Gradation of Algebras of Curves by the Winding Number

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ABSTRACT. We construct a new grading on the Goldman Lie algebra of a closed oriented surface by the winding number. This grading induces a grading on the HOMFLY-PT skein algebra and related algebras. Our work supports the conjectures of B. Cooper and P. Samuelson [CSb].

CONTENTS

1. Introduction

The Goldman Lie algebra $\mathfrak{g}(S)$ of a surface S is the Lie algebra of free homotopy classes of loops. The product is given by summing the signed

concatenations of curves over their points of intersection. This Lie algebra encodes a wealth of information about intersection numbers of curves [Cha15], maintains a close relation to skein invariants [Tur91] and invariant functions on spaces of surface group representations [Gol86]. It is also a fundamental component of string topology [CSa]. In this article, we construct a new cyclic grading on the Goldman Lie algebras and the HOMFLY-PT skein algebras of surfaces. In particular, there is a product preserving decomposition

$$\mathfrak{g}(S) = \bigoplus_{a \in \mathbb{Z}/\chi} \mathfrak{g}(S)_a$$

where $\chi = \chi(S)$ is the Euler characteristic of S. The existence of a cyclic grading by the winding number is predicted by recent work of B. Cooper and P. Samuelson [CSb]. This work relates the HOMFLY-PT skein algebra to the Hall algebra of the Fukaya category. For closed surfaces, these conjectures cannot be verified directly because the lack of $\mathbb{Z}$-grading on the Fukaya category currently impedes a rigorous study of their Hall algebras. In this way, our work constitutes new evidence for these conjectures. Presently, the only non-trivial evidence for closed surfaces S of genus greater than one. This new grading may also allow us to glean new information about these algebras and their many connections to other areas of mathematics.

In the remainder of the introduction, we explain what is meant by winding number, we discuss the conjectural context for our construction and we present an outline of our approach to the construction of the grading.

1.1. Winding numbers. The winding number of an closed oriented immersed curve in a surface is the total signed number of revolutions that its tangent vector undergoes in one traversal. In the plane this is a well-defined integer, but the generalization to closed surfaces of genus $g > 1$ has some indeterminacy. For a survey of winding number see [MC93]. A brief introduction is provided below.

In his study of regular closed curves in the plane, H. Whitney showed that the planar winding number is invariant under regular homotopy [Whi37]. In his thesis work [Sma58], S. Smale showed that for a Riemannian manifold M,

$$Imm_p(S^1, M) \simeq \Omega_{p'}S(TM),$$

the space $Imm_p(S^1, M)$ of immersed loops at the basepoint $p = (q, v) \in TM$ with initial and terminal velocity $v \in T_qM$ is weakly homotopic to the space of loops in the unit tangent bundle $S(TM)$ with basepoint $p' = (q, v/|v|)$. This equivalence determines isomorphisms

(1.1) $$\pi_0(Imm_p(S^1, M)) \cong \pi_0(\Omega_{p'}S(TM)) \cong \pi_1(S(TM), p').$$

The *regular fundamental group* $\pi_R(M, p)$ is defined to be this group

$$\pi_R(M, p) := \pi_0(Imm_p(S^1, M)).$$

In words, $\pi_R(M, p)$ is the group of regular homotopy classes of curves based at p in M under the operation of loop concatentation, see Def. 2.1. Eqn. (1.1) implies that $\pi_R(M, p)$ is independent of basepoint.

H. Seifert computed the group $\pi_1(S(TS))$ for a closed orientable surface S of genus g [Sei33]. Recall that such a surface S can be expressed as a quotient of the $4g$-gon in which the ith edge is identified with the $i + 2$nd edge for each $0 \leqslant i < 4g$ such that $i \equiv 0, 1 \,(\mathrm{mod}\,4)$. This decomposition gives us the presentation

$$(1.2) \qquad \pi_1(S) \cong \langle a_1, b_1, \dots, a_g, b_g \mid r \rangle \qquad \text{where} \qquad r = \prod_{i=1}^{g} [a_i, b_i].$$

and $[a, b] = aba^{-1}b^{-1}$. If $f = s^{-1}(q)$ denotes the homotopy class of a fiber of the unit circle bundle $s : S(TS) \to S$ then the fundamental group of the unit tangent bundle has a compatible presentation

$$(1.3) \quad \pi_1(S(TS)) \cong \left\langle a_1, b_1, ..., a_g, b_g, f \;\middle|\; \begin{array}{c} f^{2g-2}r \quad \text{and} \\ [f, a_i], [f, b_i] \text{ for } 1 \leqslant i \leqslant g \end{array} \right\rangle.$$

From this perspective, the *winding number* $\omega(\gamma)$ of a regular closed curve $\gamma \in \pi_R(S)$ is its projection onto the torsion component $\mathbb{Z}/\chi \subset \pi_R(S)^{ab}$ of the abelianization. In more detail, the abelianization can be computed from Equation (1.3) using Smale's isomorphism

$$(1.4) \qquad \pi_R(S)^{ab} \cong \pi_1(S(TS))^{ab} \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/\chi \qquad \text{where} \qquad \chi = 2g - 2$$

and the observation that $r = 1$ in the abelianization. The torsion group is generated by the fiber f. The winding number homomorphism

$$(1.5) \qquad\qquad\qquad \omega : \pi_R(S) \to \mathbb{Z}/\chi$$

first maps $\gamma \in \pi_R(S)$ to the abelianization $\pi_R(S)^{ab}$ and then extracts the coefficient of the fiber f.

There are many curves with non-zero winding number. For instance, the separating curve on the genus 2 surface featured in Figure 1 has winding number $1 \,(\mathrm{mod}\,2)$.
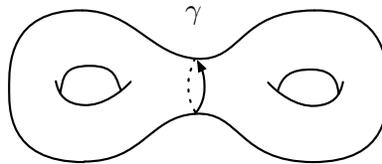

FIGURE 1. A curve γ with winding number $\omega(\gamma) = 1 \in \mathbb{Z}/2$

In his work on winding numbers of regular curves on surfaces B. L. Reinhart introduced an integral presentation for the winding number

$$(1.6) \qquad \omega(\gamma) = \frac{1}{2\pi} \int_{S^1} A^*_\gamma(d\theta) \qquad (\mathrm{mod}\,\chi)$$

as the degree of the map determined by the angle A_γ between the derivative of γ and a vector field S, see Definition 2.4 and [Rei60]. Reinhart's integral satisfies the property that $\omega(\gamma) = 0$ for each of the regular representatives of generators $\gamma \in \{a_i, b_i\}_{i=1}^g$ and $\omega(\eta) = 1$ for a small counterclockwise contractible regular loop η. This makes it compatible with the definition above, as the generators $\{a_i, b_i\}_{i=1}^g$ determine the $\mathbb{Z}^{2g}$-component of $\pi_R(S)^{ab}$, while the fiber f corresponds to a small homotopically trivial counterclockwise loop. See Section 2.4 for further discussion.

Remark 1.1. The winding number depends on the splitting isomorphism in Eqn. (1.4). Although the arguments in this paper hold for any such isomorphism, we will use the one determined by the choice of $\{a_i, b_i\}$ basis mentioned in the above. For some applications, it may be natural to grade algebras by $H_1(S(TS))$ rather than $\mathbb{Z}^{2g} \oplus \mathbb{Z}/\chi$.

1.2. Context. The work of B. Cooper and P. Samuelson establishes a relationship between the HOMFLY-PT skein algebra of a surface and the Hall algebra of the Fukaya category of the surface [CSb]. For certain surfaces with boundary, they prove that elements in the Hall algebra of the Fukaya category satisfy the HOMFLY-PT skein relation. It is natural to ask whether there is evidence for a relationship between the Hall algebra of the Fukaya category of a *closed surface S* and the skein algebra. The theorem of H. Morton and P. Samuelson which relates the skein algebra of the torus and the elliptic Hall algebra [MS17] provides some evidence in genus one when one assumes that a version of homological mirror symmetry holds over finite fields. Our construction of a grading by winding number on the HOMFLY-PT skein algebra provides evidence for this conjecture when the surfaces are closed and $\chi < 0$.

Since the Hall algebra of a category $\mathcal{C}$ is always graded by the Grothendieck group $K_0(\mathcal{C})$ of the category, the Hall algebra of the Fukaya category $\mathcal{F}(S)$ should be graded by the group $K_0(\mathcal{F}(S))$. M. Abouzaid computed this group for closed surfaces

$$K_0(\mathcal{F}(S)) \cong H_1(S(TS)) \oplus \mathbb{R} \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/\chi \oplus \mathbb{R}$$

when the Euler characteristic satisfies $\chi < 0$ [Abo08]. So if the Hall algebra of the Fukaya category of a closed surface could be defined then it would be graded by the group $H_1(S) \oplus \mathbb{Z}/\chi$. Because of these observations, it was conjectured that the HOMFLY-PT skein algebra shares this grading.

Now, it is considered well-known that one can grade skein algebras by the first homology group $H_1(S)$ of the surface. However, a grading by the winding number ω is somewhat counterintuitive because the curves which

constitute elements of the skein algebra live in the 3-dimensional space $S \times [0,1]$, so Reinhart's integral (1.6) is not applicable. It is in this sense that the $\mathbb{Z}/\chi$-grading by winding number on the HOMFLY-PT skein algebra extends and supports the conjectures of B. Cooper and P. Samuelson.

1.3. Results. The principal result of this paper is to establish a grading of the Goldman Lie algebra $\mathfrak{g}(S)$ and its quantization $H_q(S)$ the HOMFLY-PT skein algebra by the winding number. The theorem below summarizes our main results.

Theorem. *Let S be a compact connected oriented surface. There is a canonical extension of the winding number homomorphism*

$$\omega : \pi_R(S) \to \mathbb{Z}/\chi \qquad \rightsquigarrow \qquad \hat{\pi}(S) \to \mathbb{Z}/\chi$$

to a map on the set $\hat{\pi}(S)$ of free homotopy classes of loops on S. This extension determines a grading on the Goldman Lie algebra

$$\mathfrak{g}(S) = \bigoplus_{a \in \mathbb{Z}/\chi} \mathfrak{g}(S)_a \qquad \text{where} \qquad [\mathfrak{g}(S)_a, \mathfrak{g}(S)_b] \subset \mathfrak{g}(S)_{a+b}.$$

There are similar gradings on the regular Goldman Lie algebra $\mathfrak{g}_R(S)$ and the HOMFLY-PT skein algebra $H_q(S)$.

The main obstacle to the construction of the grading is the domain of definition: the notion of a winding number is defined on regular homotopy classes whereas the Goldman Lie algebra is defined on free homotopy classes of curves. The winding number of a curve is not well-defined on curves up to free homotopy because there are multiple regular homotopy classes within any given free homotopy class. More concretely, topological artifacts such as fish-tails can be introduced to artificially increase or decrease the winding number of a curve, see Section 2.5.
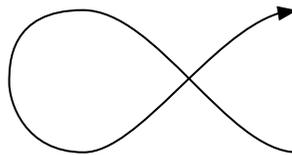

FIGURE 2. A fish-tail

Organization Section 2 reviews concepts which will be important in the main construction. In Section 3, the relationship between the set of free regular homotopy classes of curves $\hat{\pi}_R(S)$ and free homotopy classes of curves $\hat{\pi}(S)$ is articulated. In §3.1, we show that there is a surjective loop product preserving map $s_* : \hat{\pi}_R(S) \to \hat{\pi}(S)$, the fiber of which is a $\mathbb{Z}$-torsor generated by the operation of adding or removing fish-tails. In §3.2, we show that the map s_* admits a section $\Phi : \hat{\pi}(S) \to \hat{\pi}_{U,R}(S) \subset \hat{\pi}_R(S)$, taking free homotopy classes of curves to unobstructed representatives. In §4, the

winding grading on the Goldman Lie algebra $\mathfrak{g}(S)$ is introduced by setting the winding number of a non-contractible free homotopy class $[\alpha]$ to be the winding number of its unobstructed representative $\Phi([\alpha])$. We check that this definition respects concatenation of loops, extends additively over the Lie bracket and so defines a map $\omega : \mathfrak{g}(S) \to \mathbb{Z}/\chi$ which determines a grading of the Goldman Lie algebra. Section 5 introduces the regular Goldman Lie algebra $\mathfrak{g}_R(S)$. In Section 6, we prove that the winding number grading of the regular Goldman Lie algebra extends to the HOMFLY-PT skein algebra.

2. Notations and definitions

We recall a few basic definitions, the definition of the Goldman Lie algebra, Reinhart's integral definition of the winding number and some results about unobstructed curves.

2.1. Grading. If M is an R-module and A is an abelian group then a *grading* of M by A is a direct sum decomposition $M \cong \oplus_{a \in A} M_a$. If M is an algebra then we require that $mn \in M_{a+b}$ when $m \in M_a$ and $n \in M_b$. Likewise, if M is a Lie algebra then we require that $[m, n] \in M_{a+b}$ when $m \in M_a$ and $n \in M_b$.

If S is a set and $M = R\langle S \rangle$ is the free R-module on S then a grading of M by A is determined by a map $gr : S \to A$; in this case, $M_a := R\langle m \mid gr(m) = a \rangle$. If M is a Lie algebra then $[m, n] \in M_{a+b}$ when $gr(m) = a$ and $gr(n) = b$.

2.2. Surface topology. Throughout this paper, S will always be a closed connected oriented surface. A set of curves $\{\gamma_i : S^1 \to S\}_{i \in I}$ on S is said to be in general position when all curves are normal closed, all intersections are transverse and there are no triple intersections among curves. We will always assume curves are in general position.

Two curves $\alpha, \beta : S^1 \to S$ are *freely homotopic* when they are in the same path component of the free loop space $Map(S^1, S)$, they are *homotopic* when they share a basepoint p and are contained in the same path component of the based loop space $\Omega_p S = Map_p(S^1, S)$. Two embeddings α and β are *isotopic* when they are contained in the same path component of the space of embeddings $Emb(S^1, S)$. Two immersions are *freely regularly homotopic* when they are contained in the same path component of the space of immersions $Imm(S^1, S)$ and *regularly homotopic* when they are contained in the same path component of the space of immersions $Imm_p(S^1, S)$ based at p.

The set of *free homotopy classes of loops on S* will be denoted by $\hat{\pi}(S)$. A free homotopy class of map in $\hat{\pi}(S)$ can also be thought of as a conjugacy

class of $\pi_1(S)$. If $\gamma : S^1 \to S$ is a loop then we will denote by $[\gamma]$ its corresponding free homotopy class. Any other equivalence relation on curves will be indicated by a subscript on the brackets, for example

$$[\gamma]_{\pi_1} \in \pi_1(S, p)$$

means that $\gamma : S^1 \to S$ should be seen as an equivalence class $[\gamma]$ of curves in S based at p in $\pi_1(S, p)$.

2.3. The Goldman Lie algebra. Here we recall the definition of the Goldman Lie algebra. The first definition recalls how we will concatenate curves.

Definition 2.1. $(\alpha \cdot_p \beta)$ Suppose that two loops $\alpha, \beta : S^1 \to S$ cross at a point $p \in \alpha \cap \beta$ then α and β define elements $[\alpha]_{\pi_1}, [\beta]_{\pi_1} \in \pi_1(S, p)$ and their *oriented loop product* $[\alpha \cdot_p \beta]_{\pi_1} := [\alpha]_{\pi_1}[\beta]_{\pi_1}$ is just their product in $\pi_1(S, p)$. More generally, $[\alpha \cdot_p \beta]$ will denote the image of $[\alpha \cdot_p \beta]_{\pi_1}$ in $\hat\pi(S)$. The free homotopy class of $\alpha \cdot_p \beta$ does not depend on the choice of representatives $\alpha \in [\alpha]$ or $\beta \in [\beta]$.

Up to free homotopy, in a small neighborhood U of $p \in \alpha \cap \beta$, the product $\alpha \cdot_p \beta$ can be viewed as replacing the picture on the lefthand side of Figure 3 with the righthand side.
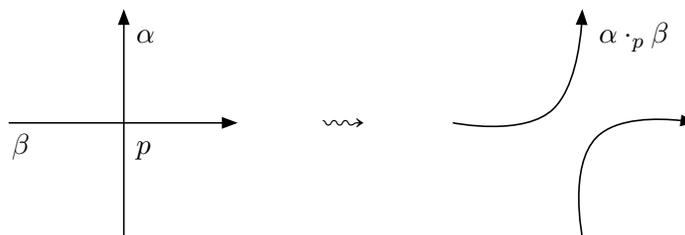

FIGURE 3. The loop product

This oriented loop product preserves immersions because the two diagrams agree on vectors in the boundary ∂U.

Definition 2.2. (ϵ_p) If two immersed curves $\alpha, \beta : (0,1) \to S$ intersect at a point p then the *sign* $\epsilon_p(\alpha, \beta)$ of their intersection is $+1$ if the ordered pair of derivatives $(\dot\alpha(p), \dot\beta(p))$ agrees with the orientation of S at p and $\epsilon_p(\alpha, \beta) := -1$ otherwise.

The loop product and the sign are important to us because they are used to define the Goldman Lie algebra of the surface.

Definition 2.3 (Goldman Lie algebra). The *Goldman Lie algebra* $\mathfrak{g}(S)$ of a surface S is the free abelian group on the set $\hat\pi(S)$ of free homotopy classes

of curves equipped with a Lie bracket

$$\mathfrak{g}(S) := \mathbb{Z}\langle\hat{\pi}(S)\rangle.$$

Given two classes of curves $[\alpha], [\beta] \in \hat{\pi}(S)$, we choose representatives $\alpha \in [\alpha]$ and $\beta \in [\beta]$ so that the Goldman Lie bracket is defined to be the signed sum of their oriented loop products at points p of intersection

$$(2.1) \qquad [[\alpha], [\beta]] := \sum_{p \in \alpha \cap \beta} \epsilon_p(\alpha, \beta)[\alpha \cdot_p \beta],$$

see [Gol86, Thm. 5.3] for further details.

2.4. Reinhart's winding number. As mentioned in the introduction, B. L. Reinhart used differential topology to give a definition of the winding number $\omega : \pi_R(S) \to \mathbb{Z}/\chi$, see [Rei60]. Here we recall a few more details.

Definition 2.4 (Winding number homomorphism)**.** Choose a Riemannian metric on S and a vector field X with isolated zeros. Let $\angle_p(Y, Z)$ denote the angle between two vectors Y and Z in T_pS at $p \in S$.

If $\gamma(\theta) : S^1 \to S$ is an immersed curve then there is a map $A_\gamma : S^1 \to S^1$ given by

$$A_\gamma(\theta) := \angle_p(\dot{\gamma}, X) \qquad \text{where} \qquad p = \gamma(\theta)$$

the angle between the tangent $\dot{\gamma} = \gamma_*(d/d\theta)$ of γ at p and the vector $X = X_p$ at the same point.

Reinhart's *winding number* $\omega(\gamma) \in \mathbb{Z}/\chi$ is the degree of the map A_γ taken modulo the Euler characteristic of S

$$\omega(\gamma) = \frac{1}{2\pi} \int_{S^1} A_\gamma^*(d\theta) \qquad (\text{mod } \chi)$$

when the vector field X is chosen so that $\omega(a_i) = 0$ and $\omega(b_i) = 0$ for $\{a_i, b_i\}$ curves in the 1-skeleton which determines Eqn. (1.2).

Reinhart showed that this formula is independent of the choice of the vector field X, within the imposed constraints, and the choice of the regular homotopy representative of γ after the degree is taken modulo the Euler characteristic of the surface [Rei60, Prop. 2]. This is due to the value of the integral shifting by $\pm\chi$ as a regular homotopy moves a segment of γ over a zero of X.

Remark 2.5. Reinhart's definition is also independent of the chosen metric. This is because the space of all Riemannian metrics on S is path-connected, if the metric is continuously varied then the value of the integral varies continuously, but since $\mathbb{Z}$ is discrete, the value must be constant.

2.5. Unobstructed curves and Abouzaid's lemma. In this section we recall the notion of unobstructed curves and Abouzaid's lemma which characterizes them in a convenient way.

Definition 2.6 (Unobstructed curve)**.** Let $\tilde{S} \to S$ be the universal covering space of S. We say that an immersed loop $\alpha : S^1 \to S$ is *unobstructed* when it lifts to a properly embedded path in $\tilde{S}$.

Abouzaid characterized unobstructed curves in terms of fish-tails and contractability [Abo08]. Roughly speaking, a fish-tail is a homotopically trivial self-intersection, see Figure 2.

Definition 2.7 (Fish-tail)**.** If $\gamma : S^1 \to S$ is an immersed curve then a *fish-tail* in γ is a disk map $D^2 \to S$ which maps the boundary of the disk to γ and is non-singular everywhere except one point on γ.

Abouzaid's lemma states that for non-nulhomotopic curves fish-tails represent the only obstruction to being unobstructed.

Lemma 2.8 (Abouzaid)**.** *A properly immersed smooth curve γ is unobstructed if and only if it is not nulhomotopic and does not bound an immersed fish-tail.*

Unobstructed curves are regular homotopy representatives of curves with no unnecessary winding. The unobstructed loops are important in what follows because they will be our canonical choice of regular representative within each free homotopy class. More precisely, if $\hat{\pi}_{U,R}(S)$ denotes the set of regular free homotopy classes of unobstructed curves on S then we will show that there is a bijection between the sets

$$\Phi : \hat{\pi}_{U,R}(S) \xrightarrow{\sim} \hat{\pi}(S) \backslash \{C\}$$

where C denotes the free homotopy class of the contractible curve. We must remove C from the righthand side since every regular representative of the contractible free homotopy class C is obstructed.

The first step in the construction of the grading will be to define the grading on unobstructed curves in $\hat{\pi}_{U,R}(S)$. Since C is contained in the center of the Goldman Lie bracket, we can define the grading on the Goldman Lie algebra modulo the center. Then the grading can be lifted by choosing any regular homotopy representative of C in $\pi_R(S)$. We will choose the one in Figure 4.
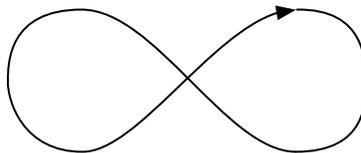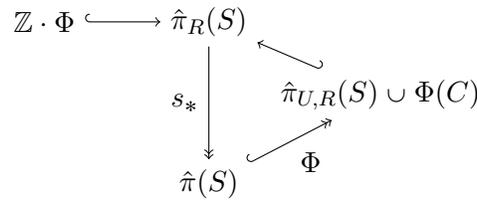

FIGURE 4. Our preferred choice of the regular representative for the free homotopy class C of the contractible loop.

3. Relations between free and regular homotopy

In this section we study the relationship between free and regular homotopy classes of curves on a surface. In Section 3.1, we prove that the map $s_* : \hat{\pi}_R(S) \to \hat{\pi}(S)$ which passes from regular homotopy classes to free homotopy classes is a surjective, oriented loop product preserving map. The fiber of s_* over any free homotopy class is identified with the $\mathbb{Z}$-orbit of any regular representative. In §3.2, a section $\Phi : \hat{\pi}(S) \to \hat{\pi}_R(S)$ is constructed by mapping free homotopy classes to unobstructed representatives. This information is summarized by the diagram below.

$$
\begin{array}{ccc}
\mathbb{Z} \cdot \Phi & \longhookrightarrow & \hat{\pi}_R(S) \\
& & \Big\downarrow{\scriptstyle s_*} \qquad \hat{\pi}_{U,R}(S) \cup \Phi(C) \\
& & \hat{\pi}(S) \qquad \Phi
\end{array}
$$

3.1. Regular curves as a $\mathbb{Z}$-torsor. Here we show that the map $s_* : \hat{\pi}_R(S) \to \hat{\pi}(S)$, which takes a free regular curve to its free homotopy class, is surjective and preserves oriented loop products. The map s_* commutes with a $\mathbb{Z}$-action on $\hat{\pi}_R(S)$ which is given by gluing fish-tails onto regular curves and there is $\mathbb{Z}$-equivariant isomorphism from the fiber over any free homotopy class to the $\mathbb{Z}$-orbit of any representative

$$
\mathbb{Z} \cdot [\gamma]_R \xrightarrow{\sim} s_*^{-1}([\gamma]) \qquad \text{for} \qquad \gamma \in [\gamma].
$$

Definition 3.1. (s_*) Composing the maps in Eqn. (1.1) with the bundle map $s : S(TS) \to S$ defines a map

$$
\pi_R(S, p) = \pi_0(Imm_p(S^1, S)) \cong \pi_1(S(TS), p') \xrightarrow{\pi_1(s)} \pi_1(S, q),
$$

which takes a based regular homotopy class of immersed curve to its corresponding homotopy class. This induces a map between sets of conjugacy classes

$$
s_* : \hat{\pi}_R(S) \to \hat{\pi}(S).
$$

This map is onto because every loop in $\pi_1(S, q)$ lifts to a loop in $\pi_1(S(TS), p')$; alternatively, $Imm_p(S^1, S) \subset Map(S^1, S)$ is dense. See also Lem. 3.5.

Definition 3.2. ($\mathbb{Z}$-action) There are maps $t, t^{-1} : \hat{\pi}_R(S) \to \hat{\pi}_R(S)$ which generate a $\mathbb{Z}$-action. If γ is an immersed curve in S then picking any point p on γ we can add a small positive fish-tail at p or a small negative fish-tail at p. In either case, this defines a new immersed curve which we call $t\gamma$ or $t^{-1}\gamma$. These operations are well-defined and independent of p because we can both shrink a fish-tail to be arbitrarily small and drag it around the

curve by regular homotopy. The maps t and t^{-1} are illustrated by
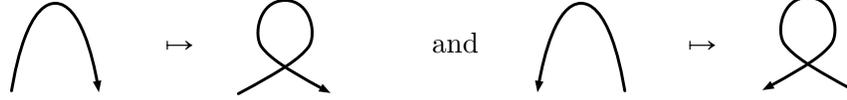
respectively. These maps satisfy $tt^{-1} = 1_{\hat{\pi}_R(S)}$ and $t^{-1}t = 1_{\hat{\pi}_R(S)}$ because, up to regular homotopy, a positive fish-tail can be cancelled with a negative fish-tail. One of the two cases is shown below.
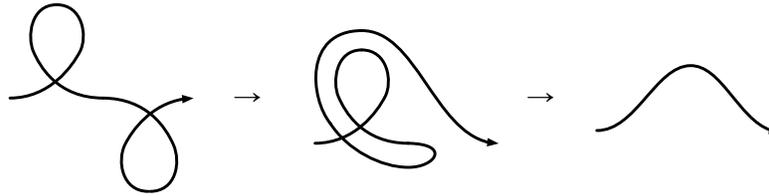
The picture proof of the other case is obtained by reversing the orientation above. So the maps t and t^{-1} define a $\mathbb{Z} \cong \langle t \rangle$ group action on $\hat{\pi}_R(S)$.

Since any additional fish-tails on a curve γ can be removed by homotopy, the $\mathbb{Z}$-action preserves the fiber

$$s_*(t[\gamma]_R) = s_*([\gamma]_R) \quad \text{and} \quad s_*(t^{-1}[\gamma]_R) = [\gamma]_R.$$

Moreover, Reinhart's integral formula shows that this shifts the winding number, $\omega(t[\gamma]_R) = \omega([\gamma]_R) + 1$, compare to Cor. 4.6.

Remark 3.3. By virtue of Smale's isomorphism $\phi : \hat{\pi}_R(S) \xrightarrow{\sim} \hat{\pi}(S(TS))$. There is an alternative way to define the $\mathbb{Z}$-action. Recall from Eqn. (1.3) that there is a central element $f \in \pi_1(S(TS), p')$ representing the fiber of the unit tangent bundle. If $[\gamma]_R \in \hat{\pi}_R(S)$ then

$$\phi(t[\gamma]_R) = f\phi([\gamma]_R) \in \hat{\pi}(S(TS))$$

so that multiplication by f determines the action of t on the set $\hat{\pi}_R(S)$. This can be seen directly by observing that a curve $t\gamma$ with fish-tail in $Map(S^1, S(TS))$ is homotopic to the curve γ in S with section vectors forming a positive full twist.

Definition 3.4. The category of pointed sets and pointed set maps is the undercategory $Set_* := \{pt\}/Set$. In more detail, an object is a pair (A, a) consisting of a set A and an element $a \in A$. A map $f : (A, a) \to (B, b)$ between two such pairs is a set map $f : A \to B$ which satisfies $f(a) = b$. A sequence of such maps

$$(A, a) \xrightarrow{f} (B, b) \xrightarrow{g} (C, c)$$

is *exact* when $g^{-1}(c) = im(f)$. There is a terminal object $1 := (\{pt\}, pt)$.

The next lemma shows that passing from groups to conjugacy classes of elements in groups is a functor from groups to pointed sets. The important part of the lemma shows that a central inclusion of groups is mapped to an injective map of pointed sets. So the functor preserves the exactness of central extensions in a strong sense.

Lemma 3.5. *Taking conjugacy classes determines a functor* $\hat{\cdot} : Grp \to Set_*$ *from groups to pointed sets. In particular, a homomorphism $a : K \to G$ of groups induces a map $\hat{a} : \hat{K} \to \hat{G}$ from conjugacy classes in K to conjugacy classes in G. This functor satisfies the three properties below.*

(1) *A short exact sequence of groups induces a short exact sequence of pointed sets.*
(2) *If $b : G \to H$ is an epimorphism then the set map $\hat{b}$ is surjective.*
(3) *If $a : K \to G$ is a monomorphism and central, i.e. $im(a) \subset Z(G)$, then the set map $\hat{a}$ is injective.*

Proof. If $a : K \to G$ then set $\hat{a}([k]) := [a(k)]$ if $k \sim k'$ so $k = rk'r^{-1}$ then $a([k]) = [a(rk'r^{-1})] = [a(r)a(k')a(r)^{-1})] = [a(k')]$. This is a functor, if $a : K \to G$ and $b : G \to H$ then $\hat{b}(\hat{a}([k])) = \hat{b}([a(k)]) = [ba(k)] = \widehat{ba}([k])$.

1. Suppose $1 \to K \xrightarrow{a} G \xrightarrow{b} H \xrightarrow{p} 1$ is a short exact sequence of groups. Then there is a sequence of maps

$$1 \to \hat{K} \xrightarrow{\hat{a}} \hat{G} \xrightarrow{\hat{b}} \hat{H} \xrightarrow{\hat{p}} 1$$

First we show that exactness holds on the left. If $[k] \in \hat{a}^{-1}([1])$ then $[a(k)] = [1]$ implies $a(k) = r1r^{-1} = 1$ for some $r \in G$. So $k \in ker(a) = \{1\}$. Therefore, $\hat{a}^{-1}([1]) = [1]$.

Exactness hold on the right because, for all $[h] \in \hat{p}^{-1}([1])$, there is a $g \in G$ such that $b(g) = h$ by surjectivity of b. So $\hat{b}([g]) = [b(g)] = [h]$. This is a special case of *2.* below.

Now we will show that $im(\hat{a}) = \hat{b}^{-1}([1_H])$. First $\hat{b}\hat{a}[k] = [b(a(k))] = [1_H]$ implies $im(\hat{a}) \subset \hat{b}^{-1}([1_H])$. Suppose $[g] \in \hat{b}^{-1}([1_H])$, so $\hat{b}([g]) = [1_H]$ then there is $h \in H$ such that $hb(g)h^{-1} = 1_H$ or $b(g) = 1_H$, so $g \in ker(b) = im(a)$ and there exists $k \in K$ such that $a(k) = g$, it follows that $\hat{a}([k]) = [g]$. Thus $\hat{b}^{-1}([1_H]) \subset im(\hat{a})$, so $im(\hat{a}) = \hat{b}^{-1}([1_H])$.

2. Assume that $b : G \to H$ is a surjective homomorphism. If $[h] \in \hat{H}$ then for any $h \in [h]$, there is a $g \in G$ such that $b(g) = h$. So $\hat{b}([g]) = [b(g)] = [h]$.

3. Assume that $a : K \to G$ is an injective homomorphism and $im(a) \subset Z(G)$. If $\hat{a}([k]) = \hat{a}([k'])$ then $[a(k)] = [a(k')]$. So there is an $r \in G$ such that $a(k) = ra(k')r^{-1}$, but $a(k') \in im(a) \subset Z(G)$ implies $a(k) = a(k')rr^{-1} = a(k')$, so $k = k'$ by injectivity of a. $\square$

Proposition 3.6. *Suppose that $[\tau]_R \in s_*^{-1}([\gamma])$ is any curve in the fiber of s_* over $[\gamma] \in \hat{\pi}(S)$. Then there is a $\mathbb{Z}$-equivariant bijection*

$$\kappa : \mathbb{Z} \cdot [\tau]_R \xrightarrow{\sim} s_*^{-1}([\gamma])$$

from the $\mathbb{Z}$-orbit of $[\tau]_R$ to the fiber.

Proof. This is because the kernel of s_* is $\langle f \rangle \cong \mathbb{Z}$ where f is the fiber. In more detail, there is a diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(S(TS)) & \xrightarrow{\pi_1(s)} & \pi_1(S) & \longrightarrow & 1 \\
& & \downarrow & & \phi_1 \big\downarrow z \;\; \pi_R(S) & & \downarrow & & \\
1 & \longrightarrow & \hat{\pi}(S^1) & \xrightarrow{\hat{\pi}(i)} & \hat{\pi}(S(TS)) & \xrightarrow{\hat{\pi}(s)} & \hat{\pi}(S) & \longrightarrow & 1 \\
& & & & \phi \;\; \hat{\pi}_R(S) & & s_* & &
\end{array}
$$

The long exact sequence of homotopy groups associated to the circle bundle $S^1 \xrightarrow{i} S(TS) \xrightarrow{s} S$ gives a short exact sequence among fundamental groups because S^1 and S are both $K(G,1)$-spaces. This gives the first row. The second row comes from Lem. 3.5 above. The map ϕ is the Smale isomorphism and the bottom triangle commutes because $s_* = \hat{\pi}(s) \circ \phi$. There is an isomorphism $\pi_1(S^1) \cong \hat{\pi}(S^1) = \langle f \rangle \cong \mathbb{Z}$ is generated by the fiber element in the Seifert presentation of $\pi_1(S(TS))$, see Eqn. (1.3).

The map κ is determined by $\mathbb{Z}$-equivariance and the assignment $0 \cdot [\tau]_R \mapsto [\tau]_R$. The two statements below are equivalent to injectivity and surjectivity of κ respectively.

(1) If $[x]_R \in s_*^{-1}([\gamma])$ then $[x]_R = t^n[\tau]_R$ in $\pi_R(S,p)$ for some $n \in \mathbb{Z}$.
(2) $t^n[\tau]_R \not\simeq [\tau]_R$ for all $n \in \mathbb{Z} \backslash \{0\}$ and $[\tau]_R \in \hat{\pi}_R(S)$.

For the first statement, fix a basepoint $p = (x(\theta_0), \dot{x}(\theta_0))$ on x for an $x \in [x]_R$ and $\theta_0 \in S^1$. In doing so $\phi[x]_R$ lifts trivially to $[x]_{\pi_1} \in z^{-1}(\phi[x]_R)$. Since $[x]_R \in s_*^{-1}([\gamma])$, $\phi[x]_R \in \hat{\pi}(s)^{-1}([\gamma])$ and $[x]_{\pi_1} \in \pi_1(s)^{-1}[\gamma]$. But $\pi_1(s)^{-1}[\gamma] = [\gamma]\mathrm{ker}(\pi_1(s)) = [\gamma]\langle f \rangle$. So $[x]_{\pi_1} = f^n[\gamma]$ for some $n \in \mathbb{Z}$. The rest follows from Rmk. 3.3.

The second statement follows from the injectivity of $\hat{\pi}(i)$ and centrality of $f \in \pi_1(S(TS), p')$. Here is a detailed argument. By Rmk. 3.3 we may identify the action of t with multiplication by the fiber element f. Notice that if the genus of S is one or $[\tau]_R = f^N C$ is homotopically trivial then the statement is trivial. So assume that the genus is at least two, τ is homotopically non-trivial and the conjugacy classes $f^n[\tau]_R = [\tau]$ are equal in $\hat{\pi}(S(TS))$ for some $n \in \mathbb{Z}$. There is a curve α such that $f^n\tau = \alpha\tau\alpha^{-1}$ or $f^n = \alpha\tau\alpha^{-1}\tau^{-1}$. Applying the map $s' := \pi_1(s)$ gives $1 = s'(\alpha\tau\alpha^{-1}\tau^{-1})$ in $\pi_1(S)$. In particular, $s'(\alpha) \in Cent_{\pi_1}(s'(\tau))$. The centralizer of a non-trivial element $s'(\tau) \in \pi_1(S)$ is cyclic $\langle \beta \rangle$ when S is hyperbolic, which is true when the genus is greater than two, see [FM11, §1.1.3]. Every element $s'(\tau)$ is

contained in it's own centralizer. So

$$s'(\alpha), s'(\tau) \in Cent_{\pi_1}(s'(\tau)) = \langle \beta \rangle$$

and $s'(\alpha) = \beta^q$ and $s'(\tau) = \beta^m$ for some $q, m \in \mathbb{Z}$ and for some $\beta \in \pi_1(S)$. By exactness of the first row, there are integers $p, \ell \in \mathbb{Z}$ such that $\alpha = f^p \beta^q$ and $\tau = f^\ell \beta^m$. Combining these observations gives

$$
\begin{aligned}
f^n &= \alpha \tau \alpha^{-1} \tau^{-1} \\
&= f^p \beta^q f^\ell b^m f^{-p} \beta^{-q} f^{-\ell} b^{-m} \\
&= \beta^q b^m \beta^{-q} b^{-m} \\
&= 1
\end{aligned}
$$

in $\pi_1(S(TS))$, which contradicts injectivity of the fiber map $\pi_1(i)$.　□

Remark 3.7. The map κ depends on the choice τ. The section Φ in §3.2 will give a canonical choice of τ.

Proposition 3.8. *The map s_* preserves the oriented loop product*

$$s_*([\alpha \cdot_p \beta]_R) = s_*([\alpha]_R) \cdot_q s_*([\beta]_R).$$

Proof. Since s_* is induced by a map $\pi_1(S(TS), p') \to \pi_1(S, q)$ between fundamental groups, it must preserve the product at any basepoint p.　□

3.2. The s_*-section Φ. Here we define a section $\Phi : \hat{\pi}(S) \to \hat{\pi}_R(S)$ of the map $s_* : \hat{\pi}_R(S) \to \hat{\pi}(S)$. The map Φ is injective and its image is the set

$$im(\Phi) = \hat{\pi}_{U,R}(S) \cup \Phi(C)$$

of unobstructed regular curves in S together with a choice $\Phi(C)$ of regular representative for the contractible curve (see Fig. 4). These properties are established in Proposition 3.12. The proof of this proposition uses Lemma 3.9 which shows that every free homotopy class contains an unobstructed representative and Lemma 3.11 which shows that any two unobstructed representatives of the same free homotopy class are freely regularly homotopic.

Lemma 3.9. *Let $\alpha : S^1 \to S$ be a non-nulhomotopic loop in S and let $[\alpha] \in \hat{\pi}(S) \backslash \{C\}$ denote its free homotopy class. Then there is a representative $\bar{\alpha} \in [\alpha]$ such that $\bar{\alpha}$ is unobstructed.*

Proof. Let $[\alpha] \in \hat{\pi}(S) \backslash \{C\}$ with α a normal representative. Assume by contradiction that α fails to be unobstructed. Since α is not nulhomotopic, Abouzaid's Lemma 2.8 implies that the image of α bounds a finite (by normality) number of fish-tails. By Definition 2.7 of fish-tail, there must be a disk through which any fish-tail can be contracted. Such disks determine a smooth convex homotopy $F : S^1 \times [0,1] \to S$ with support in a contractible neighborhood of the fish-tails that straightens them from innermost to outermost into an immersed embedded line segment. Thus the resulting loop $\bar{\alpha}$ representing $[\alpha]$ obtained via F is regular, non-trivial, has no fish-tails, and is unobstructed, again, by Abouzaid's Lemma 2.8.　□

The lemma above establishes the existence of unobstructed representatives of curves within each free homotopy class. The next lemma will show that there is a unique choice up to regular homotopy. We will use a relative version of Epstein's Theorem which we now recall.

Theorem 3.10. *([?]) Let S be a surface with boundary and α, β two embedded arcs with endpoints that are equal and contained in the boundary of S. If α and β are homotopic with endpoints fixed then α and β are isotopic with endpoints fixed.*

We are now ready to prove our lemma.

Lemma 3.11. *If two unobstructed curves α and β are freely homotopic then they are freely regularly homotopic.*

The interplay between different types of equivalence classes of loops now becomes important. Recall our notational convention from Section 2.2 of appending subscripts to brackets around loops, indicating the quotient set to which the equivalence class belongs.

Proof. Let α and β be unobstructed curves belonging to the same free homotopy class: $[\alpha]_{\hat{\pi}} = [\beta]_{\hat{\pi}}$, given in general position, in particular there are finitely many points of intersection.

First we show that we can reduce the problem to the case where α and β are homotopic at a fixed basepoint. Since α and β are freely homotopic, there are basepoints a and b for α and β and a path $\gamma : [0,1] \to S$ from a to b such that

$$[\alpha]_{\pi_1(S,a)} = [\bar{\gamma} \cdot \beta \cdot \gamma]_{\pi_1(S,a)}$$

After a small homotopy we can assume that the curves γ and $\bar{\gamma}$ are smooth, regularly parametrized, concatenate regularly with each other and β and have no fish-tails. In particular, the cyclic segment $\gamma \cdot \bar{\gamma}$ can be made to share the same tangent vector as α at a and to freely regularly retract back to the original path segment of β crossing b. In this way we obtain a new unobstructed representative $\tilde{\beta}$ of $[\bar{\gamma} \cdot \beta \cdot \gamma]_{\pi_1(S,a)}$ which is regularly freely homotopic to β and basepoint homotopic to α.

So, without loss of generality, we make the additional assumption that α and β share a basepoint and are basepoint homotopic

$$(3.1) \qquad\qquad\qquad [\alpha]_{\pi_1} = [\beta]_{\pi_1}.$$

Let $\pi : \tilde{S} \to S$ be the universal covering and, after fixing a basepoint $\tilde{p} \in \pi^{-1}(p)$, there are paths $\hat{\alpha}$ and $\hat{\beta}$ lifting the loops α and β respectively. Since α and β are unobstructed, the paths $\hat{\alpha}$ and $\hat{\beta}$ are embedded. Equation (3.1) implies that these lifts share endpoints and are homotopic rel endpoints. So by Epstein's Theorem 3.10, $\hat{\alpha}$ and $\hat{\beta}$ are isotopic rel endpoints in $\tilde{S}$. Applying the covering map π to this isotopy produces a regular homotopy between α and β. $\qquad\square$

Combining Lemma 3.9 and Lemma 3.11 shows that any two choices of unobstructed representatives for a free homotopy class are freely regularly homotopic. Thus any non-nulhomotopic free homotopy class $[\gamma]$ contains a unique unobstructed representative $\Phi([\gamma])$ up to free regular homotopy. This is recorded by the proposition below.

Proposition 3.12. *There is a bijection Φ between non-nulhomotopic free homotopy classes of curves on S and regular free homotopy classes of unobstructed curves on S*

$$\Phi : \hat{\pi}(S) \backslash \{C\} \xrightarrow{\sim} \hat{\pi}_{U,R}(S)$$

such that a free homotopy class $[\alpha]$ contains the unobstructed regular homotopy class to which it corresponds.

Proof. By Lemma 3.9, for any non-nulhomotopic free homotopy class $[\alpha]_{\hat{\pi}}$, there exists an unobstructed representative $\alpha_U \in [\alpha]_{\hat{\pi}}$. Furthermore, by Lemma 3.11, any other unobstructed representative $\alpha'_U \in [\alpha]_{\hat{\pi}}$ is freely regularly homotopic to α_U. Thus $[\alpha_U]_{\pi_{U,R}} = [\alpha'_U]_{\pi_{U,R}}$. So we define

$$\Phi([\alpha]_{\hat{\pi}}) := [\alpha_U]_{\pi_{U,R}}.$$

Now the map Φ is injective because if two loops β and γ are not freely homotopic then any choice of unobstructed representatives β_U and γ_U will, by transitivity, not be freely homotopic, and so cannot be freely regularly homotopic. The map Φ is surjective because any unobstructed curve is non-nulhomotopic and thus belongs to some non-nulhomotopic free homotopy class. $\square$

In order to define the grading, we must address additivity under the product $\alpha \cdot_p \beta$ through this correspondence. This is the content of the next section.

4. Grading the Goldman Lie algebra

We want to use the map Φ in Proposition 3.12 to define a grading on the Goldman Lie algebra. In Lemma 4.1 we show that the oriented loop product of two non-nulhomotopic unobstructed curves at a point of intersection is unobstructed when the point of intersection is not the corner of a bigon. In Proposition 4.4 we show that the winding number is additive over unobstructed curves. The section concludes with the grading on the Goldman Lie algebra in Theorem 4.8.

Lemma 4.1. *Suppose α and β are unobstructed curves intersecting at a point $p \in S$. If p is not the corner of a bigon then the oriented loop product $\alpha \cdot_p \beta$ is either unobstructed or nulhomotopic.*

Proof. Let U be a small neighborhood of the point p as in Figure 3. By contrapositive, assume $\alpha \cdot_p \beta$ is obstructed and not nulhomotopic, by Abouzaid's Lemma 2.8, the product $\alpha \cdot_p \beta$ contains a fish-tail. Observing that a self-intersection must occur at a fish-tail, and that $\alpha \cdot_p \beta$ does not self-intersect in

U by construction, we have that there must be some point of self-intersection $q \in S \backslash U$. In $S \backslash U$, connected components of $\alpha \cdot_p \beta$ belong originally to either α or β which by property of being unobstructed, cannot have fish-tails. Thus the fish-tail beginning at q must cross through U. Since the fish-tail at q bounds a disk, by undoing the resolution at p we can see that this implies that p and q are the corners of a bigon as in Figure 5 below.   □
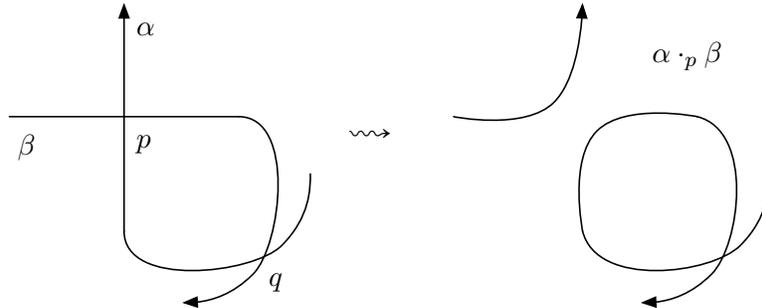

FIGURE 5. Recovering a bigon from a fish-tail that crosses U

Corollary 4.2. *If γ and τ do not represent the trivial nulhomotopic class and p is not the corner of a bigon then*

$$\Phi([\gamma \cdot_p \tau]) = \Phi([\gamma]) \cdot_p \Phi([\tau])$$

Proof. The map Φ contracts fish-tails. The equation holds when no new fish-tails are created by the loop product and Lem. 4.1 above gives this criteria.   □

Remark 4.3. Note that if p and q are the corners of a bigon between two curves α and β then

$$[\alpha \cdot_p \beta] = [\alpha \cdot_q \beta] \qquad \text{and} \qquad \epsilon_p(\alpha, \beta) = -\epsilon_q(\alpha, \beta).$$

So the two summands in $[[\alpha], [\beta]]$ corresponding to $\alpha \cdot_p \beta$ and $\alpha \cdot_q \beta$ cancel with each other. Thus for our purposes, when thinking about the Goldman Lie bracket, we do not need to concern ourselves with additivity of the winding number of curves over oriented resolution at the corners of a bigon.

We next show that ω is additive when the criteria above is satisfied.

Proposition 4.4. *Let α and β be unobstructed curves. Suppose that $p \in \alpha \cap \beta$ is not the corner of a bigon between α and β then the winding number is additive over the oriented loop product*

$$\omega(\alpha \cdot_p \beta) = \omega(\alpha) + \omega(\beta).$$

Proof. The curves α and β determine classes $[\alpha]_R$ and $[\beta]_R$ and the oriented loop product $(\alpha, \beta) \mapsto \alpha \cdot_p \beta$ is the product in the group $\pi_R(S, p)$. In Section 1.1, the winding number $\omega : \pi_R(S, p) \to \mathbb{Z}/\chi$ is defined to be a homomorphism on this group. If p is not the corner of a bigon then $\alpha \cdot_p \beta$ will be unobstructed and so it will agree up to regular homotopy with the unobstructed representative which determines the winding number. $\square$

A second geometric argument uses Reinhart's definition of winding number.

Proof. Arrange the curves as in Figure 3 and let U be a small neighborhood of p. We choose a vector field X which is constant in the horizontal direction. Let A_α, A_β and $A_{\alpha \cdot_p \beta}$ be the angle functions from Definition 2.4. We will prove that

$$(4.1) \qquad \frac{1}{2\pi} \int_{S^1} A_\alpha^*(d\theta) + \frac{1}{2\pi} \int_{S^1} A_\beta^*(d\theta) = \frac{1}{2\pi} \int_{S^1} A_{\alpha \cdot_p \beta}^*(d\theta) \qquad (\mathrm{mod}\, \chi).$$

Since the curves $\alpha \cup \beta$ and $\alpha \cdot_p \beta$ are identical outside of U, the integrals take the same values when restricted to the preimage of $S \backslash U$ in S^1.
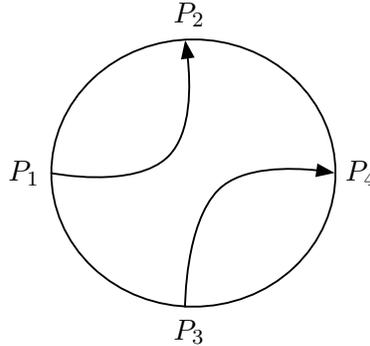

FIGURE 6. Labelled points of entry and exit of $\alpha \cdot_p \beta$ on $\partial \bar{U}$

We must examine what happens within U. We show that the component of the integrals inside the preimage of U in S^1 provides no contribution to the integrals on both sides of the equality. Without loss of generality, assume X is tangent to $\alpha \cdot_p \beta$ at P_1 and the rest of X on U is obtained by parallel translation. Then when traversing the curve $\alpha \cdot_p \beta$ from P_1 to P_2, there is a change of $\pi/2$ in angle. Similarly traversing $\alpha \cdot_p \beta$ from P_3 to P_4, the opposite change of angles between the tangent vectors and X occurs in a mirrored fashion, giving an overall contribution of zero to the total change in angles between the curve's tangent and X inside of U. Since α is perpendicular to X in U and β is parallel to X in U, the total change of angle between tangent vectors and X in U is zero for both α and β. Hence Equation (4.1) holds. $\square$

The proposition above implies the corollaries below.

Corollary 4.5. *Crossings can be resolved without changing winding number of regular homotopy classes. Crossings which do not bound bigons can be resolved without changing the winding number of free homotopy classes*

where $\gamma \sim_\omega \gamma'$ when $\omega(\gamma) = \omega(\gamma')$.

An important special case shows that fish-tails can be removed.

Corollary 4.6. *Fish-tail crossings can be resolved without changing the winding number*

where $\gamma \sim_\omega \gamma'$ when $\omega(\gamma) = \omega(\gamma')$.

Since a counterclockwise nulhomotopic curve has winding number $+1$. The two corollaries above show that the winding number of any curve can be computed in terms of its unobstructed representative by removing bigons and fish-tails.

We are ready to define the grading.

Definition 4.7 (Grading on $\mathfrak{g}(S)$)**.** Recall that $\mathfrak{g}(S) = \mathbb{Z}\langle\hat{\pi}(S)\rangle$. Let $[\alpha]$ be an element of $\hat{\pi}(S)\backslash\{C\}$. Let $\Phi([\alpha])$ be the regular homotopy class of the unobstructed representative of $[\alpha]$. The *winding number* $\omega_{\mathfrak{g}}([\alpha])$ of $[\alpha]$ is defined to be the winding number of $\Phi([\alpha])$. In the language of Proposition 3.12,

$$\omega_{\mathfrak{g}}([\alpha]_{\hat{\pi}}) := \omega(\Phi([\alpha]_{\hat{\pi}})).$$

Following discussion in Section 2.5, if C represents the free homotopy class of the contractible curve then we choose $\Phi(C)$ to be the infinity curve and set $\omega_{\mathfrak{g}}(\Phi(C)) = 0$.

The theorem below confirms that this definition gives a cyclic $\mathbb{Z}/\chi$-grading on the Goldman Lie algebra.

Theorem 4.8. *The map $\omega_{\mathfrak{g}}$ defines a grading of Goldman Lie algebra $\mathfrak{g}(S)$.*

Proof. Since the unobstructed representative of a free homotopy class of curves is well-defined up to regular free homotopy, so is its winding number. Furthermore, by Proposition 4.4, along with the fact that the sum expression (Def. 2.3) for $[[\alpha], [\beta]]$ simplifies to contain no contribution from bigon corners in $\alpha \cap \beta$, we have that $[[\alpha], [\beta]]$ only consists of summands that are oriented loop products at points where additivity holds and thus

$$\omega_{\mathfrak{g}}([[\alpha], [\beta]]) = \omega_{\mathfrak{g}}([\alpha]) + \omega_{\mathfrak{g}}([\beta]).$$

Hence the above map determines a grading of the Goldman Lie algebra
$\mathfrak{g}(S) = \mathbb{Z}\langle\hat{\pi}(S)\rangle$.   □

Recall that when $G = GL(n,\mathbb{R}), GL(n,\mathbb{C})$ or $GL(n,\mathbb{H})$, Goldman con-
structed a Poisson action of $\mathfrak{g}(S)$ on the representation variety, [Gol86, Thm.
5.4]. We ask whether there is a compatible grading on the representation
variety.

Question 4.9. *Is the action of the Goldman Lie algebra on representation
varieties graded?*

5. Regular Goldman Lie Algebra

Just as the Goldman Lie algebra $\mathfrak{g}(S)$ is defined in terms of free homotopy
classes of curves $\hat{\pi}(S)$, there is a Lie algebra $\mathfrak{g}_R(S)$ which is defined in terms
of free regular homotopy classes of immersed curves $\hat{\pi}_R(S)$.

Definition 5.1. The *regular Goldman Lie algebra* of a surface S is the free
abelian group on the set $\hat{\pi}_R(S)$ of free regular homotopy classes of immersed
curves

$$\mathfrak{g}_R(S) := \mathbb{Z}\langle\hat{\pi}_R(S)\rangle.$$

The Lie bracket is defined in precisely the same way as the Goldman Lie
bracket in Def. 2.3. Given $[\alpha]_R, [\beta]_R \in \hat{\pi}_R(S)$, the bracket is

$$(5.1) \qquad\qquad [[\alpha]_R, [\beta]_R] := \sum_{p \in \alpha \cap \beta} \epsilon_p(\alpha, \beta)[\alpha \cdot_p \beta]_R.$$

The bracket above determines a Lie algebra structure. In fact, the same
argument Goldman used to show that $\mathfrak{g}(S)$ is a Lie algebra also shows that
$\mathfrak{g}_R(S)$ is a Lie algebra [Gol86, Thm. 5.3].

All of the acrobatics related to Φ and unobstructed regular representa-
tives, used to introduce the winding number grading on $\mathfrak{g}(S)$, are unnec-
essary for the regular Goldman Lie algebra. This is because the winding
number homomorphism $\omega : \pi_R(S, p) \to \mathbb{Z}/\chi$ is defined on $\pi_R(S, p)$, so it
defines a map on conjugacy classes $\hat{\pi}_R(S)$ by Lem. 3.5. The proposition
below summarizes this discussion.

Proposition 5.2. *The regular Goldman Lie algebra $\mathfrak{g}_R(S)$ is graded by
winding number.*

In Def. 3.2 a $\mathbb{Z}$-action on $\hat{\pi}_R(S)$ was introduced. Here we note that it is
compatible with the bracket above.

Lemma 5.3. *The regular Goldman Lie bracket is $\mathbb{Z}$-equivariant*

$$t[[\alpha]_R, [\beta]_R] = [t[\alpha]_R, [\beta]_R] = [[\alpha]_R, t[\beta]_R] \qquad \text{for} \qquad t \in \mathbb{Z}.$$

Proof. A generator $t \in \mathbb{Z}$ acts on any term $[\alpha \cdot_p \beta]_R$ in Eqn. (5.1) by adding
a fish-tail somewhere along the curve. This fish-tail can be smoothly and
regularly isotoped to be small and then moved along the curve to either α
or β. This commutes with the resolution in Fig. 3.   □

The results of Section 3, relating the set of free homotopy classes of curves $\hat{\pi}(S)$ to the set of regular homotopy classes of curves $\hat{\pi}_R(S)$, can be extended to describe the relationship between the Goldman Lie algebra $\mathfrak{g}(S)$ and the regular Goldman Lie algebra $\mathfrak{g}_R(S)$.

Definition 5.4. Suppose $\mathfrak{g}$ is a Lie algebra over a commutative ring k and R is a commutative k-algebra then $\mathfrak{g} \otimes_k R$ is a Lie algebra with product

$$(5.2) \qquad [x \otimes r, y \otimes s] := [x, y] \otimes rs.$$

The *loop algebra* is

$$\mathscr{L}\mathfrak{g} := \mathfrak{g} \otimes_k k[t, t^{-1}].$$

The loop algebra also has a canonical $\mathbb{Z}$-action generated by t; $t \cdot (x \otimes t^n) = x \otimes t^{n+1}$. This suggests a relationship between the loop algebra of the Goldman Lie algebra $\mathscr{L}\mathfrak{g}(S)$ and the regular Goldman Lie algebra $\mathfrak{g}_R(S)$. The next theorem addresses this point.

Theorem 5.5. *There is a $\mathbb{Z}$-equivariant Lie algebra isomorphism*

$$\alpha : \mathscr{L}\mathfrak{g}(S) \xrightarrow{\sim} \mathfrak{g}_R(S)$$

from the loop algebra of the Goldman Lie algebra to the regular Goldman Lie algebra.

Proof. There is a map $\alpha : \mathscr{L}\mathfrak{g}(S) \to \mathfrak{g}_R(S)$ and an inverse map $\beta : \mathfrak{g}_R(S) \to \mathscr{L}\mathfrak{g}(S)$ which are determined by

$$\alpha([\gamma] \otimes t^n) := t^n \Phi([\gamma]) \quad \text{and} \quad \beta([\tau]_R) := s_*([\tau]_R) \otimes t^{d([\tau]_R)}$$

where $d([\tau]_R)$, by Prop. 3.6, is the unique function $d : \hat{\pi}_R(S) \to \mathbb{Z}$ which satisfies the equation

$$(5.3) \qquad [\tau]_R = t^{d([\tau]_R)} \Phi(s_*([\tau]_R)).$$

The value of $d([\tau]_R)$ is the number of fish-tails needed to make the unobstructed representative $\Phi(s_*[\tau]_R)$ regularly homotopic to $[\tau]_R$. In particular, $d(\Phi([\gamma])) = 0$ for all $[\gamma] \in \hat{\pi}(S)$.

The map d is $\mathbb{Z}$-equivariant because $t^n[\tau]_R = t^{d([\tau]_R)+n} \Phi(s_*([\tau]_R)) = t^{d([\tau]_R)+n} \Phi(s_*(t^n[\tau]_R))$. Which gives $d(t^n[\tau]_R) = d([\tau]_R) + n$ for $n \in \mathbb{Z}$ from Eqn. (5.3).

The map β is $\mathbb{Z}$-equivariant because $\beta(t^n[\tau]_R) = s_*(t^n[\tau]_R) \otimes t^{d(t^n[\tau]_R)} = s_*([\tau]_R) \otimes t^{d(t^n[\tau]_R)} = s_*([\tau]_R) \otimes t^n t^{d([\tau]_R)} = t^n \beta([\tau]_R)$.

Also the map α is $\mathbb{Z}$-equivariant. This is because $\alpha(t^n \cdot ([\gamma] \otimes t^m)) = \alpha([\gamma] \otimes t^{n+m}) = t^{n+m} \Phi([\gamma]) = t^n (t^m \Phi([\gamma])) = t^n \alpha([\gamma] \otimes t^m)$.

Next we show $\beta\alpha = 1_{\mathscr{L}\mathfrak{g}(S)}$.

$$
\begin{aligned}
\beta\alpha([\gamma] \otimes t^n) &= \beta(t^n \Phi([\gamma])) && \text{(Def. of } \alpha) \\
&= t^n \beta(\Phi([\gamma])) && \text{(Equivariance)} \\
&= t^n \cdot (s_*(\Phi([\gamma])) \otimes t^{d(\Phi([\gamma]))}) && \text{(Def. of } \beta) \\
&= s_*(\Phi([\gamma])) \otimes t^{n+d(\Phi([\gamma]))} && (\mathbb{Z}\text{-action}) \\
&= [\gamma] \otimes t^{n+d(\Phi([\gamma]))} && (s_*\Phi = 1) \\
&= [\gamma] \otimes t^n && (d(\Phi([\gamma])) = 0)
\end{aligned}
$$

where $d(\Phi([\gamma])) = 0$ is discussed above.

Now we show that $\alpha\beta = 1_{\mathfrak{g}_R(S)}$.

$$
\begin{aligned}
\alpha\beta([\tau]_R) &= \alpha(s_*([\tau]_R) \otimes t^{d([\tau]_R)}) && \text{(Def. of } \beta) \\
&= t^{d([\tau]_R)} \Phi(s_*([\tau]_R)) && \text{(Def. of } \alpha) \\
&= [\tau]_R && (5.3)
\end{aligned}
$$

Next we show that the map α is a Lie algebra homomorphisms. If $[\gamma]$ is contractible then $[[\gamma], [\tau]] = 0$ for any $[\tau]$, but $\alpha([\gamma] \otimes t^n) = t^n \Phi([\gamma])$ so that $[\alpha([\gamma]), \alpha([\tau])] = 0$ too. So we assume that $[\gamma]$ and $[\tau]$ are not nulhomotopic. In the equations below, all of the sums are taken over $p \in \gamma \cap \tau$ and $\epsilon_p = \epsilon_p(\gamma, \tau)$; so, without loss of generality, we assume that no p is the corner of a bigon, see Rmk. 4.3 and Lem. 5.6 below.

$$
\begin{aligned}
\alpha([[\gamma] \otimes t^n, [\tau] \otimes t^m]) &= \alpha(\Sigma_p \epsilon_p [\gamma \cdot_p \tau]) \otimes t^{n+m} && (2.1) \\
&= t^{n+m} \Phi(\Sigma_p \epsilon_p [\gamma \cdot_p \tau]) && \text{(Def. of } \alpha) \\
&= t^{n+m} \Sigma_p \epsilon_p \Phi[\gamma \cdot_p \tau] && \text{(Linearity)} \\
&= t^{n+m} \Sigma_p \epsilon_p [\Phi([\gamma]) \cdot_p \Phi([\tau])] && \text{(Cor. 4.2)} \\
&= t^{n+m} [\Phi([\gamma]), \Phi([\tau])] && (2.1) \\
&= [t^n \Phi([\gamma]), t^m \Phi([\tau])] && \text{(Lem. 5.3)} \\
&= [\alpha([\gamma] \otimes t^n), \alpha([\tau] \otimes t^m)] && \text{(Def. of } \alpha)
\end{aligned}
$$

The inverse β of a bijective Lie algebra homomorphism α is necessarily a Lie algebra homomorphism because

$$
\beta[x, y] = \beta[\alpha\beta(x), \alpha\beta(y)] = \beta\alpha[\beta(x), \beta(y)] = [\beta(x), \beta(y)].
$$

For completeness, we include a direct proof below.

$$\beta[[\gamma]_R, [\tau]_R] = \beta(\Sigma_p \epsilon_p [\gamma \cdot_p \tau]_R) \tag{5.1}$$

$$= \Sigma_p \epsilon_p \beta[\gamma \cdot_p \tau]_R \tag{Linearity}$$

$$= \Sigma_p \epsilon_p s_*([\gamma \cdot_p \tau]_R) \otimes t^{d([\gamma \cdot_p \tau]_R)} \tag{Def. of β}$$

$$= \Sigma_p \epsilon_p s_*([\gamma]_R) \cdot_p s_*([\tau]_R) \otimes t^{d([\gamma]_R) + d([\tau]_R)} \tag{Prop. 3.8 \& Lem. 5.6}$$

$$= [s_*([\gamma]_R), s_*([\tau]_R)] \otimes t^{d([\gamma]_R) + d([\tau]_R)} \tag{2.1}$$

$$= [s_*([\gamma]_R) \otimes t^{d([\gamma]_R)}, s_*([\tau]_R) \otimes t^{d([\tau]_R)}] \tag{5.2}$$

$$= [\beta[\gamma]_R, \beta[\tau]_R] \tag{2.1}$$

$$\square$$

The lemma below is included to justify a step in the proof that β is a homomorphism.

Lemma 5.6. *Suppose $[\gamma]_R, [\eta]_R \in \hat{\pi}_R(S)$, $\gamma \in [\gamma]_R$, $\eta \in [\eta]_R$ and $p \in \gamma \cap \eta$. Then*

(1) *If $\gamma \cdot_p \eta$ is not homotopic to the contractible curve and p is not the corner of a bigon then d is a homomorphism, $d([\gamma \cdot_p \eta]_R) = d([\gamma]_R) + d([\eta]_R)$.*

(2) *If γ, η or, for any $p \in \gamma \cap \eta$, $\gamma \cdot_p \eta$ is homotopic to the contractible curve then $[[\gamma]_R, [\eta]_R] = 0$.*

Proof. *1.* By equivariance, we may assume, without loss of generality, that γ is unobstructed and so $d([\gamma]_R) = 0$. Set $n = -d([\eta]_R)$ so that $d(t^n [\eta]_R) = 0$ and $t^n \eta$ is unobstructed. Now Lem. 4.1 shows $\gamma \cdot t^n \eta$ is unobstructed or $\gamma \cdot_p t^n \eta$ is nulhomotopic. So by assumption, $\gamma \cdot t^n \eta$ must be unobstructed and therefore $d([\gamma \cdot t^n \eta]_R) = 0$. Since d is $\mathbb{Z}$-equivariant, $d(t^n [\eta]_R) = d([\gamma \cdot t^n \eta]_R)$ implies $d([\gamma \cdot_p \eta]_R) = d([\gamma]_R) + d([\eta]_R)$.

2. If γ or η are contractible then we can make them small and distant, this shows that the geometric intersection number is zero. If $\gamma \cdot_p \eta$ is contractible then shrinking γ and η along concentric circles $S_r = \{z : |z| = r\}$ in the disk $D^2 = \{z \in \mathbb{C} : |z| \leqslant 1\}$ defining the nulhomotopy shows that their geometric intersection number is zero. For any other basepoint q, $\pi_R(S, q) = y^{-1} \cdot \pi_R(S, p) \cdot y$, for a path $y : q \to p$, and conjugation takes identity to identity. $\square$

Remark 5.7. If the function d was always a homomorphism then it would induce a homomorphism $\pi_R(S, p) \to \mathbb{Z}$ which, by virtue of factoring through the abelianization $\pi_R(S, p)^{ab}$, must factor through a homomorphism $\mathbb{Z}/\chi \to \mathbb{Z}$.

Here is a more concrete example. Suppose that the genus of S is 2, so $\chi = -2$. Set $x = [a_1, b_1]$ and $y = [a_2, b_2]$ in Eqn. (1.2). So that $xy = r$. Now $d(x) = 0$ and $d(y) = 0$ because they are unobstructed curves. But

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$0 = d(xy) = d(r) = d(f^2) = 2$ because of the relation $r = f^2$ from Eqn. (1.3).

Question 5.8. *Theorem* 5.5 *suggests that* $\mathfrak{g}_R(S)$ *is an analogue of the loop algebra* $\mathscr{L}\mathfrak{g}(S)$ *of the Goldman Lie algebra. Is there a central extension of* $\hat{\mathfrak{g}}_R(S)$ *corresponding to an affinization of the Goldman Lie algebra?*

6. Induced grading on HOMFLY-PT algebras

In this section we use the grading on the regular Goldman Lie algebra to construct a grading on the HOMFLY-PT skein algebra. We first recall some definitions.

Definition 6.1 (Conway triple). Let L_+, L_- and L_0 be link diagrams on a surface, the triple (L_+, L_-, L_0) is called a *Conway triple* if L_+, L_-, and L_0 are identical outside of some small neighborhood U, and within U appear as in Figure 7 below.
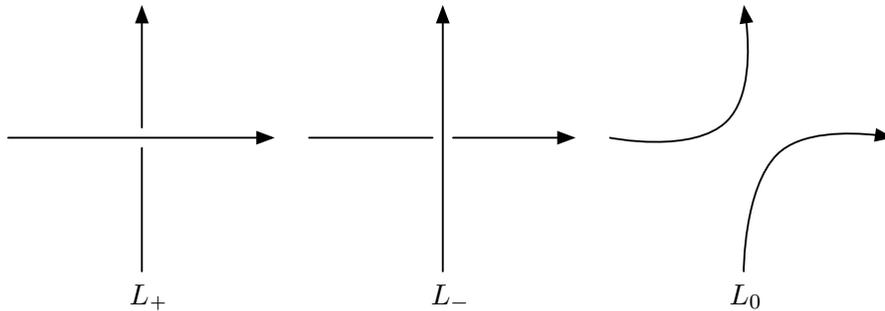

L_+ L_- L_0

FIGURE 7. Diagram showing the difference between L_+, L_-, L_0 within U

A skein algebra is usually defined as a quotient of either framed isotopy classes of framed oriented links in a thickening of the surface $S \times [0,1]$ or oriented link diagrams considered up to Reidemeister moves II and III by a relation among the Conway triples introduced above. The equivalence between these two approaches is discussed by L. Kauffman in [Kau90].

We'll stick with link diagrams.

Definition 6.2 (HOMFLY-PT skein algebra). Let $\mathcal{L}(S)$ denote the set of oriented link diagrams on a surface S. Let $R = \mathbb{Z}[x^{\pm 1}, q^{\pm 1}, v^{\pm 1}]$. The *HOMFLY-PT skein algebra* $H_q(S)$ is the quotient of the R-span of $\mathcal{L}(S)$ by the three types of relations below.

(1) When (L_+, L_-, L_0) is a Conway triple,

$$x \quad \times \quad - \quad x^{-1} \quad \times \quad = (q - q^{-1}) \quad)($$

(2) When L_2 is obtained from L_1 via a Reidemeister I untwisting move

$$\cap \quad = \quad v \quad \mathcal{Q} \qquad \text{and} \qquad \cap \quad = \quad v^{-1} \quad \mathcal{Q}$$

(3) If L_2 is obtained from L_1 via a Reidemeister II or III move then they are required to be equal: $L_1 = L_2$.

For more details, see [Mor02].

Now the natural projection $\rho : S \times [0,1] \to S$ takes framed oriented knots in $S \times [0,1]$ to immersed curves in S. A regular framed isotopy between two knots is mapped by ρ to a regular homotopy between their projections. By Kauffman's work, projection defines a map $\rho_* : \mathcal{K}(S) \to \hat{\pi}_R(S)$ from oriented knot diagrams $\mathcal{K}(S)$ on S to $\hat{\pi}_R(S)$ which preserves Reidemeister moves II and III. An oriented crossing projects to a self-intersection

$$\times \quad \rightsquigarrow \quad \times \quad .$$

This map extends to a map on links

$$\rho_* : \mathcal{L}(S) \to \mathbb{Z}\langle \hat{\pi}_R(S) \rangle$$

by setting $\rho_*(L) := \sum_i \rho_*(L_i)$ when $L = \sqcup_i L_i$ for $L_i \in \mathcal{K}(S)$.

By construction $H_q(S)$ is $\mathbb{Z}$-spanned by elements of the form

$$B(S) := \{x^n v^m q^l L : n, m, l \in \mathbb{Z} \text{ and } L \in \mathcal{L}(S)\}.$$

The winding number is defined in terms of this spanning set

$$(6.1) \qquad \omega_H(x^n v^m q^l L) := m + \omega_R(\rho_*(L))$$

where $\omega_R : \mathbb{Z}\langle \hat{\pi}_R(S) \rangle \to \mathbb{Z}/\chi$ is the linear extension of Eqn. (1.5). The proposition below shows that this definition respects the relations in Def. 6.2.

Proposition 6.3. *The map $\omega_H : B(S) \to \mathbb{Z}/\chi$ from Eqn. (6.1) extends to the skein algebra $H_q(S)$ along the map $B(S) \to H_q(S)$.*

Proof. In Def. 6.2, $H_q(S)$ is defined to be a quotient of $\mathbb{Z}\langle B(S) \rangle$ by three types of relations. In order to show that ω_H extends to the quotient, we check that it respects each of these three types of relations.

In the first case the relation involves a Conway triple (L_+, L_-, L_0). Since L_+ and L_- only differ at a crossing, their projections $\rho_*(L_+) = \rho_*(L_-)$ are identical and this implies

$$\omega_H(xL_+) = \omega_H(x^{-1}L_-).$$

The geometric argument for Prop. 4.4 shows that $\omega_R(\rho_*(L_+)) = \omega_R(\rho_*(L_0))$, so

$$\omega_H(qL_0) = \omega_H(q^{-1}L_0) = \omega_H(xL_+).$$

Thus all of terms in the skein relation are in the same graded degree.

In the second case, suppose that L_2 is obtained from L_1 via a Reidemeister I twist, so that $L_1 = vL_2$ in $H_q(S)$. By Corollary 4.6, it follows the winding number of their projections differ by 1,

$$\omega_R(\rho_*(L_1)) - 1 = \omega_R(\rho_*(L_2))$$

So $\omega_H(L_1) = \omega_H(vL_2)$.

Lastly, suppose that L_2 is obtained from L_1 by a Reidemeister II or III move. Then by Kauffman's work, L_1 and L_2 are regularly isotopic and so their projections are regularly freely homotopic. Thus in $\mathfrak{g}_R(S)$, $\rho_*(L_1) = \rho_*(L_2)$. It follows that $\omega_H(L_1) = \omega_H(L_2)$. $\square$

The theorem below uses the winding map determined by the proposition above to define a grading on the skein algebra.

Theorem 6.4. *The HOMFLY-PT skein algebra $H_q(S)$ is graded as a ring by the winding number. In more detail, there is a decomposition*

$$H_q(S) = \bigoplus_{a \in \mathbb{Z}/\chi} H_q(S)_a$$

which respects the product: if $l \in H_q(S)_a$ and $l' \in H_q(S)_b$ then $l \cdot l' \in H_q(S)_{a+b}$ via the map

$$H_q(S)_a \otimes_{\mathbb{Z}[x^{\pm 1}, q^{\pm 1}]} H_q(S)_b \dashrightarrow H_q(S)_{a+b}$$

Proof. By Prop. 6.3, the relations defining the HOMFLY-PT skein algebra respect the winding number. So for each $a \in \mathbb{Z}/\chi$, the ath homogeneous component $H_q(S)_a$ of $H_q(S)$ is given by

$$H_q(S)_a := \mathbb{Z}\langle x \in B(S) : \omega_H(x) = a \rangle.$$

We conclude by checking that the grading is additive with respect to multiplication. If $l \in H_q(S)_a$ and $l' \in H_q(S)_b$ then l is a $\mathbb{Z}$-linear combination of elements in $B(S)$ with winding number a, same for l', so it suffices to check

additivity for monomials $l = x^a q^b v^c L$ and $l' = x^\ell q^m v^n L'$ in $B(S)$. Then

$$
\begin{aligned}
\omega_H(l \cdot l') &= \omega_H(x^{a+\ell} q^{b+m} v^{c+n} L \cdot L') \\
&= c + n + \omega_R(L \cdot L') \\
&= c + n + \omega_R(L) + \omega_R(L') \\
&= \omega_H(l) + \omega_H(l')
\end{aligned}
$$

where the equality $\omega_R(L \cdot L') = \omega_R(L) + \omega_R(L')$ follows from the observation that the individual knot components in the stacking product are the same as those of the disjoint union of L and L', $L \cdot L' = (\sqcup_i L_i) \sqcup (\sqcup_j L_j)$.

Remark 6.5. Since a grading of the HOMFLY-PT skein algebra attached to the framing must respect the oriented loop product appearing in the relation 1 of Def. 6.2, it isn't a $\mathbb{Z}$-grading for the same reasons as those mentioned in Rmk. 5.7.

$\square$

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