

Braided join comodule algebras of bi-Galois objects

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ABSTRACT. A bi-Galois object A is a bicomodule algebra for Hopf-Galois coactions with trivial invariants. In the spirit of Milnor’s construction, we define the join of noncommutative bi-Galois objects (quantum torsors). To ensure that the diagonal coaction on the join algebra of the right-coacting Hopf algebra is an algebra homomorphism, we braid the tensor product $A \otimes A$ with the help of the left-coacting Hopf algebra. Our main result is that the diagonal coaction is principal. Then we show that an anti-Drinfeld double is a symmetric bi-Galois object with the Drinfeld-double Hopf algebra coacting on both left and right. In this setting, we consider a finite quantum covering as an example. Finally, we take the noncommutative torus with the natural free action of the classical torus as an example of a symmetric bi-Galois object equipped with a $*$ -structure. It yields a noncommutative deformation of a nontrivial torus bundle.

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1. Introduction and preliminaries

In algebraic topology, the join of topological spaces is a fundamental concept. In particular it is used in the celebrated Milnor's construction of a universal principal bundle [M56]. A noncommutative-geometric generalization of the n -fold join $G * \dots * G$ of a compact Hausdorff topological group G , which is the first step in Milnor's construction, was proposed in [DHH15] with G replaced by Woronowicz's compact quantum group [Wo98]. Herein our goal is to provide a more general noncommutative-geometric version of the join $G * G$ now with G replaced by a quantum torsor (bi-Galois object). In the classical setting, our construction corresponds to the join $X * X$, where X is a topological space homeomorphic with a compact Hausdorff group G .

In particular, when G is an n -element group, the join $G * G$ is the space of all points in the line segments joining every point in $\{(0, 1), \dots, (0, n)\}$ to every point in $\{(1, 1), \dots, (1, n)\}$. This is a finite Galois covering of the unreduced suspension of G .

Just as compact quantum groups are captured by cosemisimple Hopf algebras, quantum torsors are given as Galois objects [Ca98], i.e., comodule algebras with free and ergodic coactions. In particular, every Hopf algebra is a Galois object with its coproduct taken as a coaction. One can think of a Galois object over a Hopf algebra as a principal G -bundle over a point. While in the point-set topology the generalization from G to a G -bundle over one point is not significant, in the noncommutative-geometric framework it unlocks a plethora of new possibilities. Among prime examples of quantum torsors is the noncommutative 2-torus [Rie90] with the natural action of the classical 2-torus.

Better still, the richness of the realm of noncommutative Galois objects is further enhanced by Schauenburg's bi-Galois theory [Scha96]. Indeed, if A is any Galois object for a right coaction of a Hopf algebra H , then, following the geometric idea of the Ehresmann groupoid [P84], one can construct a Hopf algebra \tilde{H} coacting on A on the left, and making it a left Galois object. It is precisely the coexistence of two Galois structures on one object that is pivotal in our braided join comodule algebra construction.

To make this paper self-contained and to establish notation and terminology, we begin by recalling the basics of classical joins, Hopf–Galois coactions [SchaS05], strong connections [BrH04] and the Durdevic braiding. In [Dur96], Durdevic proved that the algebra structure on the left-hand side of the Hopf–Galois canonical map, that is induced from the tensor algebra on its right-hand side, is given by a braiding generalizing a standard Yetter–Drinfeld braiding of Hopf algebras. This generalization hides inside the natural Yetter–Drinfeld module structure, which was earlier observed by Doi and Takeuchi [DoT89] forsaking the braided algebra multiplication. It is this multiplication that we use to define a braided join algebra.

Hopf–Galois coactions that admit strong connections are called principal as they encode free actions of compact quantum groups [BaDH]. Section 2 contains the main result of this paper establishing the principality of the diagonal coaction on our braided join algebra:

Theorem 2.5 *Let H be a Hopf algebra with bijective antipode, and let A be an \tilde{H} - H bi-Galois object. Then the right diagonal coaction of H on the \tilde{H} -braided join algebra of A is principal. Furthermore, the coaction-invariant subalgebra is isomorphic to the unreduced suspension of \tilde{H} .*

Section 3 is devoted to finite-dimensional Hopf algebras, so that we can form purely algebraic Drinfeld doubles and anti-Drinfeld doubles. Anti-Drinfeld doubles were discovered as a tool for describing anti-Yetter–Drinfeld modules [HKhRS04a]. They are already right Galois objects over Drinfeld double Hopf algebras [Dri87]. Hence we only needed to determine left coactions commuting with right coactions and making anti-Drinfeld doubles bi-Galois objects. This is our second main result:

Theorem 3.1 *Let H be a finite-dimensional Hopf algebra. Then the anti-Drinfeld double $A(H)$ is a bi-Galois object over the Drinfeld double $D(H)$.*

Since modules over anti-Drinfeld doubles serve as coefficients of Hopf-cyclic homology and cohomology [HKhRS04b], we hope that the aforesaid additional structure on anti-Drinfeld doubles will be useful in Hopf-cyclic theory. Furthermore, there seems to be a clear way to generalize our braided join construction to n -fold braided joins of Galois objects, to go beyond ergodic coactions, and to replace the algebra $C([0, 1])$ of all complex-valued continuous functions on the unit interval by any algebra with an appropriate ideal structure. However, this is beyond the scope of this paper (see [DHW, DDHW]).

Recall next that *topological* infinite-dimensional Hopf algebras can be used to construct topological Drinfeld doubles [Bo94, BoFGP94, BoS05]. Since topological anti-Drinfeld doubles were successfully constructed and applied in [RaS], and our braiding is compatible with a braiding on the C^* -level

used in [NV10] to ensure that the diagonal coaction is an algebra homomorphism, it is plausible that Section 3 could be upgraded to topological infinite-dimensional Hopf algebras. However, this is also beyond the scope of this paper.

In Section 4, we consider the polynomial algebra of the aforementioned noncommutative 2-torus as an example of a bi-Galois object involving a $*$ -structure. One can view the join of the noncommutative 2-torus with itself as a field of noncommutative 4-tori over the unit interval with some collapsing at the endpoints. Since this join is a noncommutative deformation of a nontrivial 2-torus principal bundle into a 2-torus quantum principal bundle, it fits perfectly into the new framework for constructing interesting spectral triples [CoM08] proposed recently in [DS13, DSZ14, DZ].

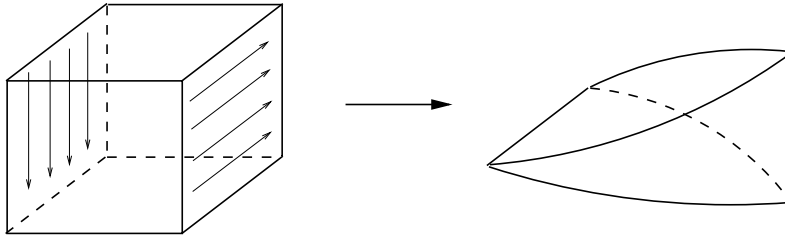
1.1. Classical principal bundles from the join construction. Let $I = [0, 1]$ be the closed unit interval and let X be a topological space. The *unreduced suspension* ΣX of X is the quotient of $I \times X$ by the equivalence relation R_S generated by

$$(1) \quad (0, x) \sim (0, x'), \quad (1, x) \sim (1, x').$$

Now take another topological space Y and, on the space $I \times X \times Y$, consider the equivalence relation R_J given by

$$(2) \quad (0, x, y) \sim (0, x', y), \quad (1, x, y) \sim (1, x, y').$$

The quotient space $X * Y := (I \times X \times Y)/R_J$ is called the *join* of X and Y . It resembles the unreduced suspension of $X \times Y$, but with only X collapsed at 0, and only Y collapsed at 1.



If G is a locally compact Hausdorff topological group acting continuously on X and Y , then it follows from [Wh48, Lemma 4] that the diagonal G -action on $X \times Y$ induces a continuous action on the join $X * Y$. Indeed, the diagonal action of G on $I \times X \times Y$ factorizes to the quotient, so that the formula

$$(3) \quad [(t, x, y)], g \longmapsto [(t, xg, yg)]$$

defines a right G -action on $X * Y$. Now, our assumption about G allows us to use [Wh48, Lemma 4] to infer that, in our setting, the product of quotient topologies is the quotient topology of the product, which implies

the continuity of the G -action on $X * Y$. Finally, if the G -actions on X and Y are free, it is immediate that so is the G -action on $X * Y$.

Next, let us take $X = Y$, and assume that we have a continuous map $X \times X \xrightarrow{\phi} X$ such that for all $x \in X$ the maps

$$(4) \quad X \ni y \longmapsto \phi(x, y) \in X \quad \text{and} \quad X \ni y \longmapsto \phi(y, x) \in X$$

are homeomorphisms. Then, by [Br93, Proposition VII.8.8], the formula

$$(5) \quad \pi: X * X \ni [(t, x, y)] \longmapsto [(t, \phi(x, y))] \in \Sigma X$$

defines a continuous surjection making the join $X * X$ a locally trivial fiber bundle over the unreduced suspension ΣX with the typical fiber X .

In particular, we can combine the above described two cases of join constructions and take $X = G = Y$, where G is a compact Hausdorff topological group. The diagonal action of G on $G \times G$ yields a free continuous G -action on $G * G$ that is automatically proper due to the compactness of G . Furthermore, taking

$$(6) \quad \phi: G \times G \ni (g, h) \longmapsto gh^{-1} \in G,$$

we conclude that $G * G$ is a locally trivial fiber bundle over the unreduced suspension ΣG with the typical fiber G . Thus the join $G * G$ is a principal G -bundle for the surjection

$$(7) \quad \pi: G * G \ni [(t, g, h)] \longmapsto [(t, gh^{-1})] \in \Sigma G.$$

It is known that such a bundle is trivializable if and only if G is contractible. Therefore, as the only contractible compact Hausdorff topological group G is the trivial group [H79], any nontrivial G yields a nontrivializable principal G -bundle over the unreduced suspension ΣG . For example, using $G = SU(2)$, $G = U(1)$ and $\mathbb{Z}/2\mathbb{Z}$, one obtains in this way the fibrations $S^7 \rightarrow S^4$, $S^3 \rightarrow S^2$ and $S^1 \rightarrow \mathbb{R}P^1$, respectively.

1.2. Left and right Hopf–Galois coactions. Let H and \tilde{H} be Hopf algebras with coproducts $\Delta, \tilde{\Delta}$, counits $\varepsilon, \tilde{\varepsilon}$, and antipodes S, \tilde{S} , respectively. Next, let $\Delta_P: P \rightarrow P \otimes H$ be a coaction making P a right H -comodule algebra, and let ${}_Q\Delta: Q \rightarrow \tilde{H} \otimes Q$ be a coaction making Q a left \tilde{H} -comodule algebra.

We shall frequently use the Heyneman-Sweedler notation (with the summation sign suppressed) for coproducts and coactions:

$$(8) \quad \begin{aligned} \Delta(h) &=: h_{(1)} \otimes h_{(2)}, & \tilde{\Delta}(k) &=: k_{(1)} \otimes k_{(2)}, \\ \Delta_P(p) &=: p_{(0)} \otimes p_{(1)}, & {}_Q\Delta(q) &=: q_{(-1)} \otimes q_{(0)}. \end{aligned}$$

Furthermore, let us define the coaction-invariant subalgebras:

$$(9) \quad \begin{aligned} B &:= P^{\text{co}H} := \{p \in P \mid \Delta_P(p) = p \otimes 1\}, \\ D &:= {}^{\text{co}\tilde{H}}Q := \{q \in Q \mid {}_Q\Delta(q) = 1 \otimes q\}. \end{aligned}$$

We call a right (respectively left) coaction *Hopf-Galois* [SchaS05] iff the right (respectively left) canonical map

$$(10) \quad \begin{aligned} \text{can}_P : P \otimes_B P \ni p \otimes p' &\longmapsto pp'_{(0)} \otimes p'_{(1)} \in P \otimes H, \\ {}_Q\text{can} : Q \otimes_D Q \ni q \otimes q' &\longmapsto q_{(-1)} \otimes q_{(0)}q' \in \tilde{H} \otimes Q, \end{aligned}$$

is a bijection. Observe that can_P is left linear over P and right linear over $P^{\text{co}H}$, whereas ${}_Q\text{can}$ is left linear over ${}^{\text{co}\tilde{H}}Q$ and right linear over Q . If the coaction-invariant subalgebras B and D are the ground field, then we call P and Q a right and a left *Galois object* respectively.

Now we focus on left Hopf-Galois coactions. First, we define the left *translation map*

$$(11) \quad \tau : \tilde{H} \longrightarrow Q \otimes_D Q, \quad \tau(h) := {}_Q\text{can}^{-1}(h \otimes 1) =: h^{[1]} \otimes h^{[2]}.$$

Note that, since ${}_Q\text{can}$ is right Q -linear, so is ${}_Q\text{can}^{-1}$. Therefore, we obtain

$$(12) \quad {}_Q\text{can}^{-1}(h \otimes q) = h^{[1]} \otimes h^{[2]}q.$$

For the sake of clarity and completeness, herein we derive basic properties of the left translation map that are well known for the right translation map (the inverse of the right canonical map restricted to H).

Proposition 1.1 (cf. Remark 3.4 in [Schn90]). *Let ${}_Q\Delta : Q \rightarrow \tilde{H} \otimes Q$ be a left Hopf-Galois coaction. Then, for all $h, k \in \tilde{H}$ and $q \in Q$, the following equalities hold:*

$$(13) \quad q_{(-1)}^{[1]} \otimes q_{(-1)}^{[2]}q_{(0)} = q \otimes 1,$$

$$(14) \quad h^{[1]}_{(-1)} \otimes h^{[1]}_{(0)} h^{[2]} = h \otimes 1,$$

$$(15) \quad h^{[1]} h^{[2]} = \tilde{\varepsilon}(h),$$

$$(16) \quad (hk)^{[1]} \otimes (hk)^{[2]} = h^{[1]}k^{[1]} \otimes k^{[2]}h^{[2]},$$

$$(17) \quad h^{[1]}_{(-1)} \otimes h^{[1]}_{(0)} \otimes h^{[2]} = h_{(1)} \otimes h_{(2)}^{[1]} \otimes h_{(2)}^{[2]},$$

$$(18) \quad h^{[1]} \otimes h^{[2]}_{(-1)} \otimes h^{[2]}_{(0)} = h_{(1)}^{[1]} \otimes \tilde{S}(h_{(2)}) \otimes h_{(1)}^{[2]}.$$

Proof. The first identity (13) follows from (12) and ${}_Q\text{can}^{-1} \circ {}_Q\text{can} = \text{id}$. The second equality (14) is an immediate consequence of ${}_Q\text{can} \circ {}_Q\text{can}^{-1} = \text{id}$. Applying $\tilde{\varepsilon} \otimes \text{id}$ to (14) yields (15). Since ${}_Q\text{can}$ is injective, applying it to both sides of (16), and using (14) twice on the right-hand side, proves (16). Transforming the left \tilde{H} -covariance of the canonical map ${}_Q\text{can}$

$$(19) \quad (\text{id} \otimes {}_Q\text{can}) \circ ({}_Q\Delta \otimes \text{id}) = (\tilde{\Delta} \otimes \text{id}) \circ {}_Q\text{can}$$

to

$$(20) \quad ({}_Q\Delta \otimes \text{id}) \circ {}_Q\text{can}^{-1} = (\text{id} \otimes {}_Q\text{can}^{-1}) \circ (\tilde{\Delta} \otimes \text{id}),$$

we obtain the left \tilde{H} -covariance (17).

To show the right \tilde{H} -covariance (18), we apply to both sides of (18) the bijective map $(\text{id} \otimes_Q \text{can}) \circ (\text{flip} \otimes \text{id})$. On the right-hand side, we get

$$(21) \quad \tilde{S}(h_{(2)}) \otimes h_{(1)}^{[1]}{}_{(-1)} \otimes h_{(1)}^{[1]}{}_{(0)} h_{(1)}^{[2]} = \tilde{S}(h_{(2)}) \otimes h_{(1)} \otimes 1.$$

Taking into account the left covariance (17), the left-hand side yields

$$(22) \quad h^{[2]}{}_{(-1)} \otimes h^{[1]}{}_{(-1)} \otimes h^{[1]}{}_{(0)} h^{[2]}{}_{(0)} = h_{(2)}^{[2]}{}_{(-1)} \otimes h_{(1)} \otimes h_{(2)}^{[1]} h_{(2)}^{[2]}{}_{(0)}.$$

Thus (18) is equivalent to the equality

$$(23) \quad \tilde{S}(h) \otimes 1 = h^{[2]}{}_{(-1)} \otimes h^{[1]} h^{[2]}{}_{(0)}.$$

Finally, using (14), we compute

$$(24) \quad \begin{aligned} h^{[2]}{}_{(-1)} \otimes h^{[1]} h^{[2]}{}_{(0)} &= \tilde{\varepsilon}(h^{[1]}{}_{(-1)}) h^{[2]}{}_{(-1)} \otimes h^{[1]}{}_{(0)} h^{[2]}{}_{(0)} \\ &= \tilde{S}(h^{[1]}{}_{(-2)}) h^{[1]}{}_{(-1)} h^{[2]}{}_{(-1)} \otimes h^{[1]}{}_{(0)} h^{[2]}{}_{(0)} \\ &= (\tilde{S}(h^{[1]}{}_{(-1)}) \otimes 1) {}_Q\Delta(h^{[1]}{}_{(0)} h^{[2]}{}_{(0)}) \\ &= \tilde{S}(h) \otimes 1 \end{aligned}$$

proving (18). □

1.3. Principal right coactions. Principal coactions are Hopf–Galois coactions with additional properties [BrH04]. One can prove (see [HKrMZ11, p. 599] and references therein) that a comodule algebra is principal if and only if it admits a strong connection. Therefore, we will treat the existence of a strong connection as a condition defining principality of a comodule algebra, and avoid the original definition of a principal coaction [BrH04]. The latter is important when going beyond coactions that are algebra homomorphisms, when we only know that the principality of a coaction implies the existence of a strong connection [BrH04].

Definition 1.2 ([BrH04]). Let H be a Hopf algebra with bijective antipode. A *strong connection* ℓ on a right H -comodule algebra P is a unital linear map $\ell : H \rightarrow P \otimes P$ satisfying:

- (1) $(\text{id} \otimes \Delta_P) \circ \ell = (\ell \otimes \text{id}) \circ \Delta$, $(\Delta_P^L \otimes \text{id}) \circ \ell = (\text{id} \otimes \ell) \circ \Delta$, where $\Delta_P^L := (S^{-1} \otimes \text{id}) \circ \text{flip} \circ \Delta_P$;
- (2) $\widehat{\text{can}} \circ \ell = 1 \otimes \text{id}$, where $\widehat{\text{can}} : P \otimes P \ni p \otimes q \mapsto (p \otimes 1)\Delta_P(q) \in P \otimes H$.

Note that Condition (2) is equivalent to the condition $m \circ \ell = \varepsilon$, where m is the multiplication map of P .

1.4. Left Durdevic braiding. Let ${}_Q\Delta : Q \rightarrow \tilde{H} \otimes Q$ be a left Hopf–Galois coaction, and D the coaction-invariant subalgebra. Using the bijectivity of the canonical map ${}_Q\text{can}$, we pullback the tensor algebra structure on $\tilde{H} \otimes Q$

to $Q \otimes_D Q$. The thus obtained algebra we shall denote by $Q \otimes_D Q$ and call a *left Hopf–Galois-braided algebra*. From the commutativity of the diagram

$$(25) \quad \begin{array}{ccc} (Q \otimes_D Q) \otimes (Q \otimes_D Q) & \xrightarrow{m_{Q \otimes_D Q}} & Q \otimes_D Q \\ \downarrow_{Q \text{can} \otimes Q \text{can}} & & \downarrow_{Q \text{can}} \\ (\tilde{H} \otimes Q) \otimes (\tilde{H} \otimes Q) & \xrightarrow{m_{\tilde{H} \otimes Q}} & \tilde{H} \otimes Q, \end{array}$$

we obtain the following explicit formula for the multiplication map $m_{Q \otimes_D Q}$:

$$(26) \quad \begin{aligned} m_{Q \otimes_D Q}(a \otimes b \otimes a' \otimes b') &:= Q \text{can}^{-1}(Q \text{can}(a \otimes b) Q \text{can}(a' \otimes b')) \\ &= Q \text{can}^{-1}(a_{(-1)} a'_{(-1)} \otimes a_{(0)} b a'_{(0)} b') \\ &= a_{(-1)}^{[1]} a'_{(-1)}^{[1]} \otimes a'_{(-1)}^{[2]} a_{(-1)}^{[2]} a_{(0)} b a'_{(0)} b' \\ &= a a'_{(-1)}^{[1]} \otimes a'_{(-1)}^{[2]} b a'_{(0)} b'. \end{aligned}$$

Here in the last equality we used (13).

Next, we show that $m_{Q \otimes_D Q}$ is the multiplication in a braided tensor algebra associated to the left-sided version of Durdevic’s braiding [Dur96, (2.2)]. Since $Q \text{can}$ is left and right D -linear, the following formula defines a left and right D -linear map:

$$(27) \quad \begin{aligned} Q \otimes_D Q \ni x \otimes y &\longmapsto \Psi(x \otimes y) \in Q \otimes_D Q, \\ \Psi(x \otimes y) &:= Q \text{can}^{-1}((1 \otimes x) Q \text{can}(y \otimes 1)) = y_{(-1)}^{[1]} \otimes y_{(-1)}^{[2]} x y_{(0)}. \end{aligned}$$

Now we can write the multiplication formula (26) as

$$(28) \quad m_{Q \otimes_D Q}(a \otimes b \otimes a' \otimes b') = a \Psi(b \otimes a') b' =: (a \otimes b) \bullet (a' \otimes b').$$

Note that when we view a Hopf algebra \tilde{H} as a left comodule algebra over itself, then the left Durdevic braiding (27) becomes the Yetter–Drinfeld braiding

$$(29) \quad \tilde{H} \otimes \tilde{H} \ni x \otimes y \longmapsto y_{(1)} \otimes \tilde{S}(y_{(2)}) x y_{(3)} \in \tilde{H} \otimes \tilde{H}.$$

Proposition 1.3 (cf. Proposition 2.1 in [Dur96]). *Let $Q \Delta: Q \rightarrow \tilde{H} \otimes Q$ be a left Hopf–Galois coaction, and D the coaction-invariant subalgebra. Then the map Ψ defined in (27) is bijective and enjoys the following properties:*

- (30) $m_Q \circ \Psi = m_Q,$
- (31) $\forall q \in Q: \Psi(q \otimes 1) = 1 \otimes q,$
- (32) $\forall q \in Q: \Psi(1 \otimes q) = q \otimes 1,$
- (33) $\Psi \circ (m_Q \otimes \text{id}) = (\text{id} \otimes m_Q) \circ (\Psi \otimes \text{id}) \circ (\text{id} \otimes \Psi),$
- (34) $\Psi \circ (\text{id} \otimes m_Q) = (m_Q \otimes \text{id}) \circ (\text{id} \otimes \Psi) \circ (\Psi \otimes \text{id}),$
- (35) $(\Psi \otimes \text{id}) \circ (\text{id} \otimes \Psi) \circ (\Psi \otimes \text{id}) = (\text{id} \otimes \Psi) \circ (\Psi \otimes \text{id}) \circ (\text{id} \otimes \Psi).$

Proof. The bijectivity of Ψ follows immediately from the fact that Q_{can} is an algebra isomorphism (25). The braided commutativity of (30) is a consequence of (15). The condition (31) is obvious, and the sibling condition (32) is implied by (13).

To prove (33), using (17) and (15), we compute

$$\begin{aligned}
 (36) \quad & ((\text{id} \otimes m_Q) \circ (\Psi \otimes \text{id}) \circ (\text{id} \otimes \Psi))(x \otimes y \otimes z) \\
 &= ((\text{id} \otimes m_Q) \circ (\Psi \otimes \text{id}))(x \otimes z_{(-1)}^{[1]} \otimes z_{(-1)}^{[2]} y z_{(0)}) \\
 &= (\text{id} \otimes m_Q)(z_{(-2)}^{[1]} \otimes z_{(-2)}^{[2]} x z_{(-1)}^{[1]} \otimes z_{(-1)}^{[2]} y z_{(0)}) \\
 &= z_{(-1)}^{[1]} \otimes z_{(-1)}^{[2]} x y z_{(0)} \\
 &= (\Psi \circ (m_Q \otimes \text{id}))(x \otimes y \otimes z).
 \end{aligned}$$

Much in the same way, to prove (34) using (16), we compute

$$\begin{aligned}
 (37) \quad & ((m_Q \otimes \text{id}) \circ (\text{id} \otimes \Psi) \circ (\Psi \otimes \text{id}))(x \otimes y \otimes z) \\
 &= ((m_Q \otimes \text{id}) \circ (\text{id} \otimes \Psi))(y_{(-1)}^{[1]} \otimes y_{(-1)}^{[2]} x y_{(0)} \otimes z) \\
 &= (m_Q \otimes \text{id})(y_{(-1)}^{[1]} \otimes z_{(-1)}^{[1]} \otimes z_{(-1)}^{[2]} y_{(-1)}^{[2]} x y_{(0)} z_{(0)}) \\
 &= (yz)_{(-1)}^{[1]} \otimes (yz)_{(-1)}^{[2]} x (yz)_{(0)} \\
 &= (\Psi \circ (\text{id} \otimes m_Q))(x \otimes y \otimes z).
 \end{aligned}$$

Finally, to show the braid relation (35), we first apply $Q_{\text{can}} \otimes \text{id}$ to its left-hand side. Then, taking advantage of the fact that above we have already computed $(\text{id} \otimes \Psi) \circ (\Psi \otimes \text{id})$, we proceed as follows:

$$\begin{aligned}
 & ((Q_{\text{can}} \otimes \text{id}) \circ (\Psi \otimes \text{id}) \circ (\text{id} \otimes \Psi) \circ (\Psi \otimes \text{id}))(x \otimes y \otimes z) \\
 &= ((Q_{\text{can}} \otimes \text{id}) \circ (\Psi \otimes \text{id}))(y_{(-1)}^{[1]} \otimes z_{(-1)}^{[1]} \otimes z_{(-1)}^{[2]} y_{(-1)}^{[2]} x y_{(0)} z_{(0)}) \\
 &= z_{(-2)} \otimes y_{(-1)}^{[1]} z_{(-1)}^{[1]} \otimes z_{(-1)}^{[2]} y_{(-1)}^{[2]} x y_{(0)} z_{(0)}.
 \end{aligned}$$

Here in the last equality we used (27).

Again much in the same way, taking advantage of the fact that above we have already computed $(\Psi \otimes \text{id}) \circ (\text{id} \otimes \Psi)$, we apply $Q_{\text{can}} \otimes \text{id}$ to the right-hand side of (35), and proceed as follows:

$$\begin{aligned}
 & ((Q_{\text{can}} \otimes \text{id}) \circ (\text{id} \otimes \Psi) \circ (\Psi \otimes \text{id}) \circ (\text{id} \otimes \Psi))(x \otimes y \otimes z) \\
 &= ((Q_{\text{can}} \otimes \text{id}) \circ (\text{id} \otimes \Psi))(z_{(-2)}^{[1]} \otimes z_{(-2)}^{[2]} x z_{(-1)}^{[1]} \otimes z_{(-1)}^{[2]} y z_{(0)}) \\
 &= (Q_{\text{can}} \otimes \text{id})(z_{(-4)}^{[1]} \otimes \tau(\tilde{S}(z_{(-2)})y_{(-1)} z_{(-1)})z_{(-4)}^{[2]} x z_{(-3)}^{[1]} z_{(-3)}^{[2]} y_{(0)} z_{(0)}) \\
 &= (Q_{\text{can}} \otimes \text{id})(z_{(-3)}^{[1]} \otimes \tau(\tilde{S}(z_{(-2)})y_{(-1)} z_{(-1)})z_{(-3)}^{[2]} x y_{(0)} z_{(0)}) \\
 &= z_{(-4)} \otimes z_{(-3)}^{[1]} (\tilde{S}(z_{(-2)})y_{(-1)} z_{(-1)})^{[1]} \\
 &\quad \otimes (\tilde{S}(z_{(-2)})y_{(-1)} z_{(-1)})^{[2]} z_{(-3)}^{[2]} x y_{(0)} z_{(0)} \\
 &= z_{(-4)} \otimes (z_{(-3)} \tilde{S}(z_{(-2)})y_{(-1)} z_{(-1)})^{[1]} \otimes (z_{(-3)} \tilde{S}(z_{(-2)})y_{(-1)} z_{(-1)})^{[2]} x y_{(0)} z_{(0)} \\
 &= z_{(-2)} \otimes y_{(-1)}^{[1]} z_{(-1)}^{[1]} \otimes z_{(-1)}^{[2]} y_{(-1)}^{[2]} x y_{(0)} z_{(0)}.
 \end{aligned}$$

Here we consecutively used (18), (15), (17) and (16). Since $Q_{\text{can}} \otimes \text{id}$ is bijective, this proves (35). \square

2. Braided principal join comodule algebras

2.1. Bi-Galois objects. Now we shall consider left and right coactions simultaneously. Let A be an \tilde{H} - H -bicomodule algebra, i.e., a left \tilde{H} -comodule algebra and a right H -comodule algebra with commuting coactions:

$$(38) \quad ({}_A\Delta \otimes \text{id}) \circ \Delta_A = (\text{id} \otimes \Delta_A) \circ {}_A\Delta.$$

This coassociativity allows us to use the Heyneman-Sweedler notation over integers:

$$(39) \quad (({}_A\Delta \otimes \text{id}) \circ \Delta_A)(a) =: a_{(-1)} \otimes a_{(0)} \otimes a_{(1)} := ((\text{id} \otimes \Delta_A) \circ {}_A\Delta)(a).$$

Since we need the left and right coaction-invariant subalgebras to coincide, we assume both of them to be the ground field: ${}^{\text{co}\tilde{H}}A = k = A^{\text{co}H}$. This brings us to the realm of Schauenburg’s bi-Galois theory [Scha96]. Recall that according to this theory, given any right Galois object A over a Hopf algebra H , there exists a unique left coaction ${}_A\Delta : A \rightarrow \tilde{H} \otimes A$ commuting with the right coaction and making A a left Galois object over \tilde{H} . An \tilde{H} - H bicomodule algebra A that is simultaneously left and right Galois object is called a *bi-Galois object*. Next, we specialize the left Durdevic braiding (27) to left Galois objects. This allows us to simplify our notation for the left Hopf-Galois-braided algebra to $A \underline{\otimes} A$.

Lemma 2.1. *Let A be an \tilde{H} - H bi-Galois object, and let $A \underline{\otimes} A$ be a left Hopf-Galois-braided algebra. Then the left canonical map (10) is an isomorphism of right H -comodule algebras intertwining the coactions given by the formulas*

$$\begin{aligned} \Delta_{A \underline{\otimes} A}(a \underline{\otimes} b) &:= a_{(0)} \underline{\otimes} b_{(0)} \otimes a_{(1)} b_{(1)}, \\ \Delta_{\tilde{H} \underline{\otimes} A}(h \otimes a) &:= (\text{id} \otimes \Delta_A)(h \otimes a) = h \otimes a_{(0)} \otimes a_{(1)}. \end{aligned}$$

Proof. To verify the commutativity of the diagram

$$(40) \quad \begin{array}{ccc} A \underline{\otimes} A & \xrightarrow{\Delta_{A \underline{\otimes} A}} & (A \underline{\otimes} A) \otimes H \\ \text{\scriptsize } {}_A\text{can} \downarrow & & \downarrow \text{\scriptsize } {}_A\text{can} \otimes \text{id} \\ \tilde{H} \otimes A & \xrightarrow{\Delta_{\tilde{H} \underline{\otimes} A}} & (\tilde{H} \otimes A) \otimes H, \end{array}$$

for any $a, a' \in A$, using (39), we compute:

$$(41) \quad \begin{aligned} (({}_A\text{can} \otimes \text{id}) \circ \Delta_{A \underline{\otimes} A})(a \underline{\otimes} a') &= ({}_A\text{can} \otimes \text{id})(a_{(0)} \underline{\otimes} a'_{(0)} \otimes a_{(1)} a'_{(1)}) \\ &= a_{(-1)} \otimes a_{(0)} a'_{(0)} \otimes a_{(1)} a'_{(1)} \\ &= (\text{id} \otimes \Delta_A)(a_{(-1)} \otimes a_{(0)} a') \\ &= (\Delta_{\tilde{H} \underline{\otimes} A} \circ {}_A\text{can})(a \underline{\otimes} a'). \end{aligned}$$

This shows that ${}_A\text{can}$ is right H -colinear. Also, since ${}_A\text{can}$ and $\Delta_{\tilde{H}\otimes A}$ are algebra homomorphisms and ${}_A\text{can}$ is bijective, we conclude from the commutativity of the diagram (40) that the diagonal coaction $\Delta_{A\otimes A}$ is an algebra homomorphism. \square

2.2. Braided join comodule algebras. To preserve the topological meaning of our join construction in the commutative setting, from now on we specialize our ground field to be the field of complex numbers.

Definition 2.2. Let A be an \tilde{H} - H bi-Galois object, and let $A \otimes A$ be a left Hopf-Galois-braided algebra. We call the unital \mathbb{C} -algebra

$$A \underset{H}{\otimes} A := \{x \in C([0, 1]) \otimes A \otimes A \mid (\text{ev}_0 \otimes \text{id})(x) \in \mathbb{C} \otimes A, (\text{ev}_1 \otimes \text{id})(x) \in A \otimes \mathbb{C}\}$$

the \tilde{H} -braided join algebra of A . Here ev_r is the evaluation map at $r \in [0, 1]$, i.e., $\text{ev}_r(f) = f(r)$.

Lemma 2.3. Let $A \underset{\tilde{H}}{\otimes} A$ be the \tilde{H} -braided join algebra of A . Then the formula

$$C([0, 1]) \otimes A \otimes A \ni f \otimes a \otimes b \longmapsto f \otimes a_{(0)} \otimes b_{(0)} \otimes a_{(1)} b_{(1)} \in C([0, 1]) \otimes A \otimes A \otimes H$$

restricts and corestricts to $\Delta_{A \underset{\tilde{H}}{\otimes} A}: A \underset{\tilde{H}}{\otimes} A \rightarrow (A \underset{\tilde{H}}{\otimes} A) \otimes H$ making $A \underset{\tilde{H}}{\otimes} A$ a right H -comodule algebra.

Proof. Let $\sum_i f_i \otimes a_i \otimes b_i \in A \underset{\tilde{H}}{\otimes} A$, i.e., $\sum_i f_i(0) a_i \otimes b_i \in \mathbb{C} \otimes A$ and $\sum_i f_i(1) a_i \otimes b_i \in A \otimes \mathbb{C}$. Then

$$(42) \quad (\text{ev}_r \otimes \text{id}) \left(\sum_i f_i \otimes (a_i)_{(0)} \otimes (b_i)_{(0)} \otimes (a_i)_{(1)} (b_i)_{(1)} \right) \\ = \sum_i (f_i(r) a_i)_{(0)} \otimes (b_i)_{(0)} \otimes (f_i(r) a_i)_{(1)} (b_i)_{(1)}.$$

For $r = 0$ the above tensor belongs to $\mathbb{C} \otimes A \otimes H$, and for $r = 1$ the above tensor belongs to $A \otimes \mathbb{C} \otimes H$. \square

2.3. Pullback structure and principality. In order to compute the coaction-invariant subalgebra, and to show that the principality of the right H -coaction on A implies the principality of the right diagonal H -coaction on $A \underset{\tilde{H}}{\otimes} A$, we present $A \underset{\tilde{H}}{\otimes} A$ as a pullback of right H -comodule algebras. Define

$$(43) \quad A_1 := \{f \in C([0, \frac{1}{2}]) \otimes A \otimes A \mid (\text{ev}_0 \otimes \text{id})(f) \in \mathbb{C} \otimes A\},$$

$$(44) \quad A_2 := \{g \in C([\frac{1}{2}, 1]) \otimes A \otimes A \mid (\text{ev}_1 \otimes \text{id})(g) \in A \otimes \mathbb{C}\}.$$

Then $A \underset{\tilde{H}}{\otimes} A$ is isomorphic to the pullback of A_1 and A_2 over $A_{12} := A \otimes A$ along the right H -colinear evaluation maps

$$(45) \quad \pi_1 := \text{ev}_{\frac{1}{2}} \otimes \text{id} : A_1 \longrightarrow A_{12}, \quad \pi_2 := \text{ev}_{\frac{1}{2}} \otimes \text{id} : A_2 \longrightarrow A_{12}.$$

Recall that, by Lemma 2.1, ${}_A\text{can}$ defines a right H -comodule algebra isomorphism ${}_A\text{can} : A \otimes A \rightarrow \tilde{H} \otimes A$. Also, we have ${}_A\text{can}(\mathbb{C} \otimes A) = \mathbb{C} \otimes A$ and ${}_A\text{can}(A \otimes \mathbb{C}) = {}_A\Delta(A)$. Next, we note that the right H -comodule algebras A_1 and A_2 are isomorphic to

$$(46) \quad B_1 := \{f \in C([0, \frac{1}{2}]) \otimes \tilde{H} \otimes A \mid (\text{ev}_0 \otimes \text{id})(f) \in \mathbb{C} \otimes A\},$$

$$(47) \quad B_2 := \{g \in C([\frac{1}{2}, 1]) \otimes \tilde{H} \otimes A \mid (\text{ev}_1 \otimes \text{id})(g) \in {}_A\Delta(A)\},$$

respectively.

Since $\Delta_A(a) = a \otimes 1$ implies that $a \in \mathbb{C}$, we obtain

$$(48) \quad B_1^{\text{co}H} = \{f \in C([0, \frac{1}{2}]) \otimes \tilde{H} \otimes \mathbb{C} \mid f(0) \in \mathbb{C}\},$$

$$(49) \quad B_2^{\text{co}H} = \{g \in C([\frac{1}{2}, 1]) \otimes \tilde{H} \otimes \mathbb{C} \mid g(1) \in \mathbb{C}\}.$$

In both cases, these algebras are isomorphic to the unreduced cone of \tilde{H} [GVF01, p. 25]. As a result, the coaction-invariant subalgebra of $A \otimes_{\tilde{H}} A$ is isomorphic to the unreduced suspension of \tilde{H} , i.e.,

$$(50) \quad (A \otimes_{\tilde{H}} A)^{\text{co}H} \cong \Sigma\tilde{H} := \{g \in C([0, 1]) \otimes \tilde{H} \mid g(0), g(1) \in \mathbb{C}\}.$$

Lemma 2.4. *Let H be a Hopf algebra with bijective antipode, and let A be an \tilde{H} - H bi-Galois object. Then the right H -comodule algebras B_1 and B_2 are principal.*

Proof. To prove the lemma, it suffices to show the existence of strong connections on B_1 and B_2 [HKrMZ11, p. 599]. Note first that the right translation map for a Galois object over a Hopf algebra with bijective antipode is a strong connection. Therefore, we will use the strong-connection notation $\text{can}_A^{-1}(1 \otimes h) =: h^{(1)} \otimes h^{(2)}$ (summation suppressed) for the right translation map. Let

$$(51) \quad \begin{aligned} \ell_1 : H &\longrightarrow B_1 \otimes B_1, \\ \ell_1(h) &:= (1 \otimes 1 \otimes h^{(1)}) \otimes (1 \otimes 1 \otimes h^{(2)}), \end{aligned}$$

$$(52) \quad \begin{aligned} \ell_2 : H &\longrightarrow B_2 \otimes B_2, \\ \ell_2(h) &:= (1 \otimes h^{(1)}_{(-1)} \otimes h^{(1)}_{(0)}) \otimes (1 \otimes h^{(2)}_{(-1)} \otimes h^{(2)}_{(0)}). \end{aligned}$$

The unitality of both ℓ_1 and ℓ_2 follows immediately from the unitality of the right translation map.

Furthermore,

$$(53) \quad \begin{aligned} (\text{can}_A \circ \ell_1)(h) &= 1 \otimes 1 \otimes h^{(1)}h^{(2)}_{(0)} \otimes h^{(2)}_{(1)} = 1 \otimes 1 \otimes 1 \otimes h, \\ (\text{can}_A \circ \ell_2)(h) &= 1 \otimes h^{(1)}_{(-1)}h^{(2)}_{(-1)} \otimes h^{(1)}_{(0)}h^{(2)}_{(0)} \otimes h^{(2)}_{(1)} \\ &= (\text{id} \otimes {}_A\Delta \otimes \text{id})(1 \otimes h^{(1)}h^{(2)}_{(0)} \otimes h^{(2)}_{(1)}) \\ &= 1 \otimes 1 \otimes 1 \otimes h. \end{aligned}$$

Finally, we verify the bilinearity of ℓ_1 and ℓ_2 . For ℓ_1 it follows immediately from the bilinearity of the right translation map. For the right

H -colinearity of ℓ_2 , we use the right H -colinearity of the right translation map to compute

$$\begin{aligned}
 (54) \quad & (\text{id} \otimes \Delta_{B_2})(\ell_2(h)) \\
 &= (1 \otimes h^{(1)}_{(-1)} \otimes h^{(1)}_{(0)}) \otimes (1 \otimes h^{(2)}_{(-1)} \otimes h^{(2)}_{(0)}) \otimes h^{(2)}_{(1)} \\
 &= (\text{id} \otimes {}_A\Delta \otimes \text{id} \otimes {}_A\Delta \otimes \text{id})(1 \otimes h^{(1)} \otimes 1 \otimes h^{(2)}_{(0)} \otimes h^{(2)}_{(1)}) \\
 &= (\text{id} \otimes {}_A\Delta \otimes \text{id} \otimes {}_A\Delta \otimes \text{id})(1 \otimes h_{(1)}^{(1)} \otimes 1 \otimes h_{(1)}^{(2)} \otimes h_{(2)}) \\
 &= (1 \otimes h_{(1)}^{(1)}_{(-1)} \otimes h_{(1)}^{(1)}_{(0)}) \otimes (1 \otimes h_{(1)}^{(2)}_{(-1)} \otimes h_{(1)}^{(2)}_{(0)}) \otimes h_{(2)} \\
 &= \ell_2(h_{(1)}) \otimes h_{(2)} = (\ell_2 \otimes \text{id})(\Delta(h)).
 \end{aligned}$$

Much in the same way, for the left H -colinearity of ℓ_2 , we use the left H -colinearity of the right translation map to compute

$$\begin{aligned}
 (55) \quad & (\Delta_{B_2}^L \otimes \text{id})(\ell_2(h)) \\
 &= S^{-1}(h^{(1)}_{(1)}) \otimes (1 \otimes h^{(1)}_{(-1)} \otimes h^{(1)}_{(0)}) \otimes (1 \otimes h^{(2)}_{(-1)} \otimes h^{(2)}_{(0)}) \\
 &= (\text{id} \otimes \text{id} \otimes {}_A\Delta \otimes \text{id} \otimes {}_A\Delta)(S^{-1}(h^{(1)}_{(1)}) \otimes 1 \otimes h^{(1)}_{(0)} \otimes 1 \otimes h^{(2)}) \\
 &= (\text{id} \otimes \text{id} \otimes {}_A\Delta \otimes \text{id} \otimes {}_A\Delta)(h_{(1)} \otimes 1 \otimes h_{(2)}^{(1)} \otimes 1 \otimes h_{(2)}^{(2)}) \\
 &= h_{(1)} \otimes (1 \otimes h_{(2)}^{(1)}_{(-1)} \otimes h_{(2)}^{(1)}_{(0)}) \otimes (1 \otimes h_{(2)}^{(2)}_{(-1)} \otimes h_{(2)}^{(2)}_{(0)}) \\
 &= h_{(1)} \otimes \ell_2(h_{(2)}) = (\text{id} \otimes \ell_2)(\Delta(h)).
 \end{aligned}$$

Summarizing, ℓ_1 and ℓ_2 are strong connections, and the lemma follows. \square

We already know that the coaction-invariant subalgebra of $A \otimes_{\tilde{H}} A$ is isomorphic to the unreduced suspension of \tilde{H} (50). Now, combining the above lemma with the right H -comodule algebra isomorphisms $B_i \cong A_i$, $i \in \{1, 2\}$, and the key fact that any two-surjective pullback of principal coactions is principal [HKrMZ11, Lemma 3.2], we arrive at the main theorem of this paper:

Theorem 2.5. *Let H be a Hopf algebra with bijective antipode, and let $A \otimes_{\tilde{H}} A$ be the \tilde{H} -braided join algebra of A . Then the coaction*

$$\Delta_{A \otimes_{\tilde{H}} A} : A \otimes_{\tilde{H}} A \longrightarrow (A \otimes_{\tilde{H}} A) \otimes H$$

is principal. Furthermore, the coaction-invariant subalgebra $(A \otimes_{\tilde{H}} A)^{\text{co}H}$ is isomorphic to the unreduced suspension of \tilde{H} (50).

3. Finite quantum coverings

In this section, first we show that for any finite-dimensional Hopf algebra H , the anti-Drinfeld double $A(H)$ is a bi-Galois object over the Drinfeld-double Hopf algebra $D(H)$. Since, by bi-Galois theory, for any right Galois object A there exists a unique left-Galois-object structure on A making A a

bi-Galois object, we conclude that the left-coacting Hopf algebra coincides with the right-coacting Hopf algebra:

$$(56) \quad \widetilde{D(H)} = D(H).$$

Then we apply our braided noncommutative join construction to the aforementioned bi-Galois object for a concrete 9-dimensional Hopf algebra H .

3.1. (Anti-)Drinfeld doubles. Recall that for any finite-dimensional Hopf algebra H , one can define the Drinfeld-double Hopf algebra $D(H) := H^* \otimes H$ by the following formulas for multiplication and comultiplication [Dri87]:

$$(57) \quad (\varphi \otimes h)(\varphi' \otimes h') := \varphi'_{(1)}(S^{-1}(h_{(3)}))\varphi'_{(3)}(h_{(1)})\varphi\varphi'_{(2)} \otimes h_{(2)}h', \\ \Delta(\varphi \otimes h) := \varphi_{(2)} \otimes h_{(1)} \otimes \varphi_{(1)} \otimes h_{(2)}.$$

Here H^* is the dual Hopf algebra, and the Heyneman-Sweedler indices refer to the coalgebra structures on H^* and H . Therefore, as a coalgebra, $D(H) = (H^*)^{\text{cop}} \otimes H$.

Much in the same way, one can define the anti-Drinfeld-double right $D(H)$ -comodule algebra $A(H) := H^* \otimes H$ by the following formulas for multiplication and coaction respectively [HKhRS04a]:

$$(58) \quad (\varphi \otimes h)(\varphi' \otimes h') := \varphi'_{(1)}(S^{-1}(h_{(3)}))\varphi'_{(3)}(S^2(h_{(1)}))\varphi\varphi'_{(2)} \otimes h_{(2)}h', \\ (59) \quad \Delta_{A(H)}(\varphi \otimes h) := \varphi_{(2)} \otimes h_{(1)} \otimes \varphi_{(1)} \otimes h_{(2)}.$$

Note that, since the formula for the right coaction is the same as the formula for the comultiplication, and $A(H) = D(H)$ as a vector space, we immediately conclude that $A(H)$ is a right $D(H)$ -Galois object. This reflects the combination of the following facts: any Yetter–Drinfeld module over H is a module over the Drinfeld double $D(H)$, any anti-Yetter–Drinfeld module over H is a module over the anti-Drinfeld double $A(H)$, and the tensor product of an anti-Yetter–Drinfeld module with a Yetter–Drinfeld module is an anti-Yetter–Drinfeld module (see [HKhRS04a] for details).

Next, let us observe that the formula

$$(60) \quad {}_{A(H)}\Delta(\psi \otimes k) := \psi_{(2)} \otimes S^2(k_{(1)}) \otimes \psi_{(1)} \otimes k_{(2)}$$

defines a left $D(H)$ -coaction on $A(H)$ which commutes with the above-defined right coaction $\Delta_{A(H)}$. Also, since the comultiplication formula (57) differs from the left-coaction formula (60) only by an automorphism $\text{id} \otimes S^2$ applied to the left tensorand, the coaction invariant subalgebra is trivial: ${}^{\text{co}D(H)}A(H) = \mathbb{C}$. By the same token, we infer that this coaction is left Hopf–Galois. Thus to arrive at the assumptions of our main result (Theorem 2.5), it suffices to show that ${}_{A(H)}\Delta$ is an algebra homomorphism. (The antipode of any finite-dimensional Hopf algebra is bijective [LS69].)

To this end, note first that φ and h' do not play an essential role in the multiplication formula (58). One can easily check that to prove that ${}_{A(H)}\Delta$

is an algebra homomorphism, one can restrict to $\varphi = \varepsilon$ and $h' = 1$. Now we compute

$$\begin{aligned}
 (61) \quad & {}_{A(H)}\Delta((\varepsilon \otimes h)(\varphi' \otimes 1)) \\
 &= {}_{A(H)}\Delta(\varphi'_{(1)}(S^{-1}(h_{(3)}))\varphi'_{(3)}(S^2(h_{(1)}))\varphi'_{(2)} \otimes h_{(2)}) \\
 &= (\varphi'_{(1)}(S^{-1}(h_{(4)}))\varphi'_{(4)}(S^2(h_{(1)}))\varphi'_{(3)} \otimes S^2(h_{(2)})) \otimes (\varphi'_{(2)} \otimes h_{(3)}).
 \end{aligned}$$

On the other hand, we compute

$$\begin{aligned}
 & {}_{A(H)}\Delta(\varepsilon \otimes h) {}_{A(H)}\Delta(\varphi' \otimes 1) \\
 &= ((\varepsilon \otimes S^2(h_{(1)}))(\varphi'_{(2)} \otimes 1)) \otimes ((\varepsilon \otimes h_{(2)})(\varphi'_{(1)} \otimes 1)) \\
 &= (\varphi'_{(2)}(S(h_{(3)}))\varphi'_{(4)}(S^2(h_{(1)}))\varphi'_{(3)} \otimes S^2(h_{(2)})) \otimes ((\varepsilon \otimes h_{(4)})(\varphi'_{(1)} \otimes 1)) \\
 &= (\varphi'_{(4)}(S(h_{(3)}))\varphi'_{(6)}(S^2(h_{(1)}))\varphi'_{(5)} \otimes S^2(h_{(2)})) \\
 &\quad \otimes (\varphi'_{(1)}(S^{-1}(h_{(6)}))\varphi'_{(3)}(S^2(h_{(4)}))\varphi'_{(2)} \otimes h_{(5)}) \\
 &= \varphi'_{(1)}(S^{-1}(h_{(6)}))\varphi'_{(3)}(S^2(h_{(4)}))\varphi'_{(4)}(S(h_{(3)}))\varphi'_{(6)}(S^2(h_{(1)}))(\varphi'_{(5)} \otimes S^2(h_{(2)})) \\
 &\quad \otimes (\varphi'_{(2)} \otimes h_{(5)}) \\
 &= \varphi'_{(1)}(S^{-1}(h_{(6)}))\varphi'_{(3)}(S(h_{(3)}S(h_{(4)})))\varphi'_{(5)}(S^2(h_{(1)}))(\varphi'_{(4)} \otimes S^2(h_{(2)})) \\
 &\quad \otimes (\varphi'_{(2)} \otimes h_{(5)}) \\
 &= \varphi'_{(1)}(S^{-1}(h_{(4)}))\varphi'_{(3)}(1)\varphi'_{(5)}(S^2(h_{(1)}))(\varphi'_{(4)} \otimes S^2(h_{(2)})) \otimes (\varphi'_{(2)} \otimes h_{(3)}) \\
 &= \varphi'_{(1)}(S^{-1}(h_{(4)}))\varphi'_{(4)}(S^2(h_{(1)}))(\varphi'_{(3)} \otimes S^2(h_{(2)})) \otimes (\varphi'_{(2)} \otimes h_{(3)}).
 \end{aligned}$$

Hence ${}_{A(H)}\Delta$ is an algebra homomorphism, as needed. Summarizing, we have arrived at:

Theorem 3.1. *Let H be a finite-dimensional Hopf algebra. Then, for coactions given by the formulas (59) and (60), the anti-Drinfeld double $A(H)$ is a bi-Galois object over the Drinfeld double $D(H)$.*

3.2. A finite quantum subgroup of $SL_{e^{2\pi i/3}}(2)$. To instantiate Theorem 2.5 and Theorem 3.1, we consider the nine-dimensional Taft Hopf algebra H [T71], which can be constructed as a quotient Hopf algebra of $\mathcal{O}(SL_{e^{2\pi i/3}}(2))$. As an algebra, H is generated by two elements a and b satisfying the relations

$$(62) \quad a^3 = 1, \quad b^3 = 0, \quad ab = qba, \quad q := e^{2\pi i/3}.$$

The comultiplication Δ , counit ε , and antipode S are respectively given by

$$\begin{aligned}
 (63) \quad & \Delta(a) := a \otimes a, & S(a) &:= a^2, & \varepsilon(a) &:= 1, \\
 & \Delta(b) := a \otimes b + b \otimes a^2, & S(b) &:= -q^2b, & \varepsilon(b) &:= 0.
 \end{aligned}$$

The set $\{b^n a^m\}_{n,m=0,1,2}$ is a linear basis of H [DHS99, Proposition 4.2].

The dual Hopf algebra H^* can be obtained as a quotient Hopf algebra of $U_{e^{2\pi i/3}}(sl(2))$. The structure of H^* and its pairing with H can be deduced from [DNS98]. We use generators k and f of H^* that in terms of generators

used in [DNS98] can be written as follows: k is the equivalence class of the grouplike generator $K \in U_{e^{2\pi i/3}}(sl(2))$ and $f := q^2 k x_-$, where x_- is the equivalence class of $X_- \in U_{e^{2\pi i/3}}(sl(2))$. Our generators satisfy the relations

$$(64) \quad k^3 = 1, \quad f^3 = 0, \quad fk = qkf.$$

The coproduct, counit and antipode are respectively given by

$$(65) \quad \begin{aligned} \Delta(k) &:= k \otimes k, & \varepsilon(k) &:= 1, & S(k) &:= k^2, \\ \Delta(f) &:= f \otimes 1 + k \otimes f, & \varepsilon(f) &:= 0, & S(f) &:= -k^2 f. \end{aligned}$$

The formulas

$$(66) \quad k(a) := q, \quad k(b) := 0, \quad f(a) := 0, \quad f(b) := 1$$

determine a nondegenerate pairing between H^* and H .

The Drinfeld double $D(H)$, as an algebra, is generated by

$$(67) \quad K := k \otimes 1, \quad F := f \otimes 1, \quad A := 1 \otimes a, \quad B := 1 \otimes b,$$

where K and F satisfy the same relations (64) as k and f , and A and B satisfy the same relations (62) as a and b . They also fulfill the cross relations

$$(68) \quad AK = KA, \quad AF = q^2 FA, \quad BK = q^2 KB, \quad BF = qFB + qKA^2 - qA.$$

The coproduct, counit and antipode are respectively determined by

$$(69) \quad \begin{aligned} \Delta(A) &:= A \otimes A, & \Delta(B) &:= A \otimes B + B \otimes A^2, \\ \Delta(K) &:= K \otimes K, & \Delta(F) &:= 1 \otimes F + F \otimes K, \\ \varepsilon(A) &:= 1 =: \varepsilon(K), & \varepsilon(B) &:= 0 =: \varepsilon(F), \\ S(A) &:= A^2, \quad S(K) &:= K^2, & S(B) &:= -q^2 B, \quad S(F) := -FK^2. \end{aligned}$$

For the anti-Drinfeld double $A(H)$, we define analogous generators:

$$(70) \quad \tilde{K} := k \otimes 1, \quad \tilde{F} := f \otimes 1, \quad \tilde{A} := 1 \otimes a, \quad \tilde{B} := 1 \otimes b.$$

It follows from (58) that \tilde{K} and \tilde{F} satisfy the same relations as k and f , and \tilde{A} and \tilde{B} fulfill the same relations as a and b . However, the cross relations now become

$$(71) \quad \tilde{A}\tilde{K} = \tilde{K}\tilde{A}, \quad \tilde{A}\tilde{F} = q^2\tilde{F}\tilde{A}, \quad \tilde{B}\tilde{K} = q^2\tilde{K}\tilde{B}, \quad \tilde{B}\tilde{F} = q\tilde{F}\tilde{B} + q^2\tilde{K}\tilde{A}^2 - q\tilde{A}.$$

The left and right $D(H)$ -coactions (60) and (59) in terms of generators are

$$(72) \quad \begin{aligned} {}_{A(H)}\Delta(\tilde{A}) &= A \otimes \tilde{A}, & \Delta_{A(H)}(\tilde{A}) &= \tilde{A} \otimes A, \\ {}_{A(H)}\Delta(\tilde{B}) &= A \otimes \tilde{B} + qB \otimes \tilde{A}^2, & \Delta_{A(H)}(\tilde{B}) &= \tilde{A} \otimes B + \tilde{B} \otimes A^2, \\ {}_{A(H)}\Delta(\tilde{K}) &= K \otimes \tilde{K}, & \Delta_{A(H)}(\tilde{K}) &= \tilde{K} \otimes K, \\ {}_{A(H)}\Delta(\tilde{F}) &= 1 \otimes \tilde{F} + F \otimes \tilde{K}, & \Delta_{A(H)}(\tilde{F}) &= 1 \otimes F + \tilde{F} \otimes K. \end{aligned}$$

Furthermore, there is an algebra isomorphism $\chi : A(H) \rightarrow D(H)$ given by

$$(73) \quad \chi(\tilde{A}) := A, \quad \chi(\tilde{B}) := qB, \quad \chi(\tilde{K}) := q^2K, \quad \chi(\tilde{F}) := q^2F.$$

A direct calculation shows that

$$(74) \quad {}_{A(H)}\Delta = (\text{id} \otimes \chi^{-1}) \circ \Delta \circ \chi.$$

Note that the existence of an equivariant algebra isomorphism between $A(H)$ and $D(H)$ means that $A(H)$ is a Hopf algebra isomorphic to $D(H)$. The existence of a finite-dimensional Hopf algebra not admitting such an equivariant algebra isomorphism is a pivotal open problem related to the famous Radford’s S^4 -formula [KR93].

It also follows from (74) that

$$(75) \quad {}_{A(H)}\text{can}^{-1}(a \otimes p) = \chi^{-1}(a_{(1)}) \otimes \chi^{-1}(S(a_{(2)}))p.$$

Indeed, applying the bijection ${}_{A(H)}\text{can}$ to the right hand side of this equality yields

$$(76) \quad a_{(1)} \otimes \chi^{-1}(a_{(2)})\chi^{-1}(S(a_{(3)}))p = a \otimes p,$$

as needed.

Our next step is to unravel the structure of the left Hopf–Galois-braided algebra $A(H) \underline{\otimes} A(H)$. To this end, we choose its generators as follows:

$$(77) \quad \begin{aligned} A_L &:= \tilde{A} \underline{\otimes} 1, & B_L &:= \tilde{B} \underline{\otimes} 1, & K_L &:= \tilde{K} \underline{\otimes} 1, & F_L &:= \tilde{F} \underline{\otimes} 1, \\ A_R &:= 1 \underline{\otimes} \tilde{A}, & B_R &:= 1 \underline{\otimes} \tilde{B}, & K_R &:= 1 \underline{\otimes} \tilde{K}, & F_R &:= 1 \underline{\otimes} \tilde{F}. \end{aligned}$$

Each of the sets of generators $\{A_L, B_L, K_L, F_L\}$ and $\{A_R, B_R, K_R, F_R\}$ satisfies the commutation relations of $A(H)$ and, from (28) and (75), we infer the cross relations:

$$\begin{aligned} A_R A_L &= A_L A_R, & B_R A_L &= q^2 A_L B_R, & K_R A_L &= A_L K_R, & F_R A_L &= q A_L F_R, \\ A_R B_L &= B_L A_R + (1 - q^2) A_L B_R, \\ B_R B_L &= q B_L B_R + (1 - q) A_L A_R B_R^2, \\ K_R B_L &= B_L K_R + (q - 1) A_L A_R^2 B_R K_R, \\ F_R F_L &= q F_L F_R + (1 - q) F_R^2, \\ A_R K_L &= K_L A_R, & B_R K_L &= q^2 K_L B_R, & K_R K_L &= K_L K_R, & F_R K_L &= q K_L F_R, \\ A_R F_L &= F_L A_R + (1 - q) A_R F_R, \\ F_R B_L &= q^2 B_L F_R - q A_L A_R K_R + A_L, \\ B_R F_L &= q^2 F_L B_R + (q - q^2) F_R B_R + q^2 A_R^2 K_R - q A_R, \\ K_R F_L &= F_L K_R + (1 - q) K_R F_R. \end{aligned}$$

Furthermore, since $A(H) \underline{\otimes} A(H) \cong H^* \otimes H \otimes H^* \otimes H$ as a vector space, the set

$$(78) \quad \{A_L^{n_1} B_L^{n_2} K_L^{n_3} F_L^{n_4} A_R^{n_5} B_R^{n_6} K_R^{n_7} F_R^{n_8} \mid n_1, \dots, n_8 \in \{0, 1, 2\}\}$$

is a linear basis of $A(H) \underline{\otimes} A(H)$. Using this basis and remembering (77), any element X of $C([0, 1]) \otimes A(H) \underline{\otimes} A(H)$ can be written as

$$(79) \quad X = \sum_{n_1, \dots, n_8=0}^2 f_{n_1, \dots, n_8} \otimes \tilde{A}^{n_1} \tilde{B}^{n_2} \tilde{K}^{n_3} \tilde{F}^{n_4} \underline{\otimes} \tilde{A}^{n_5} \tilde{B}^{n_6} \tilde{K}^{n_7} \tilde{F}^{n_8},$$

where $f_{n_1, \dots, n_8} \in C([0, 1])$. Hence

$$\begin{aligned} & A(H) \underset{D(H)}{\otimes} A(H) \\ &= \left\{ \sum_{n_1, \dots, n_8=0}^2 f_{n_1, \dots, n_8} \otimes A_L^{n_1} B_L^{n_2} K_L^{n_3} F_L^{n_4} A_R^{n_5} B_R^{n_6} K_R^{n_7} F_R^{n_8} \mid \right. \\ & \left. \text{all } f_{n_1, \dots, n_8} \in C([0, 1]), \begin{array}{l} f_{n_1, \dots, n_8}(0) = 0 \text{ for } (n_1, n_2, n_3, n_4) \neq (0, 0, 0, 0), \\ f_{n_1, \dots, n_8}(1) = 0 \text{ for } (n_5, n_6, n_7, n_8) \neq (0, 0, 0, 0) \end{array} \right\}. \end{aligned}$$

For an explicit description of the coaction-invariant subalgebra

$$(80) \quad (A(H) \underset{D(H)}{\otimes} A(H))^{\text{co}D(H)},$$

we use the fact that, by Lemma 2.1, the left canonical map ${}_{A(H)}\text{can}$ is an isomorphism of right $D(H)$ -comodule algebras. This allows us to conclude that $\{a_L^j b_L^l k_L^m f_L^n \mid j, l, m, n \in \{0, 1, 2\}\}$, where

$$(81) \quad \begin{aligned} a_L &:= {}_{A(H)}\text{can}^{-1}(A \otimes 1) = \tilde{A} \underline{\otimes} \tilde{A}^2, \\ b_L &:= {}_{A(H)}\text{can}^{-1}(B \otimes 1) = -q \tilde{A} \underline{\otimes} \tilde{B} + q^2 \tilde{B} \underline{\otimes} \tilde{A}, \\ k_L &:= {}_{A(H)}\text{can}^{-1}(K \otimes 1) = \tilde{K} \underline{\otimes} \tilde{K}^2, \\ f_L &:= {}_{A(H)}\text{can}^{-1}(F \otimes 1) = -1 \underline{\otimes} \tilde{F} \tilde{K}^2 + \tilde{F} \underline{\otimes} \tilde{K}^2, \end{aligned}$$

is a basis of the coaction-invariant subalgebra

$$(82) \quad (A(H) \underline{\otimes} A(H))^{\text{co}D(H)} \cong D(H) \otimes A(H)^{\text{co}D(H)} = D(H) \otimes \mathbb{C}.$$

Thus we obtain the following explicit description of the coaction-invariant subalgebra

$$\begin{aligned} & \left(A(H) \underset{D(H)}{\otimes} A(H) \right)^{\text{co}D(H)} \\ &= \left\{ \sum_{j, l, m, n=0}^2 g_{jlmn} \otimes a_L^j b_L^l k_L^m f_L^n \mid \text{all } g_{jlmn} \in C([0, 1]), \right. \\ & \quad \left. g_{jlmn}(0) = 0 = g_{jlmn}(1) \text{ for } (j, l, m, n) \neq (0, 0, 0, 0) \right\}. \end{aligned}$$

Since the generators a_L, b_L, k_L, f_L satisfy the same commutation relations as the generators A, B, K, F of $D(H)$, it is now evident that the coaction-invariant subalgebra is isomorphic to the unreduced suspension of $D(H)$, as claimed in Theorem 2.5.

4. *-Galois objects

To introduce a *-structure on a bi-Galois object, we need to assume that it is symmetric, i.e., that the left-coacting Hopf algebra coincides with the right-coacting Hopf algebra.

4.1. *-structure. Assume now that H is a *-Hopf algebra. This means that H is a Hopf algebra and a *-algebra such that

$$(83) \quad (* \otimes *) \circ \Delta = \Delta \circ *, \quad * \circ S \circ * \circ S = \text{id} \quad \text{and} \quad \varepsilon \circ * = \bar{} \circ \varepsilon,$$

where bar denotes the complex conjugation.

Much in the same way, we call A a right H *-comodule algebra iff it is a *-algebra and a right H -comodule algebra such that

$$(84) \quad (* \otimes *) \circ \Delta_A = \Delta_A \circ *.$$

A left *-comodule algebra is defined in the same manner.

Next, we use the algebra isomorphism ${}_{A}\text{can} : A \otimes A \rightarrow H \otimes A$ (see Lemma 2.1) to pullback the natural *-structure $h \otimes a \mapsto h^* \otimes a^*$ on $H \otimes A$ to obtain the following *-structure on the braided algebra $A \underline{\otimes} A$:

$$(85) \quad \begin{aligned} (a \underline{\otimes} b)^* &:= ({}_{A}\text{can}^{-1} \circ (* \otimes *) \circ {}_{A}\text{can})(a \underline{\otimes} b) \\ &= a^*_{(-1)}{}^{[1]} \underline{\otimes} a^*_{(-1)}{}^{[2]} b^* a^*_{(0)} = (1 \underline{\otimes} b^*) \bullet (a^* \underline{\otimes} 1). \end{aligned}$$

Our goal now is to show:

Proposition 4.1. *If A is an H *-bicomodule algebra and a bi-Galois object over H , then the H -braided join algebra $A \otimes_H A$ is a right H *-comodule algebra for the diagonal coaction.*

Proof. With the complex conjugation in the first component and the aforementioned *-structure on $A \underline{\otimes} A$, the algebra $C([0, 1]) \otimes A \underline{\otimes} A$ becomes a *-algebra. Furthermore, it follows from (85) that $(\mathbb{C} \underline{\otimes} A)^* = \mathbb{C} \underline{\otimes} A$ and $(A \underline{\otimes} \mathbb{C})^* = A \underline{\otimes} \mathbb{C}$. Therefore, as evaluation maps are *-homomorphisms, the *-structure on $C([0, 1]) \otimes A \underline{\otimes} A$ restricts to a *-structure on $A \otimes_H A$.

Next, we know from Lemma 2.1 that $\Delta_{A \underline{\otimes} A} = {}_{A}\text{can}^{-1} \circ (\text{id} \otimes \Delta_A) \circ {}_{A}\text{can}$. Since all the involved maps are *-homomorphisms, so is $\Delta_{A \underline{\otimes} A}$. Finally, since $\Delta_{A \otimes_H A}$ is a restriction of $\text{id} \otimes \Delta_{A \underline{\otimes} A}$, and $\Delta_{A \underline{\otimes} A}$ is a *-homomorphism, it follows that $\Delta_{A \otimes_H A}$ is a *-homomorphism. \square

Remark. Although it is not needed for our immediate purposes, for the sake of completeness, let us prove the left-sided version of Durdevic's formula relating the *-structure with the left translation map [Dur96, Section 2]. Let H be a *-Hopf algebra, and Q a left H *-comodule algebra such that the left canonical map (10) is bijective. Then the left translation map (see (11)) satisfies

$$(86) \quad \forall h \in H : \tau(h^*) = (h^*)^{[1]} \otimes (h^*)^{[2]} = (S^{-1}(h))^{[2]*} \otimes (S^{-1}(h))^{[1]*}.$$

To prove this, it suffices to show that Q can applied to the right hand side gives $h^* \otimes 1$. Using (18) in the second equality, we get

$$\begin{aligned}
 (87) \quad & ((S^{-1}(h))^{[2]*})_{(-1)} \otimes ((S^{-1}(h))^{[2]*})_{(0)}(S^{-1}(h))^{[1]*} \\
 &= ((S^{-1}(h))^{[2]}_{(-1)})^* \otimes ((S^{-1}(h))^{[2]}_{(0)})^*(S^{-1}(h))^{[1]*} \\
 &= (S(S^{-1}(h_{(1)})))^* \otimes (S^{-1}(h_{(2)}))^{[2]*}(S^{-1}(h_{(2)}))^{[1]*} \\
 &= h_{(1)}^* \otimes ((S^{-1}(h_{(2)}))^{[1]}(S^{-1}(h_{(2)}))^{[2]})^* \\
 &= h_{(1)}^* \otimes \overline{\varepsilon(h_2)} \\
 &= h^* \otimes 1.
 \end{aligned}$$

4.2. Noncommutative-torus algebra as a Galois object. In this subsection, we take the algebra $\mathcal{O}(\mathbb{T}^2)$ of Laurent polynomials in two variables as our $*$ -Hopf algebra H . It is generated by two commuting unitaries u and v , and the Hopf algebra structure is defined by

$$\Delta(u) := u \otimes u, \quad \Delta(v) := v \otimes v, \quad \varepsilon(u) := 1 =: \varepsilon(v), \quad S(u) := u^*, \quad S(v) := v^*.$$

Next, let $\theta \in [0, 1)$ and let $A := \mathcal{O}(\mathbb{T}_\theta^2)$ denote the polynomial $*$ -algebra of the noncommutative torus, i.e., the $*$ -algebra generated by two unitary elements U and V satisfying the relation

$$(88) \quad UV = e^{2\pi i \theta} VU.$$

We define coactions $\Delta_A : A \rightarrow A \otimes H$ and ${}_A\Delta : A \rightarrow H \otimes A$ by

$$(89) \quad \Delta_A(U) := U \otimes u, \quad \Delta_A(V) := V \otimes v, \quad {}_A\Delta(U) := u \otimes U, \quad {}_A\Delta(V) := v \otimes V.$$

These coactions turn A into an H $*$ -bicomodule algebra. As $\{U^k V^l \mid k, l \in \mathbb{Z}\}$ is a linear basis of A (by the Diamond Lemma [Be78, Theorem 1.2]), one sees immediately that ${}^{\text{co}H}A = \mathbb{C} = A^{\text{co}H}$. Furthermore, it is straightforward to check that the inverses of the left and right canonical maps are respectively given by

$${}_A\text{can}^{-1}(u^k v^l \otimes a) = U^k V^l \otimes V^{-l} U^{-k} a, \quad \text{can}_A^{-1}(a \otimes u^k v^l) = a V^{-l} U^{-k} \otimes U^k V^l.$$

Hence A is a bi-Galois object over H . As the antipode of H is bijective, A satisfies all assumptions of Theorem 2.5.

Using (88), one easily verifies that the braiding (27) reads

$$(90) \quad A \otimes A \ni U^k V^l \otimes U^m V^n \mapsto e^{2\pi i \theta(kn - lm)} U^m V^n \otimes U^k V^l \in A \otimes A.$$

Now the product (28) in $A \underline{\otimes} A$ is determined by

$$\begin{aligned}
 (91) \quad & (U^r V^s \underline{\otimes} U^k V^l) \bullet (U^m V^n \underline{\otimes} U^a V^b) \\
 &= e^{2\pi i \theta(kn - lm)} U^r V^s U^m V^n \underline{\otimes} U^k V^l U^a V^b \\
 &= e^{2\pi i \theta(kn - lm - sm - la)} U^{r+m} V^{s+n} \underline{\otimes} U^{k+a} V^{l+b},
 \end{aligned}$$

where $r, s, k, l, m, n, a, b \in \mathbb{Z}$. One readily checks that the elements

$$(92) \quad U_L := U \underline{\otimes} 1, \quad V_L := V \underline{\otimes} 1, \quad U_R := 1 \underline{\otimes} U, \quad V_R := 1 \underline{\otimes} V,$$

satisfy the relations

$$(93) \quad \begin{aligned} U_R U_L &= U_L U_R, & V_R V_L &= V_L V_R, \\ U_L V_L &= e^{2\pi i \theta} V_L U_L, & U_R V_R &= e^{2\pi i \theta} V_R U_R, \end{aligned}$$

$$(94) \quad U_R V_L = e^{2\pi i \theta} V_L U_R, \quad V_R U_L = e^{-2\pi i \theta} U_L V_R.$$

It follows from (85) that U_L, V_L, U_R, V_R are unitary. Furthermore, since they generate $A \otimes A$, any element $y \in C([0, 1]) \otimes A \otimes A$ can be written as

$$(95) \quad y = \sum_{\text{finite}} f_{klmn} \otimes U_L^k V_L^l U_R^m V_R^n, \quad f_{klmn} \in C([0, 1]).$$

From $U_L^k V_L^l U_R^m V_R^n = U^k V^l \otimes U^m V^n$, we conclude that

$$\begin{aligned} &A \otimes_H A \\ &= \left\{ \sum_{\text{finite}} f_{klmn} \otimes U_L^k V_L^l U_R^m V_R^n \in C([0, 1]) \otimes A \otimes A \mid k, l, m, n \in \mathbb{Z}, \right. \\ &\quad \left. f_{klmn}(0) = 0 \text{ for } (k, l) \neq (0, 0), f_{klmn}(1) = 0 \text{ for } (m, n) \neq (0, 0) \right\}. \end{aligned}$$

Finally, the diagonal coaction $\Delta_{A \otimes_H A} : A \otimes_H A \rightarrow (A \otimes_H A) \otimes H$ is determined by

$$(96) \quad \Delta_{A \otimes_H A}(f \otimes U_L^k V_L^l U_R^m V_R^n) = f \otimes U_L^k V_L^l U_R^m V_R^n \otimes u^{k+m} v^{l+n}.$$

By Theorem 2.5, the above coaction is principal (admits a strong connection), and the coaction-invariant subalgebra $(A \otimes_H A)^{\text{co}H}$ can be viewed as an algebra of functions on the unreduced suspension of the classical torus. Explicitly, we have

$$\begin{aligned} (A \otimes_H A)^{\text{co}H} &= \left\{ \sum_{\text{finite}} g_{kl} \otimes X^k Y^l \in A \otimes_H A \mid g_{kl}(0) = 0 = g_{kl}(1) \right. \\ &\quad \left. \text{for } (k, l) \neq (0, 0), k, l \in \mathbb{Z} \right\}, \end{aligned}$$

where $X := U_L U_R^* = U \otimes U^*$ and $Y := V_L V_R^* = V \otimes V^*$ are commuting unitaries.

To end with, let us note that, as the Hopf algebra H is commutative, the diagonal coaction $A \otimes A \rightarrow A \otimes A \otimes H$ is an algebra homomorphism already for the trivial braiding (the flip). However, for the nonbraided tensor algebra $A \otimes A$, the left canonical map ${}_{A\text{can}}$ is no longer an algebra homomorphism:

$$(97) \quad \begin{aligned} {}_{A\text{can}}((1 \otimes U)(V \otimes 1)) &= {}_{A\text{can}}(V \otimes U) = v \otimes VU \\ &\neq v \otimes UV = (1 \otimes U)(v \otimes V) \\ &= {}_{A\text{can}}(1 \otimes U) {}_{A\text{can}}(V \otimes 1). \end{aligned}$$

The braided algebra $A \otimes A$ is “more noncommutative” than $A \otimes A$ in the sense that the relations (93) among the generators are the same in both cases, but the relations (94) reduce to commutativity relations for $A \otimes A$.

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