form two sides of the boundary of a single tile. (For example, a line segment with both end points and the midpoint marked on the 2-sphere forms the boundary of a topological quadrilateral.)

Once we subdivide a tiling, we will need a map that takes open cells of the subdivision tiling homeomorphically to open cells of the original tiling.

Only when we have all three components—the initial tiling, the subdivision tiling, and a subdivision map—do we have a complete finite subdivision rule. Then, this rule can be applied recursively to yield iterated subdivisions of the original tiling.

Example 2.7. Consider Figure 6: $\hat{\mathbb{C}}$ is oriented so that the marked points $0, \pm 1$, and ∞ all lie on the equator. The equator and marked points determine a graph which yields a tiling of $\hat{\mathbb{C}}$ into two topological quadrilaterals. If we take a preimage of this structure under the map $z \mapsto z^2$, we obtain a tiling that has four quadrilaterals—each of which maps homeomorphically onto one of the quadrilaterals in the original tiling. Here, the structure on the left is our tiling, the structure on the right is the subdivided tiling, and the map $z \mapsto z^2$ is the subdivision map.

While a finite subdivision rule may be defined using analytic maps and embedded tilings as in the previous example, this is not necessary. We can use the mapping behavior of n-cells in a tiling to determine the mapping behavior of $(n+1)$-cells, thus obtaining a subdivision map based on combinatorial data. The reader may reference Cannon, Floyd, and Parry in [3] for a more detailed treatment of this topic.

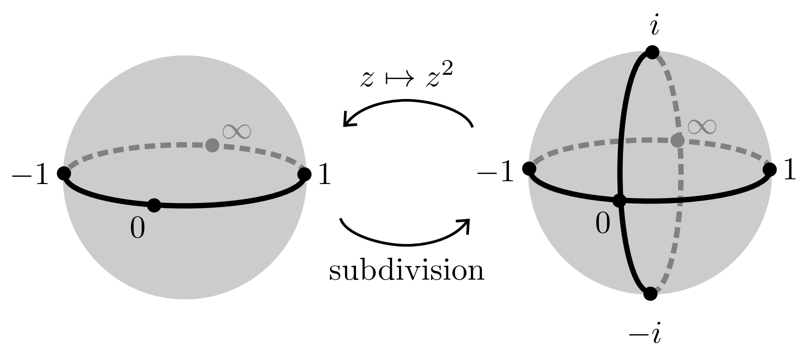

FIGURE 6. A rudimentary tile subdivision.

2.5. Hubbard trees. In order to build a finite subdivision rule later on, it will be helpful to have a finite invariant structure in mind to determine the tiling. The Julia set is invariant under iteration of its associated polynomial, but the structure of the Julia set is more complicated than we would like to use as a starting point for a finite subdivision rule. Thus, we would like to work with a discrete approximation to the Julia set: the *Hubbard tree.*

(Note: Hubbard Trees are defined in [6] using *allowable arcs.* The construction of an allowable arc is simplified considerably for the case where f has a dendritic Julia set, so for the reader's convenience we present a definition restricted to this case here.)

Definition 2.8. Let $f_\theta : \mathbb{C} \to \mathbb{C}$ be given by $f_\theta(z) = z^2 + c$ for some Misiurewicz point c, and let f_θ have Julia set $\mathcal{J}_\theta$ and postcritical set P_{f_θ}.

We say that a subset X of $\mathcal{J}_\theta$ is *allowably connected* if $x, y \in X$ implies that there is a topological arc in X that connects x and y. The *allowable hull* of a subset A in $\mathcal{J}_\theta$ is then the intersection of all allowably connected subsets of $\mathcal{J}_\theta$ which contain A. Finally, the *Hubbard tree* of f_θ is the allowable hull of P_{f_θ} in $\mathcal{J}_\theta$.

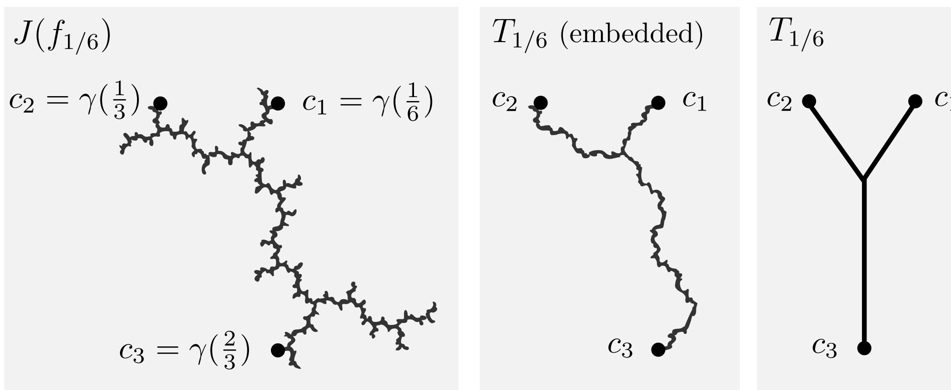

FIGURE 7. The Julia set and Hubbard trees for $f_{1/6}(z) = z^2 + i$.

The Hubbard tree as defined above is embedded in $\mathbb{C}$ and topologically equivalent to the notion of an *admissible Hubbard tree* with preperiodic critical point as discussed in [2]. The notes of Bruin and Schleicher in [2], however, emphasize the combinatorial structure of the Hubbard tree as a graph with vertices marked by elements of P_{f_θ}, rather than as an embedded object in the complex plane. (See Figure 7.) They present several explicit algorithms that can be used to construct a topological copy of T_θ from the parameter θ, building heavily on the notion that quadratic maps are local homeomorphisms off of their critical points, and degree two at their critical points. We can expand upon these observations regarding the behavior of quadratic polynomials to determine what images and preimages of the Hubbard tree T_θ under f_θ will look like: forward images are invariant and f_θ maps the tree onto itself, every point in T_θ has at most two inverse images under f_θ, f_θ acts locally homeomorphically on $T\theta$ everywhere except at the critical point, and subsequent preimages of T_θ under f_θ give discrete approximations to $\mathcal{J}_\theta$. (The nth preimage of an tree T_θ under its associated polynomial f contains 2^n miniature copies of the tree which each map homeomorphically onto the tree via $f^{\circ n}$, as in Figure 8.) In addition, Hubbard trees have many desirable characteristics that we will later require the 1-skeletons of subdivision complexes to possess—namely, being planar, finite, forward invariant, and containing the postcritical set.

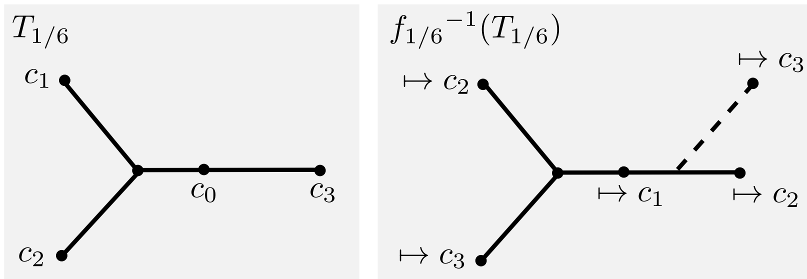

FIGURE 8. Preimages of a Hubbard tree under its associated polynomial.

3. An essential finite subdivision rule construction

Recall that the emphasis for this paper is on the nonhyperbolic case in which two postcritically finite polynomials with dendritic Julia sets are mated. If we further restrict our work to the setting where the critical values of these polynomials are not in complex conjugate limbs of the Mandelbrot set, the topological mating is Thurston-equivalent to a rational map on the Riemann sphere. In order to understand how the quotient space for the mating comes together, we will construct a combinatorial model of the mating in the form of a finite subdivision rule.

3.1. The essential construction. An ideal finite subdivision rule should be based upon a subdivision map that is dynamically similar to the topological and geometric matings. The formal mating will not always suffice: if any postcritical points of F are contained in the same equivalence class of $\sim_t$, F is not Thurston-equivalent to the topological mating. On the other hand, the essential mating *is* Thurston-equivalent to the topological and geometric matings—thus, it is a desirable subdivision map.

This leaves us to determine the tiling and subdivided tiling for a given essential mating. The Hubbard trees associated with the polynomial pair for our essential mating are a good start for a tiling 1-skeleton, as they record much of the dynamical information associated with the polynomials. However, there are two trees associated with any polynomial pair, and we need to reconcile these structures on $\mathbb{S}^2/\sim_e$. For many polynomial pairings, this problem solves itself quite naturally:

Definition 3.1 (Finite subdivision rule construction, essential type)**.** Let f_α and f_β be critically preperiodic monic quadratic polynomials such that $x \sim_e y$ for some points $x \in T_\alpha$, $y \in T_\beta$.

Give $T_\alpha \bigsqcup T_\beta / \sim_e$ a graph structure on the quotient space of the essential mating by marking all postcritical points and branched points as vertices. (If need be, mark additional periodic or preperiodic points on T_α or T_β and the points on their forward orbits to avoid tiles forming digons.) The associated two dimensional CW complex for this structure will yield the subdivision complex, $S_{\mathcal{R}}$.

Select a construction of the essential mating E and set $\mathcal{R}(S_{\mathcal{R}})$ to be the preimage of $S_{\mathcal{R}}$ under E, taking preimages of marked points of $S_{\mathcal{R}}$ to be marked points of $\mathcal{R}(S_{\mathcal{R}})$.

If $\mathcal{R}(S_{\mathcal{R}})$ is a subdivision of $S_{\mathcal{R}}$ and if the essential mating $E : \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$ is a subdivision map, then $\mathcal{R}$ is a finite subdivision rule and the above construction is labelled of *essential type*.

The central idea behind this approach is that groupings of points on the critical orbit of F which are identified under $\sim_e$ must be collapsed if we wish to use the essential mating as a subdivision map. The quotient of $T_\alpha \bigsqcup T_\beta$ under $\sim_e$ is a connected planar graph when $\sim_e$ is associated with a nontrivial essential mating, as in the example in Figure 9. If we "fill in" the faces of this graph with polygonal tiles, we obtain a subdivision complex $S_{\mathcal{R}}$ which in many cases subdivides when we consider its pullback by E. We formalize these notions with the following theorem:

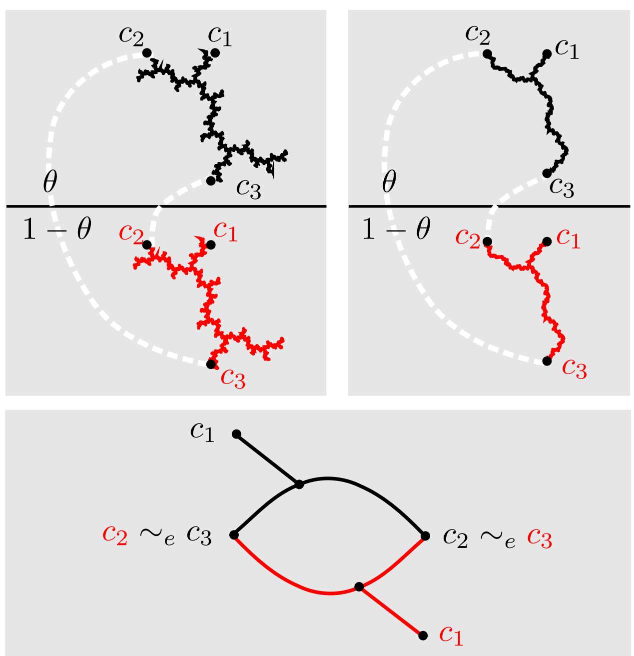

FIGURE 9. External ray-pairs which connect the periodic postcritical points of $f_{1/6} \perp\!\!\!\perp_f f_{1/6}$ also modeled on Hubbard trees. The rays shown here collapse under $\sim_e$.

Theorem 3.2. *Let F be the formal mating of f_α and f_β. The essential type construction fails to yield a finite subdivision rule generated by this* *polynomial pairing if and only if there exists some x, y in $T_\alpha \bigsqcup T_\beta$ with $x \sim_t y$, $x \nsim_e y$, and $F(x) \sim_e F(y)$.*

Proof. We prove the backward direction by contradiction. Using the notation developed in Definition 2.5 for the essential mating, if such an x and y exist, we must have that $x, y \in U_{ij}$ with $F(x), F(y) \in V_j$ for some i, j. Recall that the essential mating E is then a homeomorphism from $\pi(U_{ij})$ to $\pi(V_j)$.

Since $F(x) \sim_e F(y)$, we can choose V_j so that it contains no other marked points of our 1-skeleton, and so that $\pi(V_j)$ intersected with the 1-skeleton of $S_{\mathcal{R}}$ yields a connected subset of $\mathbb{S}^2$. E being a homeomorphism then implies that the 1-skeleton of $E^{-1}(S_{\mathcal{R}}) = \mathcal{R}(S_{\mathcal{R}})$ intersected with $\pi(U_{ij})$ is connected. $\pi(U_{ij})$ intersected with the 1-skeleton of $S_{\mathcal{R}}$, however will not be connected since $x \nsim_e y$. This suggests that at least one edge must have been added to the 1-skeleton of $\mathcal{R}(S_{\mathcal{R}})$ in this neighborhood during a subdivision of $S_{\mathcal{R}}$. Thus, the intersection of U_{ij} with $\mathcal{R}(S_{\mathcal{R}})$ should have at least two marked points corresponding to the endpoints of this edge (and potentially others) added during the subdivision of $S_{\mathcal{R}}$. This cannot be so, however, since by the construction this intersection should contain only the single marked point $E^{-1} \circ \pi \circ F(x)$. Thus, the construction does not yield a finite subdivision rule in this case.

We now prove the forward direction by contrapositive: suppose that there exist no x, y in $T_\alpha \bigsqcup T_\beta$ with $x \sim_t y$, $x \nsim_e y$, and $F(x) \sim_e F(y)$. Then for every U_{ij}, at least one of $U_{ij} \cap T_\alpha$ or $U_{ij} \cap T_\beta$ must be $\varnothing$. We will now use E to denote the essential mating formed with the additional restrictions that $E|_{U_{ij} \cap \pi(T_\alpha \bigsqcup T_\beta)} = \pi \circ F \circ \pi^{-1}$, and that E be a homeomorphism that extends continuously to this new boundary on the remainder of the $\pi(U_{ij})$. This agrees with the definition of E off $\bigcup_{i,j} U_{ij}$, and still permits E to be a homeomorphism from each U_{ij} to its respective V_j—that is, we still have that E is an essential mating as defined before; we are just being more specific regarding the homeomorphism used in the final step of its construction.

We will consider the essential type construction performed with this essential mating, E, and show that it yields a finite subdivision rule. Recall that we need three things for a finite subdivision rule: a tiling, a subdivided tiling, and a subdivision map.

For the tiling $S_{\mathcal{R}}$, note that "filling in" the faces of a finite, connected, planar graph with open 2-cell tiles guarantees a 2 dimensional CW complex. The 1-skeleton of our tiling starts with two disjoint Hubbard trees, which on their own would be finite and planar, but disconnected. The construction requires that the essential mating is nontrivial with postcritical identifications between trees on $\mathbb{S}^2/\sim_e$ though, so the 1-skeleton is connected and we obtain the desired CW complex. The final requirements for a tiling forbid monogon and digon tiles, but the construction expressly accounts for this by requiring additional marked points to fix potentially errant tiles.

For the subdivision map, we need to show that E restricted to any open cell of $\mathcal{R}(S_{\mathcal{R}})$ maps homeomorphically onto some open cell of $S_{\mathcal{R}}$. Since $\mathcal{R}(S_{\mathcal{R}})$ is obtained by pulling back the structure of $S_{\mathcal{R}}$ under E, this follows from the fact that the critical and postcritical set of E are marked as vertices in $S_{\mathcal{R}}$. Marked points of $\mathcal{R}(S_{\mathcal{R}})$ must map to marked points of $S_{\mathcal{R}}$, and since E is a branched covering it must map homeomorphically on the remaining open tiles and edges.

This leaves checking that the tiling $\mathcal{R}(S_{\mathcal{R}})$ is a tiling which is a subdivision of $S_{\mathcal{R}}$. Again, as $\mathcal{R}(S_{\mathcal{R}})$ is obtained by pulling back the structure of $S_{\mathcal{R}}$ under E, it will yield a tiling—but it is not obvious that this tiling results from a subdivision of $S_{\mathcal{R}}$. We will need to check that the open tiles and edges of $\mathcal{R}(S_{\mathcal{R}})$ resemble open tiles and edges of $\mathcal{R}(S_{\mathcal{R}})$ which have been subdivided by open edges and vertices. We will obtain this condition if the 1-skeleton of $\mathcal{R}(S_{\mathcal{R}})$ contains a subdivision of the 1-skeleton of $S_{\mathcal{R}}$. This will be true if the 1-skeleton of $S_{\mathcal{R}}$ is forward invariant under E.

By the essential construction, note that the 1-skeleton of $S_{\mathcal{R}}$ is given by points in $\pi(T_\alpha \bigsqcup T_\beta)$. The definition of our essential mating E, however, yields that $E|_{\pi(T_\alpha \bigsqcup T_\beta)} = \pi \circ F \circ \pi^{-1}$. Thus, E maps our 1-skeleton to $\pi \circ F(T_\alpha \bigsqcup T_\beta)$. Recall that the formal mating F acts as f_α on T_α and as f_β on T_β, though. Since Hubbard trees are forward invariant under their associated polynomials, F preserves $T_\alpha \bigsqcup T_\beta$, and so our 1-skeleton is mapped to itself under E.

Since we have shown that E acts as a subdivision map from the subdivided tiling $\mathcal{R}(S_{\mathcal{R}})$ to the tiling $S_{\mathcal{R}}$, the essential type construction yields a finite subdivision rule. □

In simpler words, Theorem 3.2 tells us that we will have a problem building a finite subdivision rule using the essential type construction exactly when two points are identified by $\sim_e$, but their preimages are not.

3.2. An example. To highlight a case where the essential construction yields a finite subdivision rule, we consider the essential mating $f_{1/6} \perp\!\!\!\perp_e f_{1/6}$. The essential construction prescribes that we start with the disjoint union of Hubbard trees of the two constituent polynomials in the mating, $T_{1/6}$ and $T_{1/6}$, and then take a quotient under the relation $\sim_e$ associated with this mating. The Hubbard tree is presented on the left of Figure 10, and $T_{1/6} \bigsqcup T_{1/6} / \sim_e$ is shown on the right. (Recall that a pair of external rays adjacent to the same spot on the equator of $\mathbb{S}^2$ will land at θ and $1-\theta$ on opposing Julia sets in the formal mating. Thus, if there is a θ and $1-\theta$ pairing of postcritical points on opposing trees, these points collapse under $\sim_e$.) The resulting 1-skeleton yields a 2-tile subdivision complex $S_{\mathcal{R}}$.

We now need to take the pullback of $S_{\mathcal{R}}$ under E to obtain the subdivided complex $\mathcal{R}(S_{\mathcal{R}})$. It may not be immediately obvious how to determine what the resulting 1-skeleton looks like, but the Hubbard tree structure is helpful here: the preimage of a Hubbard tree under its associated polynomial yields

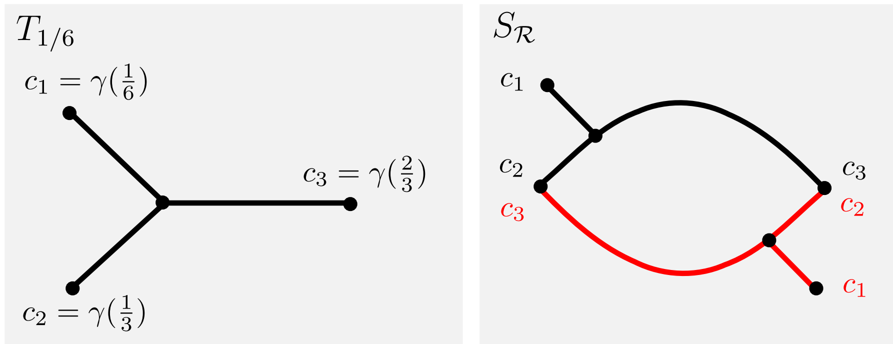

FIGURE 10. The Hubbard tree for $f_{1/6}$, and the 1-skeleton of the essential type subdivision complex, $S_{\mathcal{R}}$, for $f_{1/6} \amalg_e f_{1/6}$

two miniature copies of the tree which map homeomorphically onto the original tree, joined at the critical point. This suggests "missing limbs" that when filled in will subdivide the tiles of $S_{\mathcal{R}}$. Noting where each of the marked points maps forward shows where to embed these limbs, since the 1-skeleton of $\mathcal{R}(S_{\mathcal{R}})$ should map homeomorphically onto the 1-skeleton of $S_{\mathcal{R}}$ off of the critical point. This yields $\mathcal{R}(S_{\mathcal{R}})$, as shown in the right side of Figure 11.

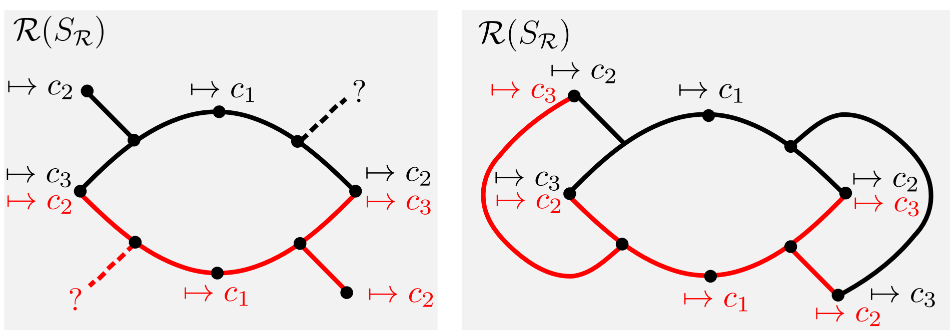

FIGURE 11. Determining the essential type subdivided complex, $\mathcal{R}(S_{\mathcal{R}})$

An important thing to note in the above example is that we can obtain up to the first subdivision utilizing the given essential mating map, but that subsequent pullbacks by E do not subdivide in the manner suggested by the original tiles. After the first subdivision we exhaust all of the equivalence classes that collapse to form the quotient space for the essential mating, meaning that the essential mating is not actually a subdivision map for these later iterations. This is precisely the problem that we want to avoid in developing a setting for the essential type construction to admit a finite subdivision rule.

Recall that finite subdivision rules do not require embedded structures or maps to yield a rule, though—combinatorially defined rules are acceptable. In this case, we can use the combinatorial rule implied by the essential construction after the first iteration. Figure 12 shows this for the $f_{1/6} \amalg_e f_{1/6}$ example mentioned above; note how the essential construction yields a 2-tile subdivision rule with a quadrilateral and an octagon. When subdividing, the quadrilateral is replaced with an octagon, and the octagon is subdivided into two quadrilaterals and a smaller octagon. This pattern continues for future subdivisions.

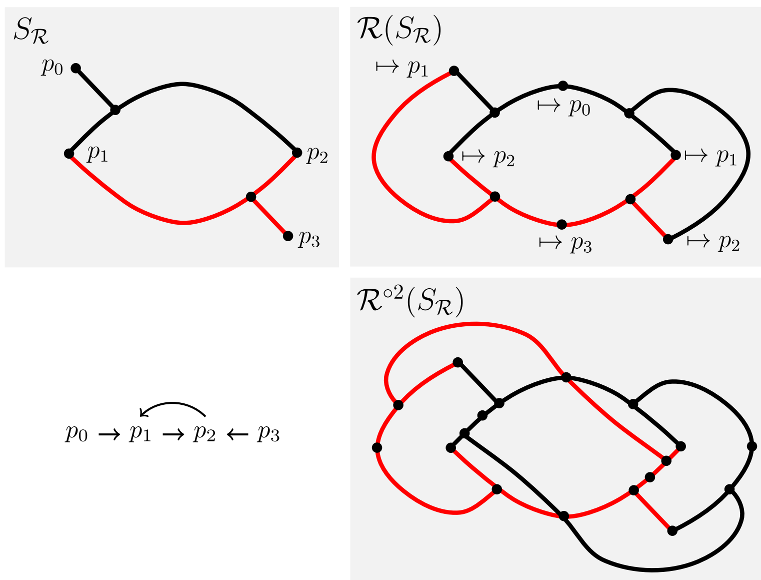

FIGURE 12. Subsequent subdivisions of of $S_{\mathcal{R}}$ for the mating $f_{1/6} \amalg_e f_{1/6}$.

While this subdivision rule will not reflect the behavior of the essential mating after the first subdivision (the subsequent subdivisions would suggest an infinite number of nontrivial equivalence classes of $\sim_e$ as we keep subdividing, which is impossible), it does show us identifications made in the topological mating. Any time the opposing Hubbard tree structures meet reflects some equivalence class of $\sim_t$ collapsing to a point.

3.3. A nonexample. To highlight a less trivial situation in which the essential construction does *not* yield a finite subdivision rule, we will consider the example $f_{7/8} \amalg_e f_{1/4}$. In Figure 13, we see the two Hubbard trees needed for the construction with postcritical points and branched points marked, along with the subdivision complex $S_{\mathcal{R}}$ associated with the essential construction for this mating. For ease of notation in the figures, we set

$\gamma(\theta) \coloneqq \gamma_{7/8}(\theta)$, and $\gamma(\theta)^* \coloneqq \gamma_{1/4}(1-\theta)$. When building $S_{\mathcal{R}}$, it will help to recall that this implies $\gamma(\theta) \sim_e \gamma(\theta)^*$.

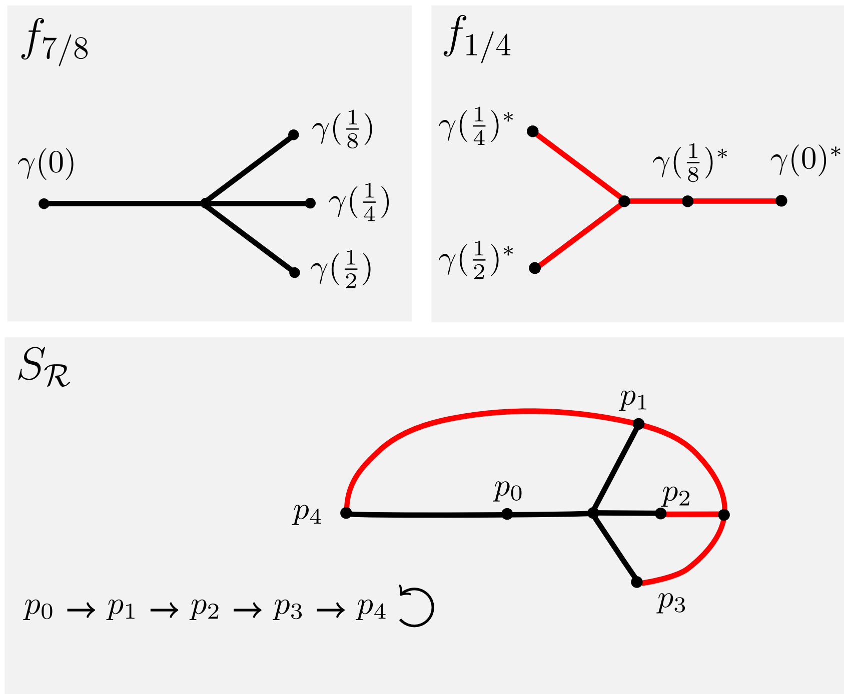

FIGURE 13. Hubbard trees for $f_{7/8}$ and $f_{1/4}$, along with $S_{\mathcal{R}}$ as suggested by the essential construction for $f_{7/8} \amalg_e f_{1/4}$.

The critical portrait for this essential mating suggests a subdivision similar to that given in Figure 14: first, we note where each of the marked points will map; and second, since we expect that the rule reflects a degree two map we should subdivide 1- and 2-cells as needed to yield a homeomorphic mapping onto $S_{\mathcal{R}}$. This forces the addition of 4 new edges and 4 new vertices to our structure—but regardless of their placement, no subdivision will have $f_{7/8} \amalg_e f_{1/4}$ serve as the subdivision map for a subdivision rule. The grey regions highlighted in Figure 14 contain points on the initial Hubbard trees which identify under $\sim_t$ but not $\sim_e$, and whose forward images identify under $\sim_e$. There are two ways to view why this is problematic: first, subdivisions of the initial tiling will not map locally homeomorphically onto $S_{\mathcal{R}}$ off of the critical points, thus any finite subdivision rule with subdivision complex $S_{\mathcal{R}}$ cannot have the essential mating as a subdivision map. Alternatively, pullbacks of $S_{\mathcal{R}}$ under the essential mating are not proper subdivisions. Instead, they possess 1-skeletons that appear to be "pinched" versions of subdivided 1-skeletons.

Experimentally, the essential construction appears most likely to falter with polynomial pairings like $f_{7/8}$ and $f_{1/4}$ where some equivalence class

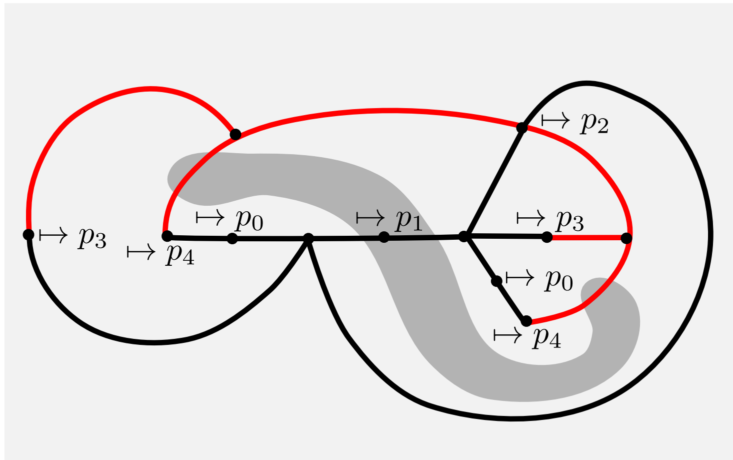

FIGURE 14. A subdivision of $S_{\mathcal{R}}$ from Figure 13 that does not map homeomorphically onto $S_{\mathcal{R}}$.

of $\sim_e$ contains two points from the same Hubbard tree. This is not to say that these kinds of matings cannot be expressed by finite subdivision rules, however. In many cases, minor adaptations can be made to the essential construction in order to produce a rule. One such adaptation is presented in Figure 15: since the full critical orbit of $f_{7/8} \perp\!\!\!\perp_e f_{1/4}$ is contained in $T_{7/8}/\sim_e$, we can use this as the 1-skeleton for a subdivision complex rather than $T_{7/8} \bigsqcup T_{1/4}/\sim_e$. The proof of Theorem 3.2 implies that if a 1-skeleton is finite, connected, planar, forward invariant, and contains the postcritical set as vertices, then filling in the 1-skeleton with tiles will yield a finite subdivision rule. The subdivision complex in this modified finite subdivision rule is then a 10-gon which is subdivided into two 10-gons when pulled back by the essential mating $f_{7/8} \perp\!\!\!\perp_e f_{1/4}$.

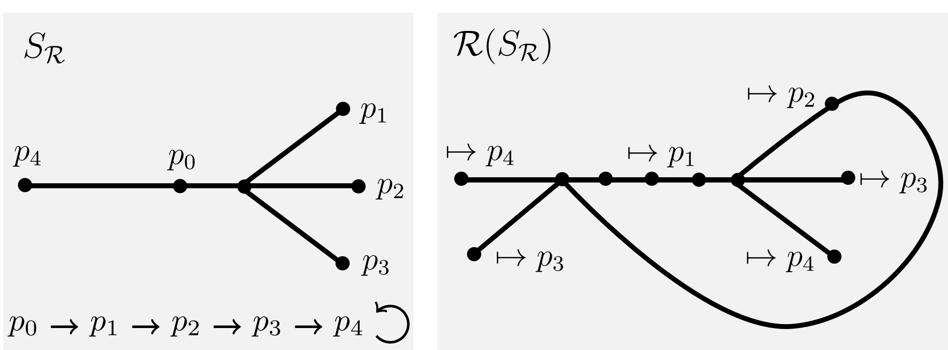

FIGURE 15. A finite subdivision rule with subdivision map given by the essential mating $f_{7/8} \perp\!\!\!\perp_e f_{1/4}$.

4. The essential construction and the pseudo-equator

In a sense, the essential construction shows us where the "most important" identifications in a mating are formed first, since we start with the essential mating and then are shown where subsequent preimage identifications must be made on polynomial Julia sets.

This section elaborates on how this technique can provide insights into other means for visualizing and understanding matings.

4.1. Meyer's pseudocircles. In [9], Meyer shows that certain postcritically finite rational maps can be viewed as matings and then decomposed into their two constituent polynomials. If the Julia set of the rational map is a 2-sphere, a sufficient condition for such a decomposition is the existence of a *pseudo-equator*:

Definition 4.1. A homotopy $H : X \times [0,1] \to X$ is a *pseudo-isotopy* if $H : X \times [0,1) \to X$ is an isotopy. We will assume $H_0 = H(x,0) = x$ for all $x \in X$.

Let f be a postcritically finite rational map, $\mathcal{C} \subseteq \hat{\mathbb{C}}$ be a Jordan curve with $P_f \subseteq \mathcal{C}$, and $\mathcal{C}^1 = f^{-1}(\mathcal{C})$. Then we say that f has a *pseudo-equator* if it has a pseudo-isotopy $H : \mathbb{S}^2 \times [0,1] \to \mathbb{S}^2$ rel. P_f with the following properties:

(1) $H_1(\mathcal{C}) = \mathcal{C}^1$.
(2) The set of points $w \in \mathcal{C}$ such that $H_1(w) \in f^{-1}(P_f)$ is finite. (We will let W denote the set of all such w.)
(3) $H_1 : \mathcal{C}\backslash W \to \mathcal{C}^1\backslash f^{-1}(P_f)$ is a homeomorphism.
(4) H deforms $\mathcal{C}$ orientation-preserving to $\mathcal{C}^1$. More specifically, H_k is orientation preserving for all k, even in the case where $k = 1$: Given an orientation on $\mathcal{C}$, both f and H_1 induce an orientation on $\mathcal{C}^1$. These orientations agree [9].

The motivation for the pseudo-equator definition appears forced when approached from the starting point of a rational map, but is quite natural when starting with the mating:

Theorem 4.2. *Let $\mathbb{S}'^2$ denote the quotient space associated with the mating $E = f_\alpha \perp\!\!\!\perp_e f_\beta$, and let P_E denote the postcritical set of E. If there exists some Jordan curve $\mathcal{C}$ on X which contains P_E and separates $(T_\alpha/\sim_e)\backslash P_E$ from $(T_\beta/\sim_e)\backslash P_E$, then E has a pseudo-equator.*

Proof. Consider the pullback of $\mathcal{C}$ under E, $\mathcal{C}^1$. Since $\mathcal{C}$ contains the critical values of E, $\mathcal{C}^1$ must pass through the two critical points of E. Locally, the pullback resembles an X at the critical points because E is a degree 2 map—and these are the only locations that the pullback has this shape, since there are only two critical points.

Since E is a branched covering map, there are a limited number of options for the topological shape of the pullback $\mathcal{C}^1$ since $\mathcal{C}^1$ may only cross itself

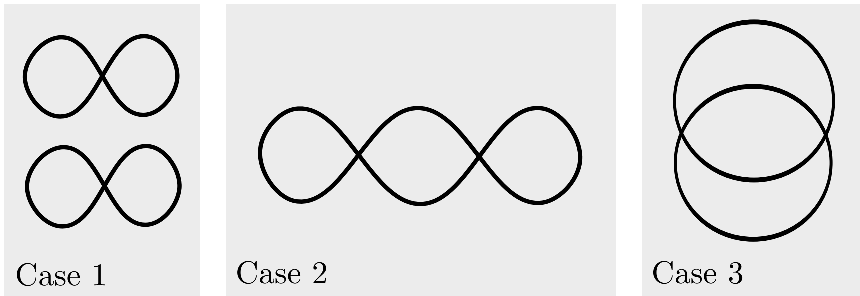

FIGURE 16. Possible pullbacks of $\mathcal{C}$ under a branched covering map.

twice. The options resemble those given in Figure 16, up to inclusion of additional components that are Jordan curves.

These are possibilities for a generic branched covering not specific to E, however. The first case in Figure 16 cannot be the pullback because $\mathbb{S}'^2\backslash\mathcal{C}^1$ contains too many components: E is a degree two map, and acts homeomorphically off the critical set. This means that we should expect $\mathbb{S}'^2\backslash\mathcal{C}^1$ to have 4 components. This line of reasoning also rules out the possibility of adjoining additional Jordan components to any of the cases in Figure 16.

The second case can be ruled out using a similar line of reasoning: we can examine where segments of the pullback will map based on where the endpoints map. The segments on either end start and end at a critical point, which means the image of these segments under E must start and end at a critical value. These end segments, when paired with their respective critical points, must map onto $\mathcal{C}$. The two segments in the middle when paired with the critical points must also map onto $\mathcal{C}$. This suggests that E is at minimum a degree 3 map, which is not the case.

We are left with the pullback resembling the the last case of Figure 16. Since E acts homeomorphically off of the critical set, we expect a mapping behavior much like that expressed in Figure 17. In this figure, blue lines denote the indicated curve and dots mark critical points. The bolded black and red lines mark Hubbard trees, with dashing to denote that we are only showing local behavior of the tree near the critical point—although we may note that the black tree $(T_\alpha/\sim_e)\backslash P_E$ and red tree $(T_\beta/\sim_e)\backslash P_E$ are separated by $\mathcal{C}$. Notice that if we 'sliced' $\mathcal{C}^1$ along the Hubbard trees, we'd obtain a curve that could be deformed in an orientation preserving manner to $\mathcal{C}$. This deformation hints at the desired pseudo-isotopy H, whose construction we sketch below.

Recall that $\mathbb{S}'^2$ is a quotient of the space $\mathbb{S}^2$, and that $\mathbb{S}^2$ was formed by identifying two copies of $\widetilde{\mathbb{C}}$ at their boundaries using opposing angles of external rays. Thus, for the formal mating we may view each point on the equator of $\mathbb{S}^2$ as the middle of an external ray pair that connects the

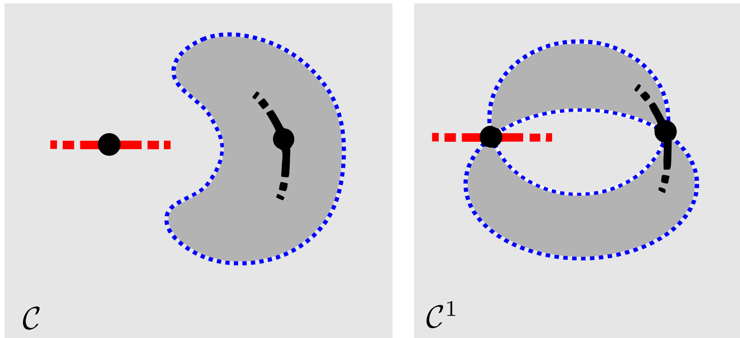

FIGURE 17. $\mathcal{C}$ and its pullback, shown with local behavior of Hubbard trees near the critical points of E.

Julia sets of the two polynomials we are mating. In forming the domain $\mathbb{S}'^2 = \mathbb{S}^2/\sim_e$ of the essential mating, this external ray structure is distorted in a few spots, but effectively preserved so that the external rays similarly connect $\mathcal{J}_\alpha/\sim_e$ to $\mathcal{J}_\beta/\sim_e$. To simplify the discussion, when referring to angles of external ray pairs, we will orient to the black polynomial f_α.

Since $\mathcal{C}$ effectively encircles the black tree, we may then assume without loss of generality that $\mathcal{C}$ was constructed in such a manner so that the curve passes through each external ray pair on $\mathbb{S}'^2$ exactly once—much like the equator on Earth passes through each line of longitude exactly once. Then, there is a natural parameterization $r_\mathcal{C} : [0,1] \to \mathbb{S}'^2$ for the curve $\mathcal{C}$ where $r_\mathcal{C}(t)$ gives the point of $\mathcal{C}$ on the external ray pair associated with angle t. Since the action of the essential mating on $\mathbb{S}'^2$ is to double angles of external rays, this suggests that the pullback curve $\mathcal{C}^1 = E^{-1}(\mathcal{C})$ can be parameterized similarly, so that the arcs of $\mathcal{C}^1$ corresponding to $r_{\mathcal{C}^1}(t)$ on $[0,1/2]$ and $[1/2,1]$ both map onto $\mathcal{C}$.

Since both $\mathcal{C}$ and $\mathcal{C}^1$ can be parameterized by angles of external ray pairs in this manner, a natural way to view H is as describing a continuous deformation which pushes $\mathcal{C}$ to $\mathcal{C}^1$ along external rays: i.e., for any $t \in [0,1]$, take $H_0(r_\mathcal{C}(t))$ to be where the t ray pair intersects $\mathcal{C}$; $H_1(r_\mathcal{C}(t))$ to be where the t ray pair intersects $\mathcal{C}^1$; and if $0 < k < 1$, $H_k(r_\mathcal{C}(t))$ is some point on the t ray pair which lies between these two points as in Figure 18.

Note that H as constructed in this manner is a pseudo-isotopy with the properties named in Definition 4.1. We clearly have that $H_1(\mathcal{C}) = \mathcal{C}^1$. Since H deforms $\mathcal{C}$ by sliding along external rays, H does not map arcs to points. In fact, the only points on $H_1(\mathcal{C})$ with two preimages on $\mathcal{C}$ are the two critical points of the mating. All other points in $H_1(\mathcal{C})$ have only one preimage in $\mathcal{C}$, and in general H_1 acts homeomorphically off of the 4 points which it sends to critical points of E. This guarantees conditions (2) and (3) in Definition 4.1. Finally, note that per our construction all deformations of $\mathcal{C}$ by H can be parameterized in terms of angle of external rays, which suggests an

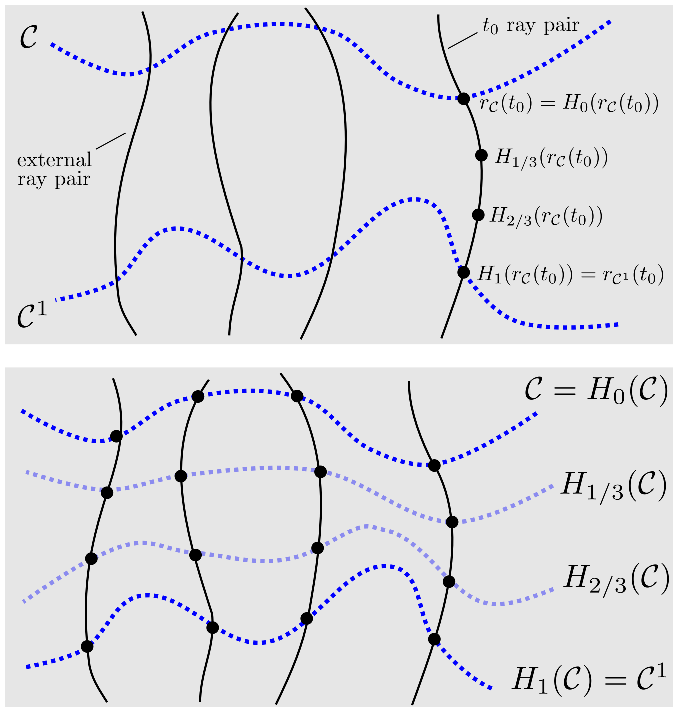

FIGURE 18. The local action of a desired pseudo-isotopy H_k on the curve $\mathcal{C}$. The top of this figure shows H_k as applied to a single marked point on $\mathcal{C}$, the bottom shows H_k as applied to the curve $\mathcal{C}$. Note that as k increases, the action of H_k is to 'push' the curve $\mathcal{C}$ along external ray pairs towards $\mathcal{C}^1$.

orientation agreeing with the orientation induced by E. Thus, H deforms $\mathcal{C}$ orientation-preserving to $\mathcal{C}^1$, and E has a pseudo-equator. $\square$

4.2. An example, continued. Theorem 4.2 implies the following method for finding pseudo-equators associated with a mating: if Γ is homotopic to the equator on $\mathbb{S}^2$ relative to T_α and T_β, then $\mathcal{C} = \Gamma/\sim_e$ generates a pseudo-equator when $\mathcal{C}$ is a Jordan curve. It is thus reasonably straightforward to visualize the pseudo-equator on particular matings by using the essential construction: form a finite subdivision rule using the essential construction, and on $S_{\mathcal{R}}$ construct a curve $\mathcal{C}$ through the postcritical points such that $\mathbb{S}'^2 \backslash \mathcal{C}$ contains two components—the closure of each containing the Hubbard tree of a polynomial in the mating. If $\mathcal{C}$ is a Jordan curve, $\mathcal{C}$ generates a pseudo-equator, and the subdivision map shows us how Meyer's 2-tiling subdivides, as in Figure 19.

With consideration for edge replacements in the pullback, the pseudo-equator provides a means for recovering information on the polynomial pair associated with the mating. Although it should be clear in this $f_{1/6} \perp\!\!\!\perp_e f_{1/6}$

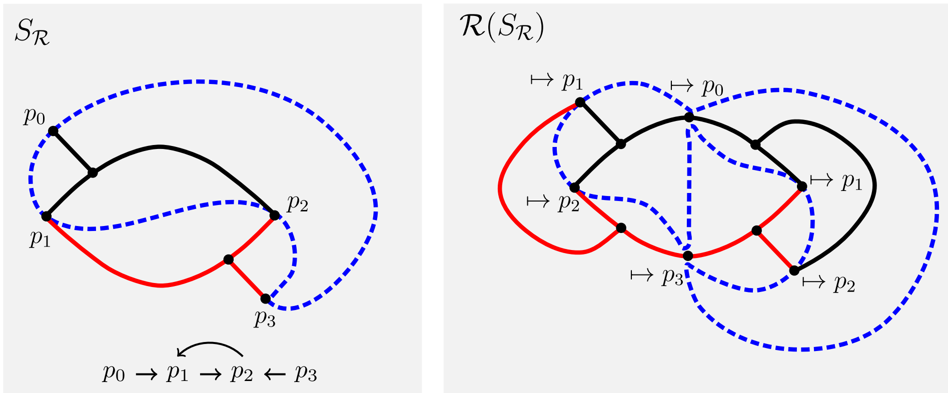

FIGURE 19. The pseudo-equator associated with $f_{1/6} \perp\!\!\!\perp_e f_{1/6}$. $\mathcal{C}$ is marked in blue on the left. The pullback of $\mathcal{C}$ under this mating is marked in blue on the right.

example that the polynomials associated with the pseudo-equator are two copies of $f_{1/6}$, we can confirm the decomposition for $\mathcal{C}$ using the methods given in [9].

First, label the postcritical vertices along the pseudo-equator as $p_0, \ldots, p_n$. We then label each edge from p_i to $p_{i+1(\text{mod } n+1)}$ as E_i, and determine the edge replacement matrix (a_{ij}) of the pseudo isotopy where a_{ij} is the number of distinct sub-edges of $H_1(E_i)$ which map to E_j . The edge replacement matrix for the example in Figure 19 is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

The degree of the mating corresponds to the leading eigenvalue of the edge replacement matrix, which is 2. When normalized so that the sum of entries is 1, the corresponding eigenvector is $v = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}^T$. The entries $v_0, v_1, \ldots$ of v then correspond to the lengths of edges $E_0, E_1, \ldots$ on the pseudo-equator, which in turn determines spacing of the marked postcritical points p_i.

Since the spacing between these points does not immediately provide information about the mating, we let the function $\theta : \{p_0, \ldots, p_n\} \to [0, 1)$ denote the external angle associated with each postcritical point with respect to one of the polynomial Hubbard trees (say, the black one in Figure 19). This function must satisfy two properties: first by tracking lengths of edges that

$$\theta(p_i) = \theta(p_0) + \sum_{k=1}^{i} v_k,$$

and second, we require that

$$\theta(p_i) = \theta \circ E(p_i) - \theta(p_i) \pmod 1$$

due to the Carathéodory semiconjugacy associated with the mating. Simple computation allows us to obtain that for the current example, $\theta(p_0) = \frac{1}{6}$, $\theta(p_1) = \frac{1}{3}$, $\theta(p_2) = \frac{2}{3}$, and $\theta(p_3) = \frac{5}{6}$. The Carathéodory semiconjugacy suggests that p_0 and p_3 are our critical values. Since the external angle is given with respect to the black polynomial, this means that only one of $\theta(p_0)$ or $\theta(p_3)$ may be taken as the correctly oriented angle associated with this polynomial, and that the other is given in reverse orientation. If we choose p_0 to have a correctly oriented angle $\frac{1}{6}$, this means that p_3 has external angle when oriented to the red polynomial of $1 - \theta(p_3) = \frac{1}{6}$. Thus, we confirm that the pseudo-equator is given by $f_{1/6}$ mated with itself.

4.3. When pseudo-equators do not exist. Not all nonhyperbolic matings have pseudo-equators. A potential reason is that the path $\mathcal{C}$ is not always a Jordan curve—any time $\sim_e$ contains equivalence classes that include multiple postcritical or critical points from one of the polynomials in the mating, the equator Γ is pinched to form $\mathcal{C}$. This falls outside of the scope of the definition for a pseudo-equator, which concerns the deformation of a Jordan curve. For instance, the example given in [9] for $f_{1/6} \perp\!\!\!\perp f_{13/14}$ presents with subdivision complex $S_{\mathcal{R}}$ and $\mathcal{C}$ as shown in Figure 20. Notice the pinching of the blue equator curve due to the postcritical identifications on $f_{13/14}$.

4.4. Implications and future work. The essential finite subdivision rule constructions provide an alternative model for matings of critically preperiodic quadratic polynomials. Further, finite subdivision rules are a useful tool for visualizing basic dynamics and modeling the mapping properties of certain matings—When paired with Bruin and Schleicher's algorithms from [2], the essential construction is simple enough that many elementary function pairings with few postcritical points can have their mapping behaviors sketched without the aid of a computer.

In addition, these constructions serve as complementary to work in the current literature: In [1], the Medusa algorithm is provided for obtaining rational maps from matings of quadratic polynomials, but the algorithm eventually diverges in the case of nonhyperbolic pairings. It is the author's belief that the finite subdivision rule constructions in this paper could be used to modify the Medusa algorithm in a way that would yield rational maps from matings of nonhyperbolic polynomials.

In [9], the relationship between rational maps and matings is only stressed with the existence of an equator or pseudo-equator, to the exclusion of structures such as those highlighted in Figure 20. As highlighted in the above examples, 2-tilings generated by the essential construction have potential to show how nonhyperbolic mated maps are related to different space-filling

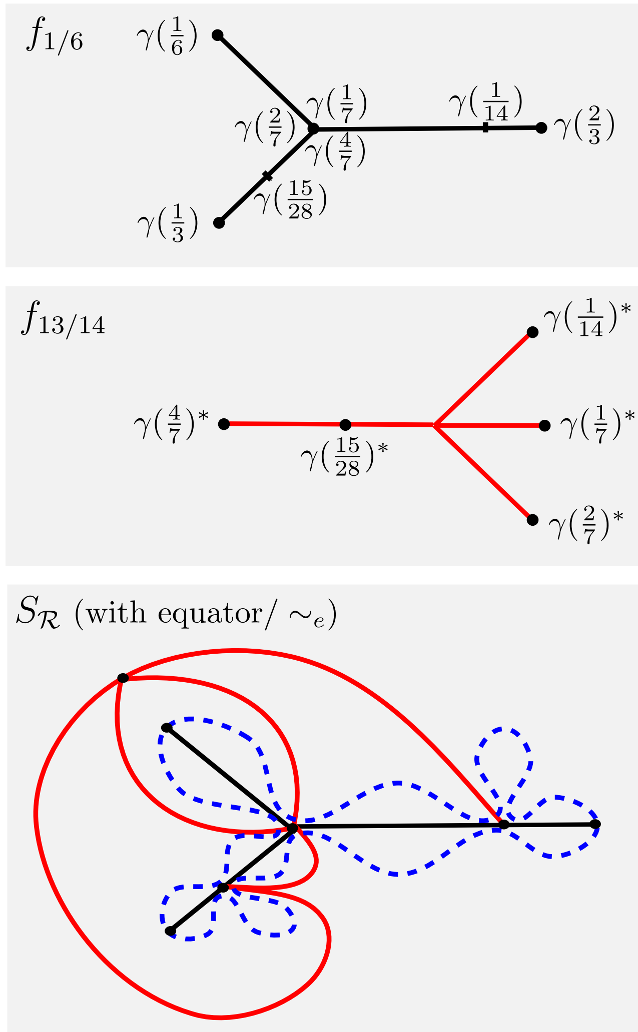

FIGURE 20. The "pseudo-equator" is pinched by $\sim_e$ into a non-Jordan curve.

curves on the 2-sphere: the quotient of the equator on $\mathbb{S}^2$ with respect to $\sim_t$ is a topological 2-sphere, and 1-skeletons of subdivisions of the 2-tiling give subsequent approximations to this quotient space. The essential construction and these 2-tilings should provide further insight on the conditions in which postcritically finite rational maps can be realizable as matings, and suggest alternative structures to consider when rational maps do not have pseudo-equators.

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