New York Journal of Mathematics

New York J. Math. **23** (2017) 505–526.

On numerical invariants for homogeneous submodules in $H^2(\mathbb{D}^2)$

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Abstract. The Hardy space $H^2(\mathbb{D}^2)$ can be viewed as a module over the polynomial ring $\mathbb{C}[z,w]$ with module action defined by multiplication of functions. The core operator is a bounded self-adjoint integral operator defined on submodules of $H^2(\mathbb{D}^2)$, and it gives rise to some interesting numerical invariants for the submodules. These invariants are difficult to compute or estimate in general. This paper computes these invariants for homogeneous submodules through Toeplitz determinants.

Contents

0. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk with boundary

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

The Hardy space $H^2(\mathbb{D}^2)$ over the bidisk is the closure of all polynomials in $L^2(\mathbb{T}^2, dm)$, where dm is the normalized Lebesgue measure on $\mathbb{T}^2$. The Hardy space $H^2(\mathbb{D}^2)$ can be viewed as a module over the polynomial ring $\mathbb{C}[z,w]$ with module action defined by multiplication of functions. Thus a closed subspace M of $H^2(\mathbb{D}^2)$ is a submodule if and only if it is invariant under multiplication by both coordinate functions z and w.

In the classical Hardy space $H^2(\mathbb{D})$ (which is a module over $\mathbb{C}[z]$), Beurling's theorem ([1]) asserts that every submodule is of the form $M = \theta H^2(\mathbb{D})$

Received February 1, 2017.

2010 *Mathematics Subject Classification.* Primary 47A13; Secondary 46E20.

Key words and phrases. Hardy space over the bidisk, submodule, core operator, fringe operator, Toeplitz matrix.

ISSN 1076-9803/2017

for some inner function $\theta(z)$. This theorem has no direct generalizations to $H^2(\mathbb{D}^2)$. As a matter of fact, in [8] Rudin constructed two somewhat "pathological" submodules: one is infinitely generated and the other contains no bounded functions other than 0. This fact seems to suggest that function-theoretic approach to characterizing submodules in $H^2(\mathbb{D}^2)$ is nearly impossible. An alternative operator-theoretic approach has proven to be successful in the past two decades. A key ingredient of this approach is the so-called core operator C^M defined for submodules M in [6]. C^M is an invariant for M in the sense that if M' is a submodule that is unitarily equivalent to M then $C^{M'}$ is unitarily equivalent to C^M. Thus numerical invariants of C^M, such as rank, eigenvalues, trace, or Hilbert–Schmidt norm, are indeed invariants for the submodule M. Although the Hilbert–Schmidtness of C^M is proved in [6] for a very large class of submodules, explicitly computing or estimating these invariants remains a challenging task. Among submodules, those generated by a single homogenous polynomial p has a relatively simple structure. This type of submodules will be called homogeneous submodules and denoted by $[p]$. In recent years, much progress has been made in understanding the essential normality of the quotient module $H^2(\mathbb{D}^2) \ominus [p]$ (cf. [4, 5]). This paper computes the numerical invariants for this type of submodules. Toeplitz determinant plays an important role in the computations.

1. Preparation

Let T_z and T_w be multiplication operators by z and w on $H^2(\mathbb{D}^2)$ respectively. We denote by R_z and R_w the restrictions of T_z and T_w to submodule M. Clearly, (R_z, R_w) is a pair of commuting isometries acting on M. The pair (S_z, S_w) is the compression of Toeplitz operators (T_z, T_w) to the quotient space $H^2(\mathbb{D}^2) \ominus M$. To be precise:

$$\begin{aligned} S_z f &= (I - P_M) z f, \\ S_w f &= (I - P_M) w f, \quad f \in H^2(\mathbb{D}^2) \ominus M, \end{aligned}$$

where P_M is the orthogonal projection from $H^2(\mathbb{D}^2)$ onto M. We denote the reproducing kernel for $H^2(\mathbb{D}^2)$ by $K(\lambda, z)$, i.e., for $\lambda, z \in \mathbb{D}^2$,

$$K(\lambda, z) = \frac{1}{(1 - \overline{\lambda_1} z_1)(1 - \overline{\lambda_2} z_2)}.$$

By $K^M(\lambda, z)$ we mean the reproducing kernel for the submodule M. The core operator on $H^2(\mathbb{D}^2)$ is given by

$$C^M(f)(z) := \int_{\mathbb{T}^2} G^M(\lambda, z) f(\lambda) dm(\lambda), \quad z \in \mathbb{D}^2$$

where $G^M(\lambda, z)$ is the core function defined as

$$G^M(\lambda, z) = \frac{K^M(\lambda, z)}{K(\lambda, z)} = (1 - \overline{\lambda_1} z_1)(1 - \overline{\lambda_2} z_2) K^M(\lambda, z)\ , \quad \lambda, z \in \mathbb{D}^2.$$

A submodule M is said to be Hilbert–Schmidt if its core operator C^M is Hilbert–Schmidt, or equivalently its core function $G^M(\lambda, z)$ is in $L^2(\mathbb{T}^2 \times \mathbb{T}^2)$.

The following relation of core operator C^M with operators R_z and R_w is shown in [6]

$$C^M = I - R_z R_z^* - R_w R^*{}_w + R_z R_w R_z^* R_w^* \quad \textit{on } M. \tag{1.1}$$

From (1.1) it is easy to see that C^M (or C for short) is a bounded self-adjoint operator on M. We denote by $[R_z^*, R_z]$ and $[R_w^*, R_w]$ the self commutators for operators R_z and R_w respectively, and for simplicity let

$$P_z := [R_z^*, R_z] \quad \text{and} \quad P_w := [R_w^*, R_w].$$

It is easy to see that P_z and P_w are orthogonal projections from M onto the defect spaces $M \ominus zM$ and $M \ominus wM$, respectively. We have the following theorem from [14].

Theorem 1.1. *If M is a submodule of $H^2(\mathbb{D}^2)$ generated by a finite number of polynomials then:*

(a) $[S_z^*, S_w]$ *is Hilbert–Schmidt on* $H^2(\mathbb{D}^2) \ominus M$*,*
(b) $[R_z^*, R_w]$ *is Hilbert–Schmidt on* M*,*
(c) $[R_z^*, R_z][R_w^*, R_w]$ *is Hilbert–Schmidt on* M*.*

For convenience we let

$$T := [R_z^*, R_z][R_w^*, R_w][R_z^*, R_z], \quad S := [R_z^*, R_w][R_w^*, R_z].$$

By the above theorem, for M generated by a finite number of polynomials T and S are trace class. We set $\Sigma_0 = tr(T)$ and $\Sigma_1 = tr(S)$. If $\{\Phi_n : n = 0, 1, \ldots, \infty\}$ is an orthonormal basis for $M \ominus zM$ and $\{\Psi_n : n = 0, 1, \ldots, \infty\}$ is an orthonormal basis for $M \ominus wM$, then by [13] we have

$$\Sigma_0 = \sum_{n=0}^{\infty} |\langle \Phi_n, \Psi_n \rangle|^2, \quad \text{and} \quad \Sigma_1 = \sum_{n=0}^{\infty} |\langle w\Phi_n, z\Psi_n \rangle|^2.$$

Furthermore, it is shown in [12] that for Hilbert–Schmidt submodules

$$\Sigma_0 - \Sigma_1 = 1.$$

The following lemma is taken from [11].

Lemma 1.2. *For every submodule M, C^2 is unitarily equivalent to the diagonal block matrix*

$$\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}. \tag{1.2}$$

So in particular, we have $tr(C^2) = \Sigma_0 + \Sigma_1$.

2. Orthonormal bases for defect spaces

The subspaces $M \ominus zM$ and $M \ominus wM$ are sometimes called defect spaces for submodule M. They capture much information about M. Except for a few submodules, the orthonormal basis for the defect spaces are impossible to compute. However, homogeneous submodule $[p]$ has a nice orthogonal decomposition, and that enables us to determine the orthonormal basis.

Let $H_n = span\{z^i w^j | i+j=n, i,j \geq 0\}$ be the space of degree n homogeneous polynomials. Clearly, zH_n is a subspace in H_{n+1} with codimension 1. Let

$$p = \sum_{j=0}^{k} c_j z^j w^{k-j}$$

be a homogeneous polynomial of degree k. Then it is easy to see that $M = [p] = \overset{\infty}{\underset{n=0}{\oplus}} pH_n$, and hence $zM = \overset{\infty}{\underset{n=0}{\oplus}} pzH_n$. Therefore,

$$M \ominus zM = \mathbb{C}p \oplus \overset{\infty}{\underset{n=1}{\oplus}} \left(pH_n \ominus pzH_{n-1}\right).$$

Likewise,

$$M \ominus wM = \mathbb{C}p \oplus \overset{\infty}{\underset{n=1}{\oplus}} \left(pH_n \ominus pwH_{n-1}\right).$$

We first set $\Phi_0 = \Psi_0 = \frac{p}{\|p\|}$.

So to find an orthonormal basis $\{\Phi_n : n = 0, 1, \ldots, \infty\}$ for $M \ominus zM$ is to find a $h_n = \sum\limits_{j=0}^{n} {c_n}^j z^j w^{n-j} \in H_n$ such that

(a) $ph_n \in pH_n \ominus pzH_{n-1}$, $n \geq 1$,
(b) $\|ph_n\| = 1$.

Since $ph_n \perp pzH_{n-1}$, we have

$$\begin{aligned} 0 &= \langle ph_n, pz^{k+1}w^{n-1-k}\rangle \\ &= \sum_{j=0}^{n} {c_n}^j \langle pz^j w^{n-j}, pz^{k+1}w^{n-k-1}\rangle \\ &= \sum_{j=0}^{n} \langle pw^{k+1-j}, pz^{k+1-j}\rangle c_n^j, \quad 0 \leq k \leq n-1. \end{aligned}$$

Replacing $k+1$ by i we have

$$\sum_{j=0}^{n} \langle pw^{i-j}, pz^{i-j}\rangle c_n^j = 0, \quad 1 \leq i \leq n. \tag{2.1}$$

(2.1) is a system of n equations with $n+1$ unknowns $c_n^0, c_n^1, ..., c_n^n$. Write

$$\overrightarrow{C} = \begin{pmatrix} c_n^0 \\ c_n^1 \\ \vdots \\ c_n^n \end{pmatrix}$$

and let A^n be the Gramian matrix $(\langle pz^j w^{n-j}, pz^i w^{n-i}\rangle)_{(n+1)\times(n+1)}$. Then by a simple calculation using the fact $z^{-1} = \bar{z}$ and $w^{-1} = \bar{w}$, we have

$$A^n = \begin{pmatrix} ||p||^2 & \overline{\langle pw, pz\rangle} & \dots & \overline{\langle pw^n, pz^n\rangle} \\ \langle pw, pz\rangle & ||p||^2 & \dots & \overline{\langle pw^{n-1}, pz^{n-1}\rangle} \\ \langle pw^2, pz^2\rangle & \langle pw, pz\rangle & \dots & \overline{\langle pw^{n-2}, pz^{n-2}\rangle} \\ \vdots & \vdots & & \vdots \\ \langle pw^n, pz^n\rangle & \langle pw^{n-1}, pz^{n-1}\rangle & \dots & ||p||^2 \end{pmatrix}.$$

Note that A^n is an $(n+1)\times(n+1)$ Toeplitz matrix! Further, since A^n is Gramian, it is positive definite. Now remove the first row in A^n and denote the resulting matrix by A_*^n. Then the system (2.1) can be written as $A_*^n \overrightarrow{C} = 0$, or more explicitly

(2.2)

$$\begin{pmatrix} \langle pw, pz\rangle & ||p||^2 & \dots & \overline{\langle pw^{n-2}, pz^{n-2}\rangle} & \overline{\langle pw^{n-1}, pz^{n-1}\rangle} \\ \langle pw^2, pz^2\rangle & \langle pw, pz\rangle & \dots & \overline{\langle pw^{n-1}, pz^{n-1}\rangle} & \overline{\langle pw^{n-2}, pz^{n-2}\rangle} \\ \vdots & \vdots & & \vdots & \vdots \\ \langle pw^n, pz^n\rangle & \langle pw^{n-1}, pz^{n-1}\rangle & \dots & \langle pw, pz\rangle & ||p||^2 \end{pmatrix} \begin{pmatrix} c_n^0 \\ c_n^1 \\ \vdots \\ c_n^0 \end{pmatrix}$$
$$= 0.$$

Since A^n is invertible, A_*^n has rank n, and hence (2.2) has a nontrivial solution and its solution space is 1-dimensional.

Writing $A^n = (a_{i,j})_{i,j=0}^n$, where $a_{i,j} = \langle pw^{i-j}, pz^{i-j}\rangle, \quad i,j = 0,1,\dots,n$, and denoting its cofactor matrix by $\left(A_{i,j}^n\right)_{i,j=0}^n$, then by cofactor theorem we have

$$a_{i,0}A_{0,0}^n + a_{i,1}A_{0,1}^n + \cdots + a_{i,n}A_{0,n}^n = 0, \quad i = 1,2,\dots,n. \tag{2.3}$$

Comparing (2.3) with (2.1), and using the fact the solution space of (2.1) is one-dimensional, we have

$$\begin{pmatrix} c_n^0 \\ c_n^1 \\ \vdots \\ c_n^0 \end{pmatrix} = k \begin{pmatrix} A_{0,0}^n \\ A_{0,1}^n \\ \vdots \\ A_{0,n}^n \end{pmatrix},$$

for some scalar k. Therefore we have

$$\begin{aligned}\Phi_n = ph_n &= p\sum_{j=0}^{n} c_n^j z^j w^{n-j} \\ &= k\sum_{j=0}^{n} pA_{0,j}^n z^j w^{n-j}\end{aligned}$$

where the constant $k \in \mathbb{C}$ is to normalize Φ_n such that $\|\Phi_n\|= 1$. Without loss of generality, we assume $k > 0$. To determine k, we consider

$$\begin{aligned}1 &= \langle \Phi_n, \Phi_n \rangle \\ &= k^2 \left\langle \sum_{i=0}^{n} pA_{0,i}^n z^i w^{n-i}, \sum_{j=0}^{n} pA_{0,j}^n z^j w^{n-j} \right\rangle \\ &= k^2 \sum_{i,j=0}^{n} A_{0,i}^n \langle pw^{j-i}, pz^{j-i} \rangle \overline{A_{0,j}^n} \\ &= k^2 \left\langle A^n \begin{pmatrix} A_{0,0}^n \\ A_{0,1}^n \\ \vdots \\ A_{0,n}^n \end{pmatrix}, \begin{pmatrix} A_{0,0}^n \\ A_{0,1}^n \\ \vdots \\ A_{0,n}^n \end{pmatrix} \right\rangle.\end{aligned} \tag{2.4}$$

By cofactor theorem

$$A^n \begin{pmatrix} A_{0,0}^n \\ A_{0,1}^n \\ \vdots \\ {A^n}_{0,n} \end{pmatrix} = \begin{pmatrix} \det A^n \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This along with (2.4) gives $k^2 A_{0,0}^n \det A^n = 1$. For simplicity, for the rest of the paper we shall denote $\det A^{n-1}$ by D_n, $n \geq 1$. Since $A_{0,0}^n = \det A^{n-1}$, by (2.4) we have

$$k = \frac{1}{\sqrt{D_{n+1} D_n}}.$$

Therefore, we conclude that

$$\Phi_n(z,w) = \frac{\sum_{j=0}^{n} pA_{0,j}^n z^j w^{n-j}}{\sqrt{D_{n+1} D_n}}. \tag{2.5}$$

Now we turn to the orthonormal basis $\{\Psi_n : n = 0, 1, \dots, \infty\}$ for $M \ominus wM$. The calculation is similar but with a slight difference at (2.7).

Define $\Psi_n = ph'_n$, where $h'_n = \sum_{j=0}^{n} c'^j_n z^j w^{n-j}$ satisfies

(a) $ph'_n \in pH_n \ominus wpH_{n-1}, \quad n \geq 1$,

(b) $||ph'_n||= 1$.

Going through similar steps as that for Φ_n, we get $c'^j_n = k' A^n_{n,j}$, $j = 0, 1, 2, \ldots, n$, where $k' > 0$, and the equation

$$k'^2 \left\langle A_n \begin{pmatrix} A^n_{n,0} \\ A^n_{n,1} \\ \vdots \\ A^n_{n,n} \end{pmatrix}, \begin{pmatrix} A^n_{n,0} \\ A^n_{n,1} \\ \vdots \\ A^n_{n,n} \end{pmatrix} \right\rangle = 1. \tag{2.6}$$

Applying cofactor theorem in (2.6) we get

$$k'^2 \left\langle \begin{pmatrix} 0 \\ 0 \\ \vdots \\ D_{n+1} \end{pmatrix}, \begin{pmatrix} A^n_{n,0} \\ A^n_{n,1} \\ \vdots \\ A^n_{n,n} \end{pmatrix} \right\rangle = k'^2 D_{n+1} A^n_{n,n} = 1. \tag{2.7}$$

Since $A^n_{n,n} = D_n$, we have

$$k' = \frac{1}{\sqrt{D_n D_{n+1}}},$$

and therefore

$$\Psi_n = \frac{\sum\limits_{j=0}^{n} p A^n_{n,j} z^j w^{n-j}}{\sqrt{D_n D_{n+1}}}. \tag{2.8}$$

We summarize (2.5) and (2.8) as:

Proposition 2.1. *Let $[p]$ be a homogeneous submodule. Then $\{\Phi_n : n \geq 0\}$ is an orthonormal basis for $M \ominus zM$ and $\{\Psi_n : n \geq 0\}$ is an orthonormal basis for $M \ominus wM$, where*

$$\Phi_0 = \Psi_0 = \frac{p}{\|p\|}, \quad \Phi_n = \frac{\sum\limits_{j=0}^{n} p A^n_{0,j} z^j w^{n-j}}{\sqrt{D_{n+1} D_n}}, \quad \Psi_n = \frac{\sum\limits_{j=0}^{n} p A^n_{n,j} z^j w^{n-j}}{\sqrt{D_{n+1} D_n}}, \quad n \geq 1.$$

Proposition 2.1 makes it possible to compute the numerical invariants Σ_0, Σ_1, as well as the eigenvalues of the core operator C.

Corollary 2.2. *For homogeneous submodule $[p]$ we have:*

(a) $\langle w\Phi_n, z\Psi_n \rangle = -\frac{A^{n+1}_{0,n+1}}{D_{n+1}}$,

(b) $\langle \Phi_n, \Psi_n \rangle = \frac{\overline{A^n_{0,n}}}{D_n}$.

Proof. First notice that $A^n_{0,0} = A^n_{n,n} = D_n$. By Proposition 2.1, we then have

$$\langle w\Phi_n, z\Psi_n \rangle = \left\langle wk \sum_{i=0}^{n} p A^n_{0,i} z^i w^{n-i}, zk' \sum_{j=0}^{n} p A^n_{n,j} z^j w^{n-j} \right\rangle \tag{2.9}$$

$$= \frac{1}{D_n D_{n+1}} \left\langle \sum_{i=0}^{n} pA^n_{0,i} z^i w^{n+1-i}, \sum_{j=0}^{n} pA^n_{n,j} z^{j+1} w^{n-j} \right\rangle$$
$$= \frac{1}{D_n D_{n+1}} \sum_{i,j=0}^{n} A^n_{0,i} \langle pw^{j+1-i}, pz^{j+1-i}\rangle \overline{A^n_{n,j}}.$$

For clarity we write the last summation as

(2.10)

$$\left\langle \begin{pmatrix} \langle pw, pz\rangle & ||p||^2 & \dots & \overline{\langle pw^{n-1}, pz^{n-1}\rangle} \\ \langle pw^2, pz^2\rangle & \langle pw, pz\rangle & \dots & \overline{\langle pw^{n-2}, pz^{n-2}\rangle} \\ \vdots & \vdots & & \vdots \\ \langle pw^{n+1}, pz^{n+1}\rangle & \langle pw^n, pz^n\rangle & \dots & \langle pw, pz\rangle \end{pmatrix} \begin{pmatrix} A^n_{0,0} \\ A^n_{0,1} \\ \vdots \\ A^n_{0,n} \end{pmatrix}, \begin{pmatrix} A^n_{n,0} \\ A^n_{n,1} \\ \vdots \\ A^n_{n,n} \end{pmatrix} \right\rangle$$
$$:= \left\langle \begin{pmatrix} E_0 \\ E_1 \\ \vdots \\ E_n \end{pmatrix}, \begin{pmatrix} A^n_{n,0} \\ A^n_{n,1} \\ \vdots \\ A^n_{n,n} \end{pmatrix} \right\rangle.$$

By cofactor theorem, we have

$$E_i = \sum_{j=0}^{n} a_{i+1,j} A^n_{0,j} = 0, \quad i = 0, 1, 2, \dots, n-1.$$

To compute E_n, we observe the matrix

$$A^{n+1} = \begin{pmatrix} ||p||^2 & \overline{\langle pw, pz\rangle} & \dots & \overline{\langle pw^n, pz^n\rangle} & \overline{\langle pw^{n+1}, pz^{n+1}\rangle} \\ \langle pw, pz\rangle & ||p||^2 & \dots & \langle pw^{n-1}, pz^{n-1}\rangle & \overline{\langle pw^n, pz^n\rangle} \\ \vdots & \vdots & & \vdots & \vdots \\ \langle pw^n, pz^n\rangle & \langle pw^{n-1}, pz^{n-1}\rangle & \dots & ||p||^2 & \overline{\langle pw, pz\rangle} \\ \langle pw^{n+1}, pz^{n+1}\rangle & \langle pw^n, pz^n\rangle & \dots & \langle pw, pz\rangle & ||p||^2 \end{pmatrix}$$

and let M be the submatrix by removing the 0-th row and $(n+2)$-th column in A^{n+1}, i.e.,

$$M = \begin{pmatrix} \langle pw, pz\rangle & ||p||^2 & \dots & \langle pw^{n-1}, pz^{n-1}\rangle \\ \vdots & \vdots & & \vdots \\ \langle pw^n, pz^n\rangle & \langle pw^{n-1}, pz^{n-1}\rangle & \dots & ||p||^2 \\ \langle pw^{n+1}, pz^{n+1}\rangle & \langle pw^n, pz^n\rangle & \dots & \langle pw, pz\rangle \end{pmatrix}.$$

Then expanding $\det M$ along its bottom row, we have

$$\det M = (-1)^{n+2}\langle pw^{n+1}, pz^{n+1}\rangle \det M_{n,0} + (-1)^{n+3}\langle pw^n, pz^n\rangle \det M_{n,1}$$
$$+ \dots + (-1)^{2n+2}\langle pw, pz\rangle \det M_{n,n},$$

where $M_{n,j}$ is the submatrix of M with n-th row and j-th column of M removed for $j = 0, 1, \ldots, n$. Observe that $M_{n,j}$ coincides with the submatrix of A^n formed by removing 0-th row and j-th column of A^n. Hence we have $A^n_{0,j} = (-1)^{j+2} \det M_{n,j}$ for each j. Therefore

$$\begin{aligned} E_n &= \langle pw^{n+1}, pz^{n+1}\rangle A^n_{0,0} + \langle pw^n, pz^n\rangle A^n_{0,1} + \cdots + \langle pw, pz\rangle A^n_{0,n} \\ &= (-1)^2 \langle pw^{n+1}, pz^{n+1}\rangle \det M_{n,0} + (-1)^3 \langle pw^n, pz^n\rangle \det M_{n,1} \\ &\quad + \cdots + (-1)^{2+n} \langle pw, pz\rangle \det M_{n,n} = (-1)^n \det M. \end{aligned} \tag{2.11}$$

Since $A^{n+1}_{0,n+1} = (-1)^{n+3} \det M$, we have

$$E_n = (-1)^n \det M = (-1)^{2n+3} A^{n+1}_{0,n+1} = -A^{n+1}_{0,n+1}. \tag{2.12}$$

Now it follows from (2.9) and (2.10) that

$$\begin{aligned} \langle w\Phi_n, z\Psi_n\rangle &= \frac{1}{D_n D_{n+1}} \left\langle \begin{pmatrix} 0 \\ 0 \\ \vdots \\ E_n \end{pmatrix}, \begin{pmatrix} A^n_{n,0} \\ A^n_{n,1} \\ \vdots \\ A^n_{n,n} \end{pmatrix} \right\rangle \\ &= \frac{-1}{D_n D_{n+1}} A^{n+1}_{0,n+1} \overline{A^n_{n,n}}. \end{aligned}$$

Since $A^n_{0,0} = A^n_{n,n} = D_n > 0$, we have that

$$\langle w\Phi_n, z\Psi_n\rangle = \frac{-A^{n+1}_{0,n+1}}{D_{n+1}}. \tag{2.13}$$

Next we compute $\langle \Phi_n, \Psi_n\rangle$ and the steps are similar. Check that

$$\begin{aligned} \langle \Phi_n, \Psi_n\rangle &= \left\langle k \sum_{i=0}^n pA^n_{0,i}\, z^i w^{n-i}, k' \sum_{j=0}^n pA^n_{n,j}\, z^j w^{n-j} \right\rangle \\ &= \frac{1}{D_n D_{n+1}} \sum_{i,j=0}^n A^n_{0,i} \langle pw^{j-i}, pz^{j-i}\rangle \overline{A^n_{n,j}} \\ &= \frac{1}{D_n D_{n+1}} \left\langle A^n \begin{pmatrix} A^n_{0,0} \\ A^n_{0,1} \\ \vdots \\ A^n_{0,n} \end{pmatrix}, \begin{pmatrix} A^n_{n,0} \\ A^n_{n,1} \\ \vdots \\ A^n_{n,n} \end{pmatrix} \right\rangle. \end{aligned} \tag{2.14}$$

Again, using cofactor theorem, we have

$$\langle \Phi_n, \Psi_n\rangle = \frac{1}{D_n D_{n+1}} \left\langle \begin{pmatrix} \det A^n \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} A^n_{n,0} \\ A^n_{n,1} \\ \vdots \\ A^n_{n,n} \end{pmatrix} \right\rangle \tag{2.15}$$

$$= \frac{1}{D_n D_{n+1}} \det A^n \overline{A^n_{n,0}} = \frac{\overline{A^n_{0,n}}}{D_n}.$$

Therefore, for $n \geq 0$ we have:

(a) $\langle w\Phi_n, z\Psi_n \rangle = -\frac{A^{n+1}_{0,n+1}}{D_{n+1}}$,

(b) $\langle \Phi_n, \Psi_n \rangle = \frac{\overline{A^n_{0,n}}}{D_n}$,

which completes the proof. □

Clearly, it follows from Corollary 2.2 that

$$|\langle w\Phi_n, z\Psi_n \rangle| = |\langle \Phi_{n+1}, \Psi_{n+1} \rangle| = \frac{|A^{n+1}_{0,n+1}|}{D_{n+1}}. \tag{2.16}$$

Corollary 2.3. *For homogeneous submodule* $[p]$*, we have*

$$\Sigma_0 = \sum_{n=0}^{\infty} \left| \frac{A^n_{0,n}}{D_n} \right|^2.$$

It is interesting to observe that it follows directly from (2.16) that

$$\Sigma_0 - \Sigma_1 = 1.$$

This in fact holds for all Hilbert–Schmidt submodules ([12]).

3. Eigenvalues of C

In general, eigenvalue problem for core operators associated with arbitrary submodules of $H^2(\mathbb{D}^2)$ is difficult to study. In this section, using the orthonormal basis in Proposition 2.1, we will compute the eigenvalues of the core operator for homogeneous submodules $[p]$. The eigenvalue formula shall depend solely on the coefficients of p. By (1.2) and results in [9], if λ is an eigenvalue of C then λ^2 is an eigenvalue of T. Moreover, if $\lambda \in (-1,1)$ is an eigenvalue of C then so is $-\lambda$. So we shall focus our attention to the eigenvalues of T.

Some preparation is needed. For an eigenvalue λ of an operator A , let $E_\lambda(A)$ denote the corresponding eigenspace. It is shown in [6] that:

$$E_1(C) = M \ominus (zM + wM) \textit{ and } E_{-1}(C) = (zM \cap wM) \ominus zwM. \tag{3.1}$$

The following proposition is from [15].

Proposition 3.1. *If* M *is a submodule such that* $[R_z^*, R_w]$ *is compact on* M*, then* (R_z, R_w) *is Fredholm and* $\dim(Ker(S_z) \cap Ker(S_w)) < \infty$ *and*

$$\operatorname{ind}(R_z, R_w) = \dim(Ker(S_z) \cap Ker(S_w)) - \dim(Ker(R_z^*) \cap Ker(R_w^*)).$$

Since

$$Ker(S_z) \cap Ker(S_w) = (zM \cap wM) \ominus zwM,$$
$$Ker(R_z^*) \cap Ker(R_w^*) = M \ominus (zM + wM),$$

(3.1) and Proposition 3.1 give:

$$\operatorname{ind}(R_z, R_w) = \dim(E_{-1}(C)) - \dim(E_1(C)).$$

Definition 3.2. For a submodule M of $H^2(\mathbb{D}^2)$ the fringe operator F on $M \ominus zM$ is defined as $Ff = P_z wf$ where $P_z := [R_z^*, R_z]$ is as in Section 1.

For further studies and insights on fringe operators see [13]. Here we quote a result to be used later.

Lemma 3.3. *Let M be a submodule. Then on $M \ominus zM$ we have:*

(a) $I - F^*F = [R_w^*, R_z]\,[R_z^*, R_w]$,
(b) $I - FF^* = [R_z^*, R_z]\,[R_w^*, R_w]$.

The following two facts are from [12].

Lemma 3.4. *If M is a submodule, then rank$(M) \geq \dim(M \ominus (zM + wM))$. In particular, if M is singly generated then $\dim(M \ominus (zM + wM)) = 1$.*

Lemma 3.5. *If M is a Hilbert–Schmidt submodule then $\operatorname{ind}(R_1, R_2) = -1$.*

The next lemma is from [9].

Lemma 3.6. *Let M be any submodule. If $\lambda \in (-1, 1)$ is an eigenvalue of core operator C then $-\lambda$ is also an eigenvalue of C.*

Proposition 2.1 enables us to compute all the eigenvalues of C. First of all 0 is clearly an eigenvalue of C. By Lemma 1.2 the eigenvalues of operator $T := [R_z^*, R_z][R_w^*, R_w][R_z^*, R_z]$ are also eigenvalues of C^2. We now check that if λ is an eignevalue of C such that $0 < |\lambda| < 1$ then λ^2 is an eigenvalue of T. In fact, Lemma 1.2 implies that λ^2 is an eigenvalue of either T or S. So it is sufficent to check that if λ^2 is an eigenvalue of S then it is also an eigenvalue of T. Let x be a corresponding eigenvector, then we have

$$[R_z^*, R_w][R_w^*, R_z]x = Sx = \lambda^2 x,$$

which implies

$$[R_w^*, R_z][R_z^*, R_w][R_w^*, R_z]x = \lambda^2 [R_w^*, R_z]x.$$

Since $\lambda \neq 0$, this, combined with Lemma 3.3(a), shows that $y := [R_w^*, R_z]x$ is an eigenvector of $I - F^*F$. By an argument similar to the ones leading to (4.4) we see that Fy is an eigenvector of $I - FF^*$ with corresponding eigenvalue λ^2. Since $Fy \in M \ominus zM$, we conclude that

$$\begin{aligned} T(Fy) &= [R_z^*, R_z][R_w^*, R_w][R_z^*, R_z]Fy \\ &= [R_z^*, R_z][R_w^*, R_w]Fy = (I - FF^*)Fy = \lambda^2 Fy. \end{aligned}$$

So now it is sufficient to compute the eigenvalues of T. Consider homogeneous submodule $M = [p]$ and use the orthonormal basis $\{\Phi_n\}$ for $M \ominus zM$. Assume $deg(p) = k$. Then by Proposition 2.1, Φ_n is homogeneous of degree $k + n$, $n = 0, 1, \ldots$. Therefore,

$$F\Phi_n = P_{M \ominus zM} w \Phi_n \tag{3.2}$$

$$= \langle w\Phi_n, \Phi_{n+1}\rangle \Phi_{n+1}, \quad n \geq 0.$$

Hence F is a weighted shift. Further,

$$F^*\Phi_0 = P_{M\ominus zM}\bar{w}\Phi_0 = P_{M\ominus zM}\bar{w}\frac{p}{\|p\|} = 0,$$

and

$$\begin{aligned} F^*\Phi_n &= P_{M\ominus zM}\bar{w}\Phi_n \\ &= \langle \bar{w}\Phi_n, \Phi_{n-1}\rangle\Phi_{n-1} = \langle \Phi_n, w\Phi_{n-1}\rangle \Phi_{n-1}, \quad n \geq 1. \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we compute that $T\Phi_0 = \Phi_0$, and

$$\begin{aligned} T\Phi_n &= (I - FF^*)\Phi_n \\ &= (1 - |\langle w\Phi_{n-1}, \Phi_n\rangle|^2)\Phi_n, \quad n \geq 1. \end{aligned}$$

Thus T is a diagonal operator with eigenvalues $1, 1 - |\langle w\Phi_{n-1}, \Phi_n\rangle|^2$, $n = 1, 2, \ldots$. From (3.1), Lemma 3.4 and Lemma 3.5. We know -1 is not an eigenvalues of C in this case. Thus by Lemma 1.2 and Lemma 3.6, the core operator C has eigenvalues

$$0, 1, \ \pm\sqrt{1 - |\langle w\Phi_{n-1}, \Phi_n\rangle|^2}, \ n \geq 1.$$

We set $D_0 = 1$. By Proposition 2.1, we see that for $n \geq 1$,

$$\begin{aligned} \langle w\Phi_{n-1}, \Phi_n\rangle &= \frac{\langle \sum_{j=0}^{n-1} pA_{0,j}^{n-1} z^j w^{n-j}, \sum_{i=0}^{n} pA_{0,i}^{n} z^i w^{n-i}\rangle}{D_n\sqrt{D_{n-1}D_{n+1}}} \\ &= \frac{\sum_{i=0}^{n}\sum_{j=0}^{n-1} A_{0,j}^{n-1}\langle pw^{i-j}, pz^{i-j}\rangle \overline{A_{0,i}^{n}}}{D_n\sqrt{D_{n-1}D_{n+1}}}. \end{aligned} \tag{3.4}$$

One verifies that the numerator in (3.4) can be written as

(3.5)

$$\left\langle \begin{pmatrix} ||p||^2 & \overline{\langle pw, pz\rangle} & \ldots & \overline{\langle pw^{n-1}, pz^{n-1}\rangle} \\ \langle pw, pz\rangle & \|p\|^2 & \ldots & \overline{\langle pw^{n-2}, pz^{n-2}\rangle} \\ \vdots & \vdots & \vdots & \vdots \\ \langle pw^{n-1}, pz^{n-1}\rangle & \vdots & \vdots & \|p\|^2 \\ \langle pw^{n}, pz^{n}\rangle & \langle pw^{n-1}, pz^{n-1}\rangle & \ldots & \langle pw, pz\rangle \end{pmatrix} \begin{pmatrix} A_{0,0}^{n-1} \\ A_{0,1}^{n-1} \\ \vdots \\ A_{0,n-1}^{n-1} \end{pmatrix}, \begin{pmatrix} A_{0,0}^{n} \\ A_{0,1}^{n} \\ \vdots \\ A_{0,n}^{n} \end{pmatrix} \right\rangle$$

$$:= \left\langle \begin{pmatrix} E_0 \\ E_1 \\ \vdots \\ E_n \end{pmatrix}, \begin{pmatrix} A_{0,0}^{n} \\ A_{0,1}^{n} \\ \vdots \\ A_{0,n}^{n} \end{pmatrix} \right\rangle.$$

Observe that the the matrix in (3.5) is $(n+1) \times n$, and the first n rows of which is precisely A^{n-1}. Denoting A^{n-1} by $(a_{i,j})$ and using the cofactor theorem, we have $E_0 = D_n$ and

$$E_i = \sum_{j=0}^{n-1} a_{i,j}A_{0,j}^{n-1} = 0, \quad i = 1, 2, \ldots, n-1.$$

Moreover, in this case

$$E_n = \langle pw^n, pz^n\rangle A^{n-1}_{0,0} + \langle pw^{n-1}, pz^{n-1}\rangle A^{n-1}_{0,1} + \cdots + \langle pw, pz\rangle A^{n-1}_{0,n-1}.$$

Comparing this summation with (2.11) and (2.12), we get $E_n = -A^n_{0,n}$. Thus by (3.5) we conclude that

$$\langle w\Phi_{n-1}, \Phi_n\rangle = \frac{|D_n|^2 - |A^n_{0,n}|^2}{D_n\sqrt{D_{n-1}D_{n+1}}}.$$

We therefore have the following theorem.

Theorem 3.7. *For homogeneous submodule* $[p]$*, the core operator has eigenvalues*

$$0, 1, \ \pm\left(1 - \frac{(|D_n|^2 - |A^n_{0,n}|^2)^2}{D_{n-1}D_n^2D_{n+1}}\right)^{1/2}, \quad n \geq 1.$$

The eigenvalue formula in Theorem 3.7 is rather complicated. For some simple submodules, the inner product $\langle w\Phi_{n-1}, \Phi_n\rangle$ can be evaluated more explicitly.

Example 3.8. We consider $p = z - \lambda w$, where $0 < \lambda \leq 1$. Since $\Sigma_0 - \Sigma_1 = 1$, the Hilbert–Schmidt norm

$$||C||^2_{HS} = \Sigma_0 + \Sigma_1 = 2\Sigma_0 - 1.$$

We now compute the eigenvalues of C and Σ_0. It is easy to check that

$$||p||^2 = 1 + \lambda^2 \quad \text{and} \quad \langle pw, pz\rangle = -\lambda,$$

and

$$A^n = \begin{pmatrix} 1+\lambda^2 & -\lambda & 0 & \dots & 0 & 0 \\ -\lambda & 1+\lambda^2 & -\lambda & \dots & \dots & 0 \\ 0 & -\lambda & 1+\lambda^2 & \ddots & 0 & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & -\lambda \\ 0 & 0 & \dots & 0 & -\lambda & 1+\lambda^2 \end{pmatrix}_{(n+1)\times(n+1)}$$

$$A^n_{0,n} = (-1)^{2n+2}\lambda^n = \lambda^n.$$

Recall that $D_0 = 1$ and $D_n = detA^{n-1}$, $n \geq 1$. In this case, $D_1 = 1 + \lambda^2$. By cofactor expansion of D_{n+1} along the first column, we have the recursion relation

$$D_{n+1} = (1+\lambda^2)D_n - \lambda^2 D_{n-1}, \quad n \geq 1. \tag{3.6}$$

Then we have

$$D_n - D_{n-1} = \lambda^2(D_{n-1} - D_{n-2}) = (\lambda^2)^2(D_{n-2} - D_{n-3})$$

$$\vdots$$

$$
\begin{aligned}
&= (\lambda^2)^{n-1}(D_1 - D_0)\\
&= (\lambda^2)^{n-1}\lambda^2 = \lambda^{2n}, \quad n \geq 1,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
D_n &= D_n - D_{n-1} + D_{n-1} + \cdots + D_1 - D_0 + D_0\\
&= (\lambda^2)^n + (\lambda^2)^{n-1} + \cdots + \lambda^2 + 1.
\end{aligned}
$$

Hence

$$
\frac{|A^n_{0,n}|}{|A^n_{n,n}|} = \frac{\lambda^n}{(\lambda^2)^n + (\lambda^2)^{n-1} + \cdots + \lambda^2 + 1} = \frac{\lambda^n}{D_n},
$$

and therefore by Corollary 2.3 we have

$$
\begin{aligned}
\Sigma_0 &= \sum_{n=0}^{\infty} |\langle \Phi_n, \Psi_n \rangle|^2 = \sum_{n=0}^{\infty} \frac{|A^n_{0,n}|^2}{|A^n_{n,n}|^2}\\
&= \sum_{n=0}^{\infty} \left(\frac{\lambda^n}{D_n}\right)^2.
\end{aligned}
$$

By Theorem 3.7, C has eigenvalues

$$
1,\ \pm\left(1 - \frac{(D_n^2 - \lambda^{2n})^2}{D_{n-1}D_n^2 D_{n+1}}\right)^{1/2}, \quad n \geq 1.
$$

It is interesting to look at two particular cases.

1. When $n = 1$, we have

$$
\begin{aligned}
\left(1 - \frac{(D_1^2 - \lambda^2)^2}{D_0 D_1^2 D_2}\right)^{1/2} &= \left(1 - \frac{((1+\lambda^2)^2 - \lambda^2)^2}{(1+\lambda^2)^2(1+\lambda^2+\lambda^4)}\right)^{1/2}\\
&= \left(1 - \frac{1+\lambda^2+\lambda^4}{(1+\lambda^2)^2}\right)^{1/2}\\
&= \frac{\lambda}{1+\lambda^2}.
\end{aligned}
$$

Hence $\pm\lambda\ (1+\lambda^2)^{-1}$ are eigenvalues of C.

2. When $p = z - w$, we have $A^n_{0,n} = 1$, $A^n_{n,n} = D_n = n+1$. Therefore

$$
\left(1 - \frac{(D_n^2 - \lambda^{2n})^2}{D_{n-1}D_n^2 D_{n+1}}\right)^{1/2} = \left(1 - \frac{((n+1)^2 - 1)^2}{n(n+1)^2(n+2)}\right)^{1/2} = \frac{1}{n+1}.
$$

Hence C's eigenvalues are $1,\ \pm\frac{1}{n+1},\ n \geq 1$. Furthermore,

$$
\Sigma_0 = \sum_{n=0}^{\infty} \frac{|A^n_{0,n}|^2}{|A^n_{n,n}|^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6}.
$$

These facts were shown in [13].

4. The second largest eigenvalue of C

It is known that for every submodule, the core operator C is a contraction and 1 is always an eigenvalue of C. The second largest eigenvalue of C (s.l.e(C)) is particulalry interesting to us, as it usually reveals subtler information about the submodule. This section takes a closer look at $s.l.e(C)$ for homogeneous submodule $[p]$.

Proposition 4.1. *If M is a singly generated Hilbert–Schmidt submodule, then*

$$s.l.e(C) = \|[R_w^*, R_z]\|.$$

Proof. For any function $h \in H^2(\mathbb{D}^2)$ the submodule generated by h is denoted by $[h]$ and it is the closure of $h\mathbb{C}[z,w]$ in $H^2(\mathbb{D}^2)$. If $[h]$ is Hilbert–Schmidt, then by Proposition 3.1 and Lemma 3.5 the pair (R_z, R_w) is Fredholm and

$$\operatorname{ind}(R_z, R_w) = \dim(E_{-1}(C)) - \dim(E_1(C)). \tag{4.1}$$

Also since $M = [h]$ is singly generated, Lemma 3.4 gives

$$\dim E_1(C) = \dim(M \ominus (zM + wM)) = 1. \tag{4.2}$$

From (4.1) and (4.2) we get $\dim E_{-1}(C) = 0$. In other words -1 is not an eigenvalue for C. Since 1 is always an eigenvalue for T, by Lemma 1.2, 1 is not an eigenvalue for S. Since S is a positive compact contraction, $\|S\| < 1$ and $\|S\| = \|[R_w^*, R_z]\|^2$ is an eigenvalue of S. By Lemma 1.2, $\|[R_w^*, R_z]\|^2$ is an eigenvalue of C^2. This implies that $\|[R_w^*, R_z]\|$ is an eigenvalue of C. These observations conclude that

$$1 > s.l.e(C) \geq \|[R_w^*, R_z]\|.$$

To prove the other direction, we make a use of Lemma 3.3. First, observe that

$$I - FF^* = [R_z^*, R_z]\ [R_w^*, R_w]\ [R_z^*, R_z] = T. \tag{4.3}$$

But be aware that $I - F^*F = [R_w^*, R_z][R_z^*, R_w]$, which is not S!

By Lemma 1.2, $s.l.e(C)^2$ is an eigenvalue for S or T. If $s.l.e(C)^2$ is an eigenvalue for S then clearly

$$s.l.e(C) = \|[R_w^*, R_z]\|,$$

and we complete the proof.

Now suppose $\lambda = s.l.e(C)$ and λ^2 is an eigenvalue for T with corresponding eigenfunction x. Then

$$\begin{aligned} (I - FF^*)x = \lambda^2 x &\Leftrightarrow F^*(I - FF^*)x = \lambda^2 F^* x \\ &\Leftrightarrow (F^* - F^*FF^*)x = \lambda^2 F^* x \\ &\Leftrightarrow (I - F^*F)F^*x = \lambda^2 F^* x. \end{aligned} \tag{4.4}$$

If $F^*x = 0$, then by (4.3) we see that x is an eigenfunction for T with corresponding eigenvalue 1, which contradicts with the assumption that x

is an eigenfunction of λ. Therefore, λ^2 is an eigenvalue of $I - F^*F$ with corresponding eigenfunction F^*x. Thus we have

$$\lambda^2 \leq \|I - F^*F\| = \|[R_z^*, R_w]\|^2,$$

and hence $\lambda \leq \|[R_z^* R_w]\| = \|[R_w^* R_z]\|$. This completes the proof. $\square$

Theorem 4.2. *For a homogenous submodule* $[p]$,

$$s.l.e(C) = \max_{n\geq 1} \frac{|A_{0,n}^n|}{D_n}.$$

Proof. We first check that $[R_z^*, R_w] = P_M \bar{z} w f$ on $M \ominus zM$, and its range lies inside $M \ominus wM$. To see this, since $[R_z^*, R_w]f = (R_z^* R_w - R_w R_z^*)f = 0$ for every $f \in zM$ it is enough to look at $[R_z^*, R_w]$ on $M \ominus zM$. For $f \in M \ominus zM$, we have $R_z^* f = 0$, hence

$$[R_z^*, R_w]f = (R_z^* R_w - R_w R_z^*)f = R_z^* R_w f.$$

For every $g \in wM$, one verifies that

$$\langle g, P_M \bar{z}\, w f\rangle = \langle P_M g, \bar{z}\, w f\rangle = \langle zg, wf\rangle = 0,$$

i.e., wM is perpendicular to $P_M \bar{z} w f$. This shows $P_M \bar{z} w \in M \ominus wM$. Therefore,

(4.5)

$$\begin{aligned}
&\|[R_z^*, R_w]\| \\
&= \sup\{|\langle [R_z^*, R_w]f,\ g\rangle| \colon f \in M \ominus zM,\ g \in M \ominus wM,\ ||f|| = ||g|| = 1\} \\
&= \sup\{|\langle P_M \bar{z} w f, g\rangle| \colon f \in M \ominus zM,\ g \in M \ominus wM,\ ||f|| = ||g|| = 1\} \\
&= \sup\{|\langle wf, zg\rangle| \colon f \in M \ominus zM,\ g \in M \ominus wM,\ ||f|| = ||g|| = 1\} \\
&\geq \sup_{n\geq 0} |\langle w\Phi_n, z\Psi_n\rangle|.
\end{aligned}$$

Since $[p]$ is Hilbert–Schmidt,

$$\sum_{n\geq 0} |\langle w\Phi_n, z\Psi_n\rangle|^2 = \Sigma_1 < \infty,$$

which implies that $\langle w\Phi_n, z\Psi_n\rangle \to 0$ as $n \to \infty$. So the sup in (4.5) is in fact the max.

For the other direction, we assume $f = \sum_{i=0}^{\infty} \alpha_i \Phi_i$ is an arbitrary element in $M \ominus zM$ and $g = \sum_{i=0}^{\infty} \beta_i \Psi_i$ is an arbitrary element in $M \ominus wM$ such that $\|f\| = \|g\| = 1$. Then using Cauchy–Schwarz inequality and the orthogonality of $w\Phi_i$ and $z\Psi_j$ for $i \neq j$ (because they are homogeneous of different degree), one checks that

$$\begin{aligned}
|\langle [R_z^*, R_w]f,\ g\rangle| = |\langle wf, zg\rangle| &= \left|\sum_{i,j=0}^{\infty} \alpha_i \bar{\beta}_j \langle w\Phi_i, z\Psi_j\rangle\right| \\
&= \left|\sum_{i=0}^{\infty} \alpha_i \bar{\beta}_i \langle w\Phi_i, z\Psi_i\rangle\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \max_{n\geq 0}\{|\langle w\Phi_n, z\Psi_n\rangle|\}\sum_{i=0}^{\infty}|\alpha_i||\bar{\beta}_i| \\
&\leq \max_{n\geq 0}|\langle w\Phi_n, z\Psi_n\rangle|\|f\|\|g\|\leq \max_{n\geq 0}|\langle w\Phi_n, z\Psi_n\rangle|.
\end{aligned}
$$

This implies

$$\|[R_z^*, R_w]\|\leq \max_{n\geq 0}|\langle w\Phi_n, z\Psi_n\rangle|.$$

The theorem then follows from Corollary 2.2, Proposition 4.1 and the simple fact that $[R_z^*, R_w] = [R_w^*, R_z]^*$. ☐

Now we assume

$$p(z,w) = \sum_{j=0}^{k} c_j z^j w^{k-j}. \tag{4.6}$$

Since $\Phi_0 = \Psi_0 = \frac{p}{\|p\|}$, and

$$
\begin{aligned}
\langle pw, pz\rangle &= \left\langle w\left(\sum_{i=0}^{k} c_i z^i w^{k-i}\right), z\left(\sum_{j=0}^{k} c_j z^j w^{k-j}\right)\right\rangle \\
&= \left\langle \sum_{i=0}^{k} c_i z^i w^{k+1-i}, \sum_{j=0}^{k} c_j z^{j+1} w^{k-j}\right\rangle \\
&= \sum_{j=0}^{k-1}\langle c_{j+1} z^{j+1} w^{k-j}, c_j z^{j+1} w^{k-j}\rangle = \sum_{j=0}^{k-1}\bar{c_j}\, c_{j+1}.
\end{aligned}
$$

Corollary 2.2 and Theorem 4.2 give rise to the following simple estimate.

Corollary 4.3. *For a homogenous polynomial $p = \sum_{j=0}^{k} c_j z^j w^{k-j}$, then on $[p]$ we have*

$$s.l.e(C) \geqslant \frac{\left|\sum_{j=0}^{k-1}\bar{c_j}\; c_{j+1}\right|}{\sum_{j=0}^{k}|c_j|^2}. \tag{4.7}$$

Example 4.4. We consider two simple cases.

(a) For $p = z - \lambda w,\ 0 \leq |\lambda|\leq 1$, we have $c_0 = -\lambda, c_1 = 1$ and

$$s.l.e(C) \geq \frac{|\lambda|}{1+|\lambda|^2}.$$

In particular, for $p = z - w$ we observe that $s.l.e(C)$ is in fact equal to $\frac{1}{2}$. This indicates that the estimate in Corollary 4.3 is sharp.

(b) *For $p = z^2 + 2wz + w^2$, $c_0 = 1, c_1 = 2, c_2 = 1$ and $s.l.e(C) \geq \frac{4}{6}$.*

The proof of Theorem 4.2 shows that $s.l.e(C) = \max_{n\geq 0}|\langle w\Phi_n, z\Psi_n\rangle|$. So by (2.16) we have:

Corollary 4.5. *For homogeneous submodule* $[p]$,

$$s.l.e(C) = \max_{n\geq 1}|\langle \Phi_n, \Psi_n\rangle|.$$

5. Matrix A^n and Toeplitz determinant

The quantities D_n and $A^n_{0,n}$ have shown up in Corollary 2.2, Theorem 3.7 and Theorem 4.2. As we have remarked before, both quantities are in fact Toeplitz determinants. This section uses some known tools to make a study on the two quantities.

Given a sequence of complex numbers $t_k,\ k \in \mathbb{Z}$, the associated $n\times n$ Toeplitz matrix is of the form

$$T_n = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \dots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \dots & t_{-(n-2)} \\ t_2 & t_1 & t_0 & \dots & t_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \dots & t_0 \end{pmatrix},$$

and the associated trignometric polynomial is

$$f_n(\lambda) = \sum_{k=-(n-1)}^{n-1} t_k\, e^{ik\lambda}, \quad \lambda \in [0, 2\pi]. \tag{5.1}$$

The associated Fourier series is defined formally as

$$f(\lambda) = \sum_{k=-\infty}^{\infty} t_k\, e^{ik\lambda}, \quad \lambda \in [0, 2\pi]. \tag{5.2}$$

Computation of the determinant of T_n in general is rather complicated (cf. [7]). The following first Szegö limit theorem tells about the asymptotic behavior of $\det T_n$ as $n\to\infty$. For details, we refer readers to [2].

Theorem 5.1. *If f is in $L^1(\mathbb{T}, \frac{d\lambda}{2\pi})$ and is positive a.e., on $\mathbb{T}$, then*

An equivalent formulation is

$$\lim_{n\to\infty} \frac{\det T_n}{\det T_{n-1}} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log(f(\lambda))d\lambda\right). \tag{5.3}$$

We assume

$$p(z, w) = c_0 z^k + c_1 z^{k-1} w + \cdots + c_{k-1} z w^{k-1} + c_k w^k.$$

For $T_n = A^{n-1}$, by (5.1) we have

$$\begin{aligned} f_n(\lambda) = {} & \overline{\langle pw^{n-1}, pz^{n-1}\rangle} e^{-i(n-1)\lambda} + \cdots + \overline{\langle pw, pz\rangle} e^{-i\lambda} \\ & + ||p||^2 + \langle pw, pz\rangle e^{i\lambda} + \cdots + \langle pw^{n-1}, pz^{n-1}\rangle e^{-i(n-1)\lambda}. \end{aligned}$$

For $A^n_{0,n}$, the associated trigonometric polynomial is

$$\begin{aligned} g_n(\lambda) = {} & \overline{\langle pw^{n-2}, pz^{n-2}\rangle} e^{i(n-1)\lambda} + \cdots + \overline{\langle pw, pz\rangle} e^{-i2\lambda} + ||p||^2 e^{-i\lambda} \\ & + \langle pw, pz\rangle + \langle pw^2, pz^2\rangle e^{i\lambda} + \cdots + \langle pw^n, pz^n\rangle e^{i(n-1)\lambda}. \end{aligned}$$

It is easy to see that

$$f_n(\lambda) + \langle pw^n, pz^n\rangle e^{in\lambda} = e^{i\lambda} g_n(\lambda) + \overline{\langle pw^{n-1}, pz^{n-1}\rangle} e^{-i(n-1)\lambda}.$$

But we have $\langle pw^n, pz^n\rangle = 0$ for all $n > \deg(p) = k$. Therefore

$$f_n(\lambda) = e^{i\lambda} g_n(\lambda)$$

when $n > k$.

For convenience, we set $c_n = 0$ for all $n > k$. Then by direct calculation, we have

$$A^n = \begin{pmatrix} \sum_{j=0}^k |c_j|^2 & \sum_{j=0}^k \bar{c}_j c_{j+1} & \sum_{j=0}^k \bar{c}_j c_{j+2} & \cdots & \sum_{j=0}^k \bar{c}_j c_{j+n} \\ \sum_{j=0}^k c_j \bar{c}_{j+1} & \sum_{j=0}^k |c_j|^2 & \sum_{j=0}^k \bar{c}_j c_{j+1} & \cdots & \sum_{j=0}^k \bar{c}_j c_{j+(n-1)} \\ \sum_{j=0}^k c_j \bar{c}_{j+2} & \sum_{j=0}^k c_j \bar{c}_{j+1} & \sum_{j=0}^k |c_j|^2 & \cdots & \sum_{j=0}^k \bar{c}_j c_{j+(n-2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{j=0}^k c_j \bar{c}_{j+n} & \sum_{j=0}^k c_j \bar{c}_{j+(n-1)} & \sum_{j=0}^k c_j \bar{c}_{j+(n-2)} & \cdots & \sum_{j=0}^k |c_j|^2 \end{pmatrix}.$$

Note that by the assumption that $c_n = 0$ for all $n > deg(p) = k$, many terms in the summations in A^n are in fact 0! Now if we let

$$C(n) = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_k & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & c_0 & c_1 & \dots & c_{k-1} & c_k & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & c_0 & \dots & c_{k-2} & \bar{c}_{k-1} & \bar{c}_k & 0 & \dots & 0 & 0 \\ & & & & & \vdots & \vdots & & & \vdots & \vdots \\ & & & & \ddots & \vdots & \vdots & & & \vdots & \vdots \\ \vdots & & & & & \bar{c}_0 & \bar{c}_1 & \dots & \dots & \bar{c}_{k-1} & \bar{c}_k \end{pmatrix}_{(n+1)\times(n+k+1)},$$

and

$C_o(n)$

$$= \begin{pmatrix} c_1 & c_2 & \dots & c_k & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ c_0 & c_1 & \dots & c_{k-1} & c_k & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & c_0 & \dots & c_{k-2} & \bar{c}_{k-1} & \bar{c}_k & 0 & \dots & 0 & 0 & 0 \\ & & & & \vdots & \vdots & & & \vdots & \vdots & 0 \\ & & & \ddots & \vdots & \vdots & & & \vdots & \vdots & 0 \\ \vdots & & & & \bar{c}_0 & \bar{c}_1 & \dots & \dots & \bar{c}_{k-1} & \bar{c}_k & 0 \end{pmatrix}_{(n+1)\times(n+k+1)}.$$

Let $C^H(n)$ and $C_o^H(n)$ denote the conjugate transpose of $C(n)$ and $C_o(n)$ respectively, then one verifies that

$$A^n = C(n)C^H(n), \qquad M = C(n)C_o^H(n),$$

where M is the submatrix of A^{n+1} with 0-th row and (n+2)-th column of A^{n+1}removed as in Section 2. Replacing $\langle pw^j, pz^j\rangle$ by the corresponding entries in A^n above, we check that for each $n \geq 0$

$$\begin{aligned} (5.4)\quad & f_{n+1}(\lambda\ ) \\ & = (c_0 e^{ni\lambda} + c_1 e^{(n-1)i\lambda} + \cdots + c_n)(\bar{c}_0 e^{-ni\lambda} + \bar{c}_1 e^{-(n-1)i\lambda} + \cdots + \bar{c}_n), \end{aligned}$$

where $\lambda \in [0, 2\pi]$. Since $c_n = 0$ when $n > k$, we see that

$$f_{n+1}(\lambda) = |p(e^{i\lambda}, 1)|^2, \quad \forall n > k, \tag{5.5}$$

For convenience, we let $p_*(z) = p(z, 1)$.

For a complex polynomial $q(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, we let $Z(q)$ be the set of its zeros. q's Mahler measure is defined as

$$\mathcal{M}(q) = |a_0| \prod_{z \in Z(q), |z| \geq 1} |z|.$$

It follows from Jensen's formula that

$$\mathcal{M}(q) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|q(e^{i\theta})| d\theta\right).$$

For more information about Mahler measure we refer readers to [3]. By (5.3) and (5.5), we have the following:

Proposition 5.2. *For a homogeneous polynomial p,*

$$\lim_{n\to\infty} \frac{D_{n+1}}{D_n} = \mathcal{M}^2(p_*).$$

Now we turn to $A^{n+1}_{0,n+1}$, and we shall use Cauchy–Binet formula to give an estimate of it by D_n. Fix two natural numbers $n \geq k$. Let $\mathcal{J}$ be the set of tuples $J = (j_1, j_2, \dots, j_k)$ of natural numbers such that $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. Clearly $|\mathcal{J}|= \frac{n!}{k!(n-k)!}$. For any $n \times k$ matrix A, $A(J)$ will denote the $k \times k$ matrix formed using rows J (in that order), and $A^H(J)$ means $(A(J))^H$. For two $n \times k$ matrices A and B, by Cauchy–Binet formula

$$\det(B^H A) = \sum_{J \in \mathcal{J}} \det B^H(J) \det A(J).$$

Then by Cauchy–Schwarz inequality we have

$$|\det(B^H A)| \leq \sqrt{\det(B^H B)}\sqrt{\det(A^H A)}. \tag{5.6}$$

Since $M = C(n)C_o^H(n)$ and $A_{0,n+1}^{n+1} = (-1)^{n+3} \det M$, we have

$$\begin{aligned} |A_{0,n+1}^{n+1}| &= |\det M| = |\det C(n)C_o^H(n)| \\ &\leq \sqrt{\det(C(n)C^H(n))}\sqrt{\det(C_o(n)C_o^H(n))} \\ &= \sqrt{D_{n+1}}\sqrt{\det(C_o(n)C_o^H(n))}. \end{aligned}$$

Now we take a closer look at $\det(C_o(n)C_o^H(n))$. By Cauchy–Binet formula, we have

$$\begin{aligned} \det(C_o(n)C_o^H(n)) &= \sum_{J \in \mathcal{J}} \det(C_o(J)C_o^H(J)) \\ &= \sum_{J \in \mathcal{J}, j_1 \geq 2} \det(C(J)C^H(J)) \\ &= \sum_{J \in \mathcal{J}} \det(C(J)C^H(J)) - \sum_{J \in \mathcal{J}, j_1 = 1} \det(C_o(J)C_o^H(J)) \\ &= \det(C(n)C^H(n)) - \sum_{J \in \mathcal{J}, j_1 = 1} \det(C_o(J)C_o^H(J)) \\ &= D_{n+1} - \sum_{J \in \mathcal{J}, j_1 = 1} \det(C_o(J)C_o^H(J)). \end{aligned} \tag{5.7}$$

For the second term in (5.7) we observe from the matrix $C(n)$ that

$$\sum_{J \in \mathcal{J}, j_1 = 1} \det(C_o(J)C_o^H(J)) = |c_0|^2 \det(C(n-1)C^H(n-1)).$$

Therefore we have the following inequality to conclude this paper.

Corollary 5.3. $|A_{0,n+1}^{n+1}| \leq \sqrt{D_{n+1}}\sqrt{D_{n+1} - |c_0|^2 D_n}$.

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This paper is available via `http://nyjm.albany.edu/j/2017/23-23.html`.