1-complemented subspaces of Banach spaces of universal disposition

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ABSTRACT. We first unify all notions of partial injectivity appearing in the literature —(universal) separable injectivity, (universal) $\aleph$-injectivity — in the notion of (α, β)-injectivity ($(\alpha, \beta)_\lambda$-injectivity if the parameter λ has to be specified). Then, extend the notion of space of universal disposition to space of universal (α, β)-disposition. Finally, we characterize the 1-complemented subspaces of spaces of universal (α, β)-disposition as precisely the spaces $(\alpha, \beta)_1$-injective.

CONTENTS

The purpose of this paper is to establish the connection between spaces (universally) 1-separably injective and the spaces of universal disposition. A Banach space E is said to be λ-*separably injective* if for every separable Banach space X and each subspace $Y \subset X$, every operator $t : Y \to E$ extends to an operator $T : X \to E$ with $\|T\| \leq \lambda \|t\|$. Gurariy [10] introduced the property of a Banach space to be of universal disposition with respect to a given class $\mathfrak{M}$ of Banach spaces (or $\mathfrak{M}$-universal disposition, in short): U is said to be of universal disposition for $\mathfrak{M}$ if given $A, B \in \mathfrak{M}$ and into isometries $u : A \to U$ and $\imath : A \to B$ there is an into isometry $u' : B \to U$ such that $u = u'\imath$. We are particularly interested in the choices $\mathfrak{M} \in \{\mathfrak{F}, \mathfrak{S}\}$, where $\mathfrak{F}$ (resp $\mathfrak{S}$) denote the classes of finite-dimensional (resp. separable) Banach spaces.

The paper [1] started the study of spaces of universal disposition after Gurariy [10]. All available results and several new were then presented in Chapter 4 of the monograph [4]. Several overlooked and/or left open problems in the monograph were solved in [8]. An up-dated account of the facts known so far include the following:

- Spaces of S-universal disposition are 1-separably injective [1] and [4, Thm. 3.5].
- There are spaces of $\mathfrak{F}$-universal disposition that are not separably injective. Consequently, there are spaces of $\mathfrak{F}$-universal disposition that are not of S-universal disposition [8].
- 1-separably injective are Lindenstrauss and Grothendieck spaces containing c_0 [2].
- There are spaces of $\mathfrak{F}$-universal disposition containing complemented copies of c_0. Consequently, there are spaces of $\mathfrak{F}$-universal disposition that are not Grothendieck [1] and [4, Prop. 3.16].
- Under the Continuum Hypothesis CH, 1-separably injective spaces contain ℓ_∞ [2, 4]. This can no longer be true without CH [5].

Our first objective is to connect the notions of 1-separably injectivity and universal disposition. We display the basic construction technique already used in different versions in [1, 4, 9, 8]:

The push-out construction

Recall first the push-out construction; which, given an isometry $u : A \to B$ and an operator $t : A \to E$ provides us with an extension of t through u at the cost of embedding E in a larger space PO as is shown in the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ u\ } & B \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\
E & \xrightarrow{\ u'\ } & \text{PO}
\end{array}
$$

where $t'u = u't$. The PO space is defined as the quotient space $(E \oplus_1 B)/\Delta$ where $\Delta = \{(ua, -ta) : a \in A\}$. It is important to note that u' is again an isometry and that t' is a contraction or an isometry if t is. The space PO is "essentially unique" in the sense that any other space C endowed with an isometry u'' and an operator t'' such that $t''u = u''t$ and $C/u''(E) \sim B/u(A)$ is isomorphic to PO. See the Appendix in [4] for details.

The basic multiple push-out construction

To perform our basic construction we need the following data:

- A starting Banach space X.
- A class $\mathfrak{M}$ of Banach spaces.
- The family $\mathfrak{J}$ of all isometries acting between the elements of $\mathfrak{M}$.

- A family $\mathfrak{L}$ of norm one X-valued operators defined on elements of $\mathfrak{M}$.
- The smallest ordinal $\omega(\aleph)$ with cardinality $\aleph$.

For any operator $s : A \to B$, we write $\operatorname{dom}(s) = A$ and $\operatorname{cod}(s) = B$. Notice that the codomain of an operator is usually larger than its range, and that the unique codomain of the elements of $\mathfrak{L}$ is X. Set

$$\Gamma = \{(u,t) \in \mathfrak{J} \times \mathfrak{L} : \operatorname{dom} u = \operatorname{dom} t\}$$

and consider the Banach spaces $\ell_1(\Gamma, \operatorname{dom} u)$ and $\ell_1(\Gamma, \operatorname{cod} u)$ of summable families. We have an obvious isometry

$$\oplus \mathfrak{J} : \ell_1(\Gamma, \operatorname{dom} u) \longrightarrow \ell_1(\Gamma, \operatorname{cod} u)$$

defined by $(x_{(u,t)})_{(u,t)\in\Gamma} \longmapsto (u(x_{(u,t)}))_{(u,t)\in\Gamma}$; and a contractive operator

$$\Sigma\mathfrak{L} : \ell_1(\Gamma, \operatorname{dom} u) \longrightarrow X,$$

given by $(x_{(u,t)})_{(u,t)\in\Gamma} \longmapsto \sum_{(u,t)\in\Gamma} t(x_{(u,t)})$. Observe that the notation is slightly imprecise since both $\oplus\mathfrak{J}$ and $\Sigma\mathfrak{L}$ depend on Γ. We can form their push-out diagram

$$
\begin{array}{ccc}
\ell_1(\Gamma, \operatorname{dom} u) & \xrightarrow{\;\oplus\mathfrak{J}\;} & \ell_1(\Gamma, \operatorname{cod} u) \\
{\scriptstyle \Sigma\mathfrak{L}}\downarrow & & \downarrow \\
X & \xrightarrow{\;\imath_{(0,1)}\;} & \mathrm{PO}
\end{array}
$$

Thus, if we call $\mathcal{S}^0(X) = X$, then we have constructed the space $\mathcal{S}^1(X) = \mathrm{PO}$ and an isometric enlargement of $\imath_{(0,1)} : \mathcal{S}^0(X) \to \mathcal{S}^1(X)$ such that for every $t : A \to X$ in $\mathfrak{L}$, the operator $\imath t$ can be extended to an operator $t' : B \to \mathrm{PO}$ through any embedding $u : A \to B$ in $\mathfrak{J}$ provided

$$\operatorname{dom} u = \operatorname{dom} t = A.$$

In the next step we keep the family $\mathfrak{J}$ of isometries, replace the starting space X by $\mathcal{S}^1(X)$ and $\mathfrak{L}$ by a family $\mathfrak{L}_1$ of norm one operators

$$\operatorname{dom} u \to \mathcal{S}^1(X), \quad u \in \mathfrak{J},$$

and proceed by induction. The inductive step is as follows. Suppose we have constructed the directed system $(\mathcal{S}^\alpha(X))_{\alpha<\beta}$ for all ordinals $\alpha < \beta$, including the corresponding linking maps $\imath_{(\alpha,\gamma)} : \mathcal{S}^\alpha(X) \longrightarrow \mathcal{S}^\gamma(X)$ for $\alpha < \gamma < \beta$. To define $\mathcal{S}^\beta(X)$ and the maps $\imath_{(\alpha,\beta)} : \mathcal{S}^\alpha(X) \longrightarrow \mathcal{S}^\beta(X)$ we consider separately two cases, as usual: if β is a limit ordinal, then we take $\mathcal{S}^\beta(X)$ as the direct limit of the system $(\mathcal{S}^\alpha(X))_{\alpha<\beta}$ and $\imath_{(\alpha,\beta)} : \mathcal{S}^\alpha(X) \longrightarrow \mathcal{S}^\beta(X)$ the natural inclusion map. Otherwise $\beta = \alpha + 1$ is a successor ordinal and we construct $\mathcal{S}^\beta(X)$ by applying the push-out construction as above with the following data: $\mathcal{S}^\alpha(X)$ is the starting space, $\mathfrak{J}$ continues to be the set of all isometries acting between the elements of $\mathfrak{M}$, and $\mathfrak{L}_\alpha$ is the family of all isometries $t : S \to \mathcal{S}^\alpha(X)$, where $S \in \mathfrak{M}$. We then set

$$\Gamma_\alpha = \{(u,t) \in \mathfrak{J} \times \mathfrak{L}_\alpha : \operatorname{dom} u = \operatorname{dom} t\}$$

and form the push-out

(1)
$$
\begin{array}{ccc}
\ell_1(\Gamma_\alpha, \operatorname{dom} u) & \xrightarrow{\oplus \mathfrak{I}_\alpha} & \ell_1(\Gamma_\alpha, \operatorname{cod} u) \\
{\scriptstyle \Sigma \mathfrak{L}_\alpha} \downarrow & & \downarrow \\
\mathcal{S}^\alpha(X) & \longrightarrow & \mathrm{PO}
\end{array}
$$

thus obtaining $\mathcal{S}^{\alpha+1}(X) = \mathrm{PO}$. The embedding $\imath_{(\alpha,\beta)}$ is the lower arrow in the above diagram; by composition with $\imath_{(\alpha,\beta)}$ we get the embeddings $\imath_{(\gamma,\beta)} = \imath_{(\alpha,\beta)}\imath_{(\gamma,\alpha)}$, for all $\gamma < \alpha$.

Proposition 1. *A Banach space is 1-separably injective if and only if it is a 1-complemented subspace of a space of $\mathcal{S}$-universal disposition.*

Proof. Since spaces of $\mathcal{S}$-universal disposition are 1-separably injective [1], their 1-complemented subspaces also are 1-separably injective. To show the converse, let Θ be a 1-separably injective space. Set now the following input data in the basic construction:

- $X = \Theta$.
- $\mathfrak{M} = \mathcal{S}$.
- The family $\mathfrak{J}$ of all isometries acting between the elements of $\mathcal{S}$.
- The family $\mathfrak{L}$ of into Θ-valued isometries defined on elements of $\mathcal{S}$.
- The first uncountable ordinal ω_1.

From the construction we get the space $\mathcal{S}^{\omega_1}(\Theta)$. This space is of $\mathcal{S}$-universal disposition (cf. [4, Prop. 3.3]).

Claim 1. *If Θ is 1-separably injective then it is 1-complemented in $\mathcal{S}^{\omega_1}(\Theta)$.*

Proof of the Claim. It is clear operators from separable spaces into Θ extend with the same norm to separable superspaces. Thus occurs to the operators forming $\Sigma\mathfrak{L}$. Pick the identity 1_Θ and observe that the composition $1_\Theta \Sigma\mathfrak{L}$ extends to an operator $\ell_1(\Gamma, \operatorname{cod} u) \to \Theta$; hence, by the push-out property, there is an operator $\mathcal{S}^1(\Theta) \to \Theta$ that extends 1_Θ through

$$
\imath_1 : \Theta \to \mathcal{S}^1(\Theta).
$$

In other words, Θ is 1-complemented in $\mathcal{S}^1(\Theta)$. The argument can be transfinitely iterated to conclude that Θ is 1-complemented in $\mathcal{S}^{\omega_1}(\Theta)$. $\square$

This completes the proof of Proposition 1. $\square$

A basic form of Theorem 1 appears in [12, Thm. 3.22] as: *Assume the continuum hypothesis. Let V be the unique Banach space of density $\aleph_1$ that is of universal disposition for separable spaces. A Banach space of density $\leq \aleph_1$ is isometric to a 1-complemented subspace of V if and only if it is 1-separably injective.*

The paper [3] (see also [4, Chapter 5]) contains the higher cardinal generalization of the notion of separable injectivity as follows. A Banach space E is said to be $\aleph$-*injective* if for every Banach space X with density character

$< \aleph$ and each subspace $Y \subset X$ every operator $t : Y \to E$ can be extended to an operator $T : X \to E$. When for every operator t there exists some extension T such that $\|T\| \leq \lambda \|t\|$ we say that E is $(\lambda, \aleph)$-*injective*.

The case $\aleph = \aleph_1$ corresponds to separable injectivity and thus the resulting name for separable injectivity is $\aleph_1$-injectivity (not $\aleph_0$-injectivity), which is perhaps surprising. Nevertheless, we have followed the uses of set theory where properties labeled by a cardinal $\aleph$ always indicate that something happens for sets whose cardinality is strictly less than $\aleph$. Let $\mathcal{S}_\aleph$ be the class of all Banach spaces having density character less than $\aleph$ (thus, $\mathcal{S} = \mathcal{S}_{\aleph_1}$). The general version of Proposition 1 is:

Proposition 2. *A Banach space is $(1, \aleph)$-injective if and only if it is a 1-complemented subspace of a space of $\mathcal{S}_\aleph$-universal disposition.*

Proof. The proof that spaces of $\mathcal{S}_\aleph$-universal disposition are $(1, \aleph)$-injective is the same as that for $\mathcal{S}$ ([1] and [4, Thm. 3.5]); thus, their 1-complemented subspaces also are $(1, \aleph)$-injective. To show the converse, let Θ be a $(1, \aleph)$-injective space. Set now the input data as:

- $X = \Theta$.
- $\mathfrak{M} = \mathcal{S}_\aleph$.
- The family $\mathfrak{J}$ of all isometries acting between the elements of $\mathcal{S}_\aleph$.
- The family $\mathfrak{L}$ of into Θ-valued isometries defined on elements of $\mathcal{S}_\aleph$.
- The ordinal $\omega(2^\aleph)$.

Claim 2. *The space $\mathcal{S}^{\omega(2^\aleph)}(X)$ is a space of $\mathcal{S}_\aleph$-universal disposition.*

Proof of the Claim. Let $A, B \in \mathcal{S}_\aleph$ and $i : A \to B$ be an into isometry. Let $j : A \to X$ an into isometry. Since the cofinality of $2^\aleph$ is greater than $\aleph$, the image of j must fall into some $\mathcal{S}^\alpha(X)$ for some $\alpha < 2^\aleph$. Now, since both j and i are part of the amalgam of operators respect to with one does push out, there is an into isometry $j' : B \to \mathcal{S}^{\alpha+1}(X)$ extending j. In particular, there is an into isometry $j' : B \to \mathcal{S}^{\omega(2^\aleph)}(X)$ extending j. $\qquad \square$

Claim 3. *If Θ is $(1, \aleph)$-injective then it is 1-complemented in $\mathcal{S}^{\omega(2^\aleph)}(\Theta)$.*

The proof is exactly as the one we did for $\mathcal{S}$. This concludes the proof of the proposition. $\qquad \square$

We want now to handle a different set of ideas. The papers [2, 4] also introduce the notion of universally separably injective space as follows: a Banach space E is said to be *universally λ-separably injective* if for every Banach space X and each separable subspace $Y \subset X$, every operator $t : Y \to E$ extends to an operator $T : X \to E$ with $\|T\| \leq \lambda \|t\|$. It turns out that there are examples of universally separably injective spaces appearing in nature, such as:

- The Pełczyński–Sudakov [15] spaces $\ell_\infty^\aleph(\Gamma)$.

- The space ℓ_∞/c_0 and, in general, quotients of injective spaces by separably injective spaces. [2].
- Ultrapowers of $\mathcal{L}_\infty$-spaces with respect to countably incomplete ultrafilters on $\mathbb{N}$. [4, Thm.4.4].

The higher cardinal generalization of this notion was studied in [3] (see also [4, Chapter 5]). The space E is said to be *universally $\aleph$-injective* if for every space X and each subspace $Y \subset X$ with density character $< \aleph$, every operator $t : Y \to E$ can be extended to an operator $T : X \to E$. When for every such operator t there exists some extension T such that $\|T\| \leq \lambda\|t\|$ we say that E is universally $(\lambda, \aleph)$-*injective*. To obtain analogues to Propositions 1 and 2 we need to find a common generalization of the notions of $\aleph$-injectivity and universal $\aleph$-injectivity and then adapt the notion of universal disposition.

Definition. Let (α, β) two infinite cardinals, $\alpha \leq \beta$. The space E is said to be (α, β)-injective if every operator $t : A \to E$ from a subspace $A \subset B$ with density character $< \alpha$ can be extended to an operator $T : B \to E$. When for every such operator t there exists some extension T such that $\|T\| \leq \lambda\|t\|$ we say that E is $(\alpha, \beta)_\lambda$-injective.

Observe that $\aleph$-injectivity corresponds to $(\aleph, \aleph)$-injectivity while universal $\aleph$-injectivity corresponds to $(\aleph, 2^\aleph)$-injectivity since every space with density character $\aleph$ can be embedded into an injective space with density character $2^\aleph$.

Definition. Let (α, β) two infinite cardinals, $\alpha \leq \beta$. The space E is said to be of (α, β)-universal disposition if given spaces A, B with dens $A < \alpha$ and dens $B < \beta$ and into isometries $u : A \to E$ and $\imath : A \to B$ there is an into isometry $u' : B \to E$ such that $u = u'\imath$.

One has:

Theorem 1. *A Banach space is $(\alpha, \beta)_1$-injective if and only if it is a 1-complemented subspace of a space of (α, β)-universal disposition.*

Proof. The proof is a variation of those of Propositions 1 and 2. The three claims involved are:

Claim 4. *Spaces of (α, β)-universal disposition are $(\alpha, \beta)_1$-injective.*

with the obvious proof as before. Therefore, 1-complemented subspaces of spaces of (α, β)-universal disposition also are $(\alpha, \beta)_1$-injective. To show show the converse, let Θ be a $(\alpha, \beta)_1$-injective space. Set now the input data as:

- $X = \Theta$.
- $\mathfrak{M} = \mathcal{S}_\alpha$.
- The family $\mathfrak{J}$ of all into isometries acting from spaces in $\mathcal{S}_\alpha$ into spaces in $\mathcal{S}_\beta$.
- The family $\mathfrak{L}$ of into Θ-valued isometries defined on elements of $\mathcal{S}_\alpha$.

- The ordinal $\omega(2^\alpha)$.

Let us call $\mathcal{S}^{\alpha,\beta}(\Theta)$ the resulting space $\mathcal{S}^{\omega(2^\aleph)}(\Theta)$.

Claim 5. *For any choice of X, the space $\mathcal{S}^{\alpha,\beta}(X)$ is a space of (α,β)-universal disposition.*

Proof of the Claim. Let $A \in \mathcal{S}_\alpha$, $B \in \mathcal{S}_\beta$ and $i : A \to B$ an into isometry. Let $j : A \to X$ an into isometry. Since the cofinality of 2^α is greater than α, the image of j must fall into $\mathcal{S}^\gamma(X)$ for some $\gamma < 2^\alpha$. Now, since both j and i are part of the amalgam of operators respect to with one does push out, there is an into isometry $j' : B \to \mathcal{S}^{\gamma+1}(X)$ extending j. In particular, there is an into isometry $j' : B \to \mathcal{S}^{\alpha,\beta}(X)$ extending j. $\qquad\square$

Claim 6. *If Θ is $(\alpha,\beta)_1$-injective then it is 1-complemented in $\mathcal{S}^{\alpha,\beta}(\Theta)$.*

Proof of the Claim. It is clear operators from $\mathcal{S}_\alpha$ into Θ extend with the same norm to superspaces in $\mathcal{S}_\beta$. This occurs to the operators forming $\Sigma\mathfrak{L}$. Pick the identity 1_Θ and observe that the composition $1_\Theta\Sigma\mathfrak{L}$ extends to an operator $\ell_1(\Gamma, \mathrm{cod}\,u) \to \Theta$; hence, by the push-out property, there is an operator $\mathcal{S}^1(\Theta) \to \Theta$ that extends 1_Θ through $\iota_1 : \Theta \to \mathcal{S}^1(\Theta)$. In other words, Θ is 1-complemented in $\mathcal{S}^1(\Theta)$. The argument can be transfinitely iterated to conclude that Θ is 1-complemented in $\mathcal{S}^{\alpha,\beta}(\Theta)$ $\qquad\square$

This concludes the proof of the theorem. $\qquad\square$

The spaces $\mathcal{S}^{\alpha,\beta}(X)$ are, moreover, the first examples of spaces of (α,β)-universal disposition for $\alpha \neq \beta$. Replacing isometries by isomorphisms one would get the very different notion of space that is (α,β)-automorphic (see [4]). It is straightforward that every Banach space of (α,β)-universal disposition is (α,β)-automorphic ; while the converse fails: A good classical example is the space ℓ_∞, which has the property that every isomorphism between two separable subspaces can be extended to an automorphism of ℓ_∞ [14] (see also [4, Prop. 2.5.2]): or, alternatively, it is $(\aleph_1, \mathfrak{c}^+)$-automorphic. However, it is not of $(\aleph_1, \mathfrak{c}^+)$-universal disposition since not every isometry between two finite dimensional subspaces can be extended to an isometry of ℓ_∞ (see [1] and [4, Section 3.3.4]).

To complete our analysis it only remains to analyze the case $\aleph_0$. The spaces of universal $\aleph_0$-disposition are obviously the spaces of $\mathfrak{F}$-universal disposition; while the spaces $\aleph_0$-injective are called in [6] and [4, Def.1.3] locally injective spaces and shown to be the $\mathcal{L}_\infty$-spaces. The meaning of *locally λ-injective* should be clear; as should the fact that $\mathcal{L}_{\infty,\lambda}$-spaces are locally λ-injective. The following result should be clear by now:

Proposition 3. *A Banach space is locally 1-injective if and only if it is a 1-complemented subspace of a space of $\mathfrak{F}$-universal disposition.*

The list of locally 1-injective spaces however is not simple: 1-separably injective spaces are obviously locally 1-injective, but c_0, which is only 2-separably injective, is also locally 1-injective. The case of c_0 can be generalized. Recall that a Banach space is said to be polyhedral if the closed unit ball of every finite dimensional subspace is the closed convex hull of a finite number of points. Then every Lindenstrauss polyhedral space is locally 1-injective as it follows from [13, Prop. 7.2]. However, spaces of $\mathfrak{F}$-universal disposition cannot be polyhedral since they contain isometric copies of all separable spaces. On the other hand, even c, which is isomorphic to c_0, is not locally 1-injective: to show this, observe that, in the spirit of Lindenstrauss [13] one has:

Lemma 1. *The following properties are equivalent:*

(1) *X is locally 1-injective.*
(2) *Let $F \subset G$ be Banach spaces so that $\dim G/F < +\infty$. Every finite rank operator $\tau : F \to X$ can be extended to an operator $T : G \to X$ with the same norm.*
(3) *Let $F \subset G$ be Banach spaces so that $\dim G/F < +\infty$. Compact operators $\tau : F \to X$ can be extended to operators $T : G \to X$ with the same norm.*

Proof. To show that (1) implies (2), consider the push-out diagram

$$
\begin{array}{ccccc}
F & \longrightarrow & G & \longrightarrow & G/F \\
\tau\downarrow & & \downarrow\tau' & & \| \\
\tau(F) & \longrightarrow & \mathrm{PO} & \longrightarrow & G/F \\
\imath\downarrow & & & & \\
X & & & &
\end{array}
$$

where $\imath$ is the canonical inclusion. The local 1-injectivity allows to extend $\imath$ to a norm one operator $I : \mathrm{PO} \to X$ since $\dim \mathrm{PO}/\tau(F) = \dim G/F < +\infty$, which shows that PO is finite-dimensional. Therefore $I\tau'$ is an extension of τ with the same norm. We show now that (2) implies (3): as we have already mentioned, locally injective spaces are $\mathcal{L}_\infty$-spaces, and thus they have the Bounded Approximation Property, which makes any compact operator approximable by finite rank operators. Now, if every finite rank operator admits an equal norm extension, every operator that is in the closure of finite-rank operators also admits an equal norm extension. Finally, that (3) implies (1) is obvious. □

Corollary 1. *The space c is not locally 1-injective.*

Proof. This is consequence of an example in [11] of a compact operator $H \to c$ that cannot be extended with the same norm to one more dimension. □

To extend the results in this paper to the category of p-Banach spaces, a few items have to be taken into account. On one hand, the multiple push-out construction requires only minor adjustments to work (these can be seen explicitly in [6]) while the notions of p-Banach space universal disposition and separably injective can be translated verbatim. The fact that a space of universal disposition for separable p-Banach spaces is also separably injective is straightforward and can be found in [4, Prop. 3.50 (2)]. Since, as we have already shown, spaces of (α, β)-universal disposition are $(\alpha, \beta)_1$-injective, one faces an apparent contradiction with the fact that no universally separably injective p-Banach spaces exist at all (cf. [4, Prop. 3.45]. The key point is that $(\alpha, 2^\alpha)$-injective p-Banach spaces are no longer universally α-injective since no injective p-Banach spaces exist at all. It is however still true that:

Theorem 2. *A p-Banach space is $(\alpha, \beta)_1$-injective if and only if it is a 1-complemented subspace of a space of (α, β)-universal disposition.*

With the remarks above in mind, the proof goes exactly as that of Theorem 1.

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