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On postcritically finite unicritical polynomials

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Abstract. In this article, we first study arithmetical properties of postcritically finite unicritical polynomials $f_a : z \mapsto az^D + 1$ with $D \geq 2$. In particular, we answer a question of Milnor, showing that there exist non-Galois conjugate parameters $a_1 \in \mathbb{C}$ and $a_2 \in \mathbb{C}$ such that f_{a_1} and f_{a_2} have critical orbits periodic with the same period. We also answer a question of Baker and DeMarco, proving that the set of parameters $a \in \mathbb{C}$ such that 0 and 1 are simultaneously (pre)periodic for $q_a : w \mapsto w^2 + a$ is equal to $\{0, -1, -2\}$.

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Introduction

We study polynomials $f : \mathbb{C} \to \mathbb{C}$ of degree $D \geq 2$ from a dynamical point of view, i.e., we consider sequences $\{z_n\}_{n\geq 0}$ defined by iteration:

$$z_0 \in \mathbb{C} \quad \text{and} \quad z_n := f(z_{n-1}) = f^{\circ n}(z_0).$$

This sequence is called the *orbit* of z_0 for f.

The point z_0 is *periodic* if there is an integer $n \geq 1$ such that $f^{\circ n}(z_0) = z_0$. If p is the smallest integer with this property, we call it the *period* of z_0. The point z_0 is *(pre)periodic* if there exists a (smallest) integer $k \geq 0$ such that $f^{\circ k}(z_0)$ is periodic of period p. We say that k is the preperiod and that p is the *period*.

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Consider the polynomials f_a defined by

$$f_a(z) = az^D + 1, \quad a \in \mathbb{C}.$$

For $a \neq 0$, those are polynomials of degree D with a unique critical point at 0. We are interested in the sets $\mathcal{A}_D \subset \mathcal{M}_D$ defined by

$$\mathcal{A}_D := \{a \in \mathbb{C} \smallsetminus \{0\} \ ; \ 0 \text{ is (pre)periodic for } f_a\}$$

and

$$\mathcal{M}_D := \{a \in \mathbb{C} \ ; \ \text{the orbit of } 0 \text{ for } f_a \text{ is bounded}\}.$$

If $a \in \mathcal{A}_D$, we say that f_a is *postcritically finite*. The set $\mathcal{A}_D$ is the set of *Misiurewicz parameters* and the set $\mathcal{M}_D$ is the *Multibrot set* (a generalization of the Mandelbrot set in degree D).

We shall first prove a Kronecker type result, where the set of roots of unity is replaced by $\mathcal{A}_D$, and the unit disk is replaced by $\mathcal{M}_D$.

Proposition 1. *If a is an algebraic integer such that a and all its Galois conjugates are contained in $\mathcal{M}_D$, then $a \in \mathcal{A}_D \cup \{0\}$.*

Conversely, according to Milnor [M2, Theorem 3.2], if $a \in \mathcal{A}_D$, then

- a is an algebraic integer,
- its Galois conjugates are in $\mathcal{A}_D$,
- the product of a and its Galois conjugates divides D, and
- if 0 is periodic for f_a with period $p \geq 2$, then a is an algebraic unit.

We prove that for the last statement, one can get rid of the assumption that 0 is periodic.

Proposition 2. *If $a \in \mathcal{A}_D$ and 0 is preperiodic for f_a with preperiod $k \geq 2$ and period $p \geq 2$, then a is an algebraic unit.*

In §2 we study the *Gleason polynomials* $\{F_p \in \mathbb{Z}[a]\}_{p \geq 1}$ defined by

$$F_p(a) := f_a^{\circ p}(0).$$

In §3, we study the *Misiurewicz polynomials* $\{F_{k,p} \in \mathbb{Z}[a]\}_{k \geq 2, p \geq 1}$ defined by

$$F_{k,p} := \frac{F_{k+p-1}^D - F_{k-1}^D}{F_{k+p-1} - F_{k-1}} = \sum_{i+j=D-1} F_{k+p-1}^i F_{k-1}^j.$$

The parameters in $\mathcal{A}_D$ are the roots of the Gleason and Misiurewicz polynomials. The proof of the preceding proposition is based on the following two lemmas. The first lemma is due to Gleason. Our proof of the second lemma corrects the one given in the appendix of [E].

Lemma 3 (Gleason)**.** *For $p \geq 2$, the polynomial F_p has simple roots.*

Lemma 4. *If $K > k \geq 1$ and $\omega^D = 1$ with $\omega \neq 1$, then $F_K - \omega F_k$ has simple roots.*

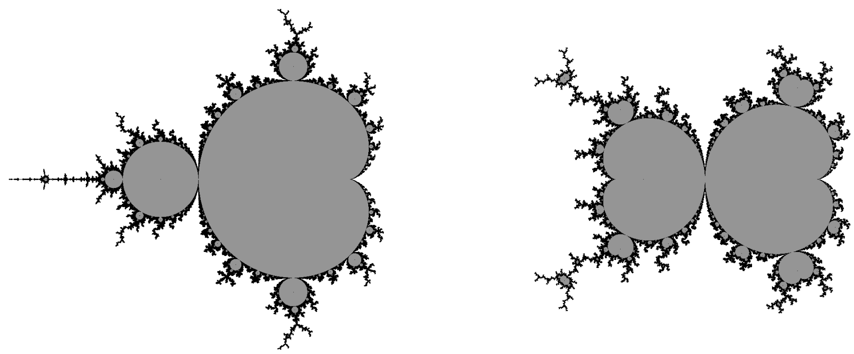

FIGURE 1. Left: the set $\mathcal{M}_2$. Right: the set $\mathcal{M}_3$

When q divides p, the polynomial F_q divides F_p. Since the roots are simple,

$$F_p = \prod_{q|p} G_q \quad \text{with} \quad G_p := \prod_{q|p} F_q^{\mu(p/q)} \in \mathbb{Z}[a],$$

where μ is the Möbius function defined by $\mu(n) = (-1)^m$ if n is the product of m distinct primes with $m \geq 0$ and $\mu(n) = 0$ otherwise. It is tempting to conjecture that the polynomials G_p are irreducible over $\mathbb{Q}$ (see [M2, Remark 3.5]). We show that this is not true in general.

Proposition 5. *The polynomial G_3 is reducible over $\mathbb{Q}$ if and only if $D \equiv 1 \mod 6$. In this case, G_3 has exactly two irreducible factors, one of which is $1 + a + a^2$.*

Note that for $D = 2$, the linear map $z \mapsto w = az$ conjugates the quadratic polynomial f_a to the monic centered polynomial $q_a : w \mapsto w^2 + a$. We conclude the article with a proof of the following result, which answers a question of Baker and DeMarco [BD].

Proposition 6. *The set of parameters $a \in \mathbb{C}$ such that 0 and 1 are simultaneously (pre)periodic for q_a is $\{0, -1, -2\}$.*

Acknowledgments. The results presented here were inspired by fruitful discussions with Adam Epstein and Sarah Koch.

1. A Kronecker type result

We first prove Proposition 1. Our treatment is largely inspired by Kronecker's proof that if an algebraic integer and all its Galois conjugates are contained in the closed unit disk, then this algebraic integer is either 0 or a root of unity.

Lemma 7. *Assume $a \in \mathbb{C}$ and $\{z_n\}$ is a bounded orbit for f_a. Then*
- *either $|a| \le 2$ and $|az_n^{D-1}| \le 2$ for all $n \ge 0$,*
- *or $|a| > 2$ and $|z_n| < 1$ for all $n \ge 0$.*

Proof. Set $w_n := az_n^{D-1}$. First, observe that if $|z_n| \ge 1$ and $|w_n| > 2$, then

$$|z_{n+1}| = \big|f_a(z_n)\big| \ge |az_n^D| - 1 \ge |az_n^D| - |z_n| = |z_n|\big(|w_n| - 1\big).$$

Now assume $|a| > 2$ and set $\kappa := |a| - 1 > 1$. If $|z_{n_0}| \ge 1$ for some n_0, it follows by induction that for $n \ge n_0$, $|z_n| \ge \kappa^{n-n_0} \ge 1$ and $|w_n| \ge \kappa + 1 \ge 2$. Indeed, for $n = n_0$, we have that $|z_{n_0}| \ge 1$ and $|w_{n_0}| \ge |a| = \kappa + 1 \ge 2$. And if the property holds for some $n \ge n_0$, then

$$|z_{n+1}| \ge |z_n|\big(|w_n| - 1\big) \ge \kappa |z_n| \ge \kappa^n \ge 1$$

and

$$|w_{n+1}| = |az_{n+1}^{D-1}| \ge |a| = \kappa + 1 \ge 2.$$

So, the orbit $\{z_n\}$ is not bounded, which contradicts our assumptions.

Finally, assume $|a| \le 2$ and $|w_{n_0}| > 2$ with $n_0 \ge 1$. Set $\kappa := |w_{n_0}| - 1 > 1$. It follows by induction that for $n \ge n_0$,

$$|z_n| > \kappa^{n-n_0} \ge 1 \quad \text{and} \quad |w_n| \ge \kappa + 1 > 2.$$

Indeed, for $n = n_0$, we have that $|w_{n_0}| = \kappa + 1 > 2$ and $|z_{n_0}^{D-1}| = |w_{n_0}/a| > 1$, so that $|z_{n_0}| > 1$. Now, if the property holds for some $n \ge n_0$, then

$$|z_{n+1}| \ge |z_n|\big(|w_n| - 1\big) \ge \kappa |z_n| \ge \kappa^n \ge 1$$

and

$$|w_{n+1}| = |az_{n+1}^{D-1}| \ge \kappa^{D-1}|w_n| > |w_n| \ge \kappa + 1 > 2.$$

So, the orbit $\{z_n\}$ is not bounded, which contradicts our assumptions. $\square$

Corollary 8. *If $a \in \mathcal{M}_D$, then $|a| \le 2$.*

Proof. By definition, if $a \in \mathcal{M}_D$, the orbit of 0 for f_a is bounded. Since $f_a(0) = 1$, we necessarily have $|a| \le 2$. $\square$

So, assume $a_1 \in \mathcal{M}_D$ is an algebraic integer whose Galois conjugates $a_2, \dots, a_d$ are in $\mathcal{M}_D$. For $j \in [\![1, d]\!]$ and $n \ge 0$, set $z_{j,n} := f_{a_j}^{\circ n}(0)$ and $w_{j,n} := a_j z_{j,n}^{D-1}$. In order to prove that f_{a_1} is postcritically finite, we must show that the sequence $\{z_{1,n}\}_{n \ge 0}$ is finite. Equivalently, we shall prove that the sequence $\{w_{1,n}\}_{n \ge 0}$ is finite.

The points $w_{j,n}$ are algebraic integers. Let $Q_n \in \mathbb{Z}[w]$ be their minimal polynomials. The Galois conjugates of $w_{1,n}$ are $w_{2,n}, \dots w_{d,n}$. According to the previous lemma, those Galois conjugates all have modulus at most 2. It follows that the coefficients of the polynomials Q_n are uniformly bounded, independently on $n \ge 1$. There is a finite number of such polynomials. So, the set $\{w_{j,n}\}_{j \in [[1,d]], n \ge 0}$ is finite.

2. Gleason polynomials

As in the introduction, for $p \geq 1$, define $F_p \in \mathbb{Z}[a]$ recursively by

$$F_1 := 1 \quad \text{and} \quad F_{p+1} := aF_p^D + 1,$$

so that $F_p(a) = f_a^{\circ p}(0)$. Those polynomials are called *Gleason polynomials.*

Example. We have that

$$F_2 = a + 1 \quad \text{and} \quad F_3 = a(a+1)^D + 1.$$

We now prove Lemma 3, i.e., that the roots of the Gleason polynomials are simple.

Proof of Lemma 3. For $p \geq 1$, we have

$$F_{p+1} = aF_p^D + 1 \quad \text{and} \quad F'_{p+1} = F_p^D + DF_p^{D-1}F'_p \equiv F_p^D \mod D.$$

Since F_p is monic,

$$\text{discriminant}(F_{p+1}) \equiv \text{resultant}(aF_p^D + 1, F_p^D) \mod D \equiv 1 \mod D.$$

In particular, the discriminant does not vanish and F_{p+1} has simple roots. □

For $p \geq 1$, let $\mathcal{A}_D^p$ be the set of parameters $a \in \mathcal{A}_D$ such that 0 is periodic for f_a with period p. Moreover, let G_p be the monic polynomial which has simple roots exactly at the points $a \in \mathcal{A}_D^p$.

Lemma 9. *For $p \geq 1$, the constant coefficient of G_p is 1 and*

$$F_p = \prod_{q|p} G_q.$$

Proof. For $p = 1$, we have that $G_1 = F_1 = 1$. For $p \geq 2$, the roots of F_p are exactly the parameters $a \in \mathcal{A}_D^q$ with q dividing p. Since F_p has simple roots and all polynomials are monic, we have the required factorization.

For $p \geq 1$, the constant coefficient of F_p is 1. In addition, $G_1 = 1$. It follows by induction on $p \geq 1$ that the constant coefficient of G_p is 1. □

Milnor [M2] asked whether the polynomials G_p are irreducible over $\mathbb{Q}$. Proposition 5 asserts that this is not true in general. We shall now prove this proposition. Note that

$$G_3 = a(a+1)^D + 1.$$

We must prove that G_3 is reducible over $\mathbb{Q}$ if and only if $D \equiv 1 \mod 6$ and that in this case, G_3 has exactly two irreducible factors, one of which is $1 + a + a^2$. This is in fact a result of Selmer [S] that we reproduce here.

Proof of Proposition 5. On the one hand, if $D \equiv 1 \mod 6$, then a^2+a+1 divides G_3. Indeed, let $\omega \neq 1$ be a cube-root of unity. Then $\omega + 1$ is a 6-th root of unity and

$$G_3(\omega) = \omega(\omega+1)^D + 1 = \omega(\omega+1) + 1 = \omega^2 + \omega + 1 = 0.$$

On the other hand, observe that

$$G_3(a) = P(a+1) \quad \text{with} \quad P(x) := x^D(x-1)+1 = x^{D+1} - x^D + 1.$$

If G_3 is reducible over $\mathbb{Q}$, then P is reducible over $\mathbb{Q}$ and we may write $P = P_1P_2$ with $P_1 \in \mathbb{Z}[x]$ and $P_2 \in \mathbb{Z}[x]$ monic polynomials of respective degree $D_1 \geq 1$ and $D_2 \geq 1$. The product of the constant coefficients of P_1 and P_2 is equal to 1, so that both are equal to $\varepsilon \in \{-1, +1\}$. Set

$$R(x) := \varepsilon x^{D_2} P_1(x) P_2(1/x) \quad \text{and} \quad S(x) := \varepsilon x^{D_1} P_1(1/x) P_2(x).$$

Note that $R \in \mathbb{Z}[x]$ and $S \in \mathbb{Z}[x]$ are monic polynomials with constant coefficient equal to 1. In addition, $R(x) = x^{D+1}S(1/x)$. It follows that if $R(x) = \displaystyle\sum_{j=0}^{D+1} c_j x^j$, then $S(x) = \displaystyle\sum_{j=0}^{D+1} c_j x^{D+1-j}$. Moreover,

$$RS = PQ \quad \text{with} \quad Q(x) = x^{D+1}P(1/x) = x^{D+1} - x + 1.$$

Identifying the coefficients of x^{D+1} on both sides yields $\displaystyle\sum_{j=0}^{D+1} c_j^2 = 3$. Thus, there are exactly three coefficients c_j which are non zero, and they are equal to ± 1. We already know that $c_{D+1} = c_0 = 1$. So, there exist $j \in [\![2, D]\!]$ and $c_j \in \{-1, +1\}$ such that

$$R(x) = x^{D+1} + c_j x^j + 1 \quad \text{and} \quad S(x) = 1 + c_j x^{D+1-j} + x^{D+1}.$$

Comparing RS to PQ again, we see that we necessarily have $j = 1$ or $j = D$ and $c_j = -1$. In other words, either $R = P$ and $P_2(x) = \varepsilon x^{D_2} P_2(1/x)$, or $S = P$ and $P_1(x) = \varepsilon x^{D_1} P_1(1/x)$. In the first case, the roots of P_2 are common roots of P and Q. In the second case, the roots of P_1 are common roots of P and Q.

To complete the proof, observe that if

$$x^{D+1} - x^D + 1 = x^{D+1} - x + 1 = 0,$$

then $x^D - x = 0$ and $x \neq 0$, so that $x^{D-1} = 1$. As a consequence, we have that $x^2 - x + 1 = x^{D+1} - x + 1 = 0$. This shows that x is a 6-th root of unity; in particular $D = 1 \mod 6$. In addition, P has two irreducible factors, one of which is $x^2 - x + 1$. Thus, G_3 has two irreducible factors, one of which is

$$(a+1)^2 - (a+1) + 1 = a^2 + a + 1.$$

$\square$

It might be interesting to study whether there are other values of D and p for which the polynomial S_p is not irreducible over $\mathbb{Q}$. Adam Epstein would probably call those *algebraic conspiracies*.

3. Misiurewicz polynomials

We now prove Lemma 4, i.e., if $K > k \geq 1$ and $\omega^D = 1$ with $\omega \neq 1$, then the polynomial $F_K - \omega F_k$ has simple roots.

Proof of Lemma 4. We first do a preliminary comment. Let $P_\alpha \in \mathbb{Z}[x]$ be the minimal polynomial of $\alpha := 1 - \omega$. Observe that

$$x^D - 1 = (x-1)(1 + x + \cdots + x^{D-1}) = (x-1) \cdot \prod_\zeta (x - \zeta),$$

where ζ ranges in the set of D-th roots of unity different from 1. The constant coefficient $c_\alpha \in \mathbb{Z}$ of P_α is the product of α and its Galois conjugates. It divides

$$\prod_\zeta (1 - \zeta) = 1 + 1^1 + \cdots + 1^{D-1} = D.$$

Now, assume a_0 is a root of

$$F_K - \omega F_k = a \cdot (F_{K-1}^D - \omega F_{k-1}^D) + \alpha,$$

with the convention $F_0 := 0$. The monic polynomial $F_K^D - F_k^D \in \mathbb{Z}[a]$ vanishes at a_0, so that, a_0 is an algebraic integer. Observe that

$$F_K' - \omega F_k' = F_{K-1}^D - \omega F_{k-1}^D + Da \cdot (F_{K-1}^{D-1} F_{K-1}' - \omega F_{k-1}^{D-1} F_{k-1}').$$

So, if a_0 were a root of $F_K' - \omega F_k'$, then we would have $\alpha = D\beta$, for some algebraic integer

$$\beta := a_0^2 \cdot \big(F_{K-1}^{D-1} F_{K-1}' - \omega F_{k-1}^{D-1} F_{k-1}'\big)(a_0).$$

Let $P_\beta \in \mathbb{Z}[y]$ be the minimal polynomial of β and let $c_\beta \in \mathbb{Z}$ be its constant coefficient. Then,

$$P_\alpha(x) = D^m P_\beta(x/D) \quad \text{with} \quad m := \deg(P_\alpha).$$

As a consequence, $D^m c_\beta = c_\alpha$ divides D, so that $m = 1$. This can occur only if $\alpha \in \mathbb{Q}$, i.e., only if $\omega = -1$. In that case, we have $2 = D\beta$ and so, $D = 2$ and $\beta = 1$. This proves that when $D \neq 2$ and $\omega \neq -1$, the roots of $F_N - \omega F_n$ are simple.

It remains to prove that when $D = 2$, the roots of $F_K + F_k$ are simple. Since $F_K(0) + F_k(0) = 2$, it is equivalent to prove that $aF_K + aF_k$ has simple roots. Set $Q_0 := 0$ and for $p \geq 1$, $Q_p := aF_p$. Then, for $p \geq 1$,

$$Q_p = aF_p = a \cdot (aF_{p-1}^2 + 1) = Q_{p-1}^2 + a$$

and

$$Q_p' = 2Q_{p-1}Q_{p-1}' + 1 = 1 \mod 2.$$

In particular,

$$\frac{Q_K' + Q_k'}{2} = 1 + Q_{K-1}Q_{K-1}' + Q_{k-1}Q_{k-1}' \in \mathbb{Z}[a].$$

We have that

$$Q_{K-1}' \equiv 1 \mod 2 \quad \text{and} \quad Q_{k-1}' \equiv 1 \mod 2,$$

so that

$$1+Q_{K-1}Q'_{K-1}+Q_{k-1}Q'_{k-1}\equiv 1+Q_{K-1}+Q_{k-1}\mod 2.$$

We also have

$$Q_K+Q_k\equiv Q_{K-1}^2+Q_{k-1}^2\mod 2\equiv (Q_{K-1}+Q_{k-1})^2\mod 2.$$

Since $(Q_{K-1}+Q_{k-1})^2$ is monic and since $1+Q_{K-1}+Q_{k-1}$ takes the value 1 at the roots of $(Q_{K-1}+Q_{k-1})^2$, we have that

$$\operatorname{resultant}\left(Q_K+Q_k,\frac{Q'_K+Q'_k}{2}\right)\equiv 1\mod 2.$$

It follows that this resultant is non zero, and that the polynomials Q_K+Q_k and F_K+F_k have simple roots. $\square$

We may now prove Proposition 2, i.e., if $a\in\mathcal{A}_D$ and 0 is preperiodic for f_a with preperiod $k\geq 2$ and period $p\geq 2$, then a is an algebraic unit.

For $k\geq 2$ and $p\geq 1$, let $\mathcal{A}_D^{k,p}$ be the set of parameters $a\in\mathcal{A}_D$ such that 0 is preperiodic for f_a with preperiod k and period p. Moreover, let $G_{k,p}$ be the monic polynomial which has simple roots exactly at the points $a\in\mathcal{A}_D^{k,p}$. Finally, following Milnor [M1], set

$$F_{k,p}:=\frac{F_{k+p-1}^D-F_{k-1}^D}{F_{k+p-1}-F_{k-1}}=\sum_{i+j=D-1}F_{k+p-1}^iF_{k-1}^j.$$

The polynomials $F_{k,p}$ are called *Misiurewicz polynomials*. Proposition 2 is a corollary of the following lemma which asserts that for $p\geq 2$, the constant coefficient of $G_{k,p}$ is equal to 1.

Lemma 10. *For $k\geq 2$ and $p\geq 1$,*

$$F_{k,p}=F_{\gcd(p,k-1)}^{D-1}\cdot\prod_{q|p}G_{k,q}.$$

The constant coefficient of $G_{k,p}$ is equal to D if $p=1$ and is equal to 1 if $p\geq 2$.

Proof. First, observe that the roots of $F_{k,p}$ are the parameters $a\in\mathcal{A}_D$ such that $F_{k+p-1}(a)=\omega F_{k-1}(a)$ for some D-th root of unity $\omega\neq 1$. According to Lemma 4, the polynomial $F_{k+p-1}-\omega F_{k-1}$ has simple roots. It follows that if $F_{k+p-1}(a)=\omega F_{k-1}(a)=0$, then a is a root of multiplicity $D-1$ of $F_{k,p}$. Otherwise, a is a simple root of $F_{k,p}$.

Second, $F_{k+p-1}(a)=\omega F_{k-1}(a)=0$ if and only if $a\in\mathcal{A}_D^q$ for some q dividing $k-1$ and p. And $F_{k+p-1}(a)=\omega F_{k-1}(a)\neq 0$ if and only if $f_a^{\circ k}(0)$ is periodic with period dividing p, but $f_a^{\circ(k-1)}(0)$ is not periodic, i.e., if and only if $a\in\mathcal{A}_D^{k,q}$ for some q dividing p.

Third, the constant coefficient of $F_{k,p}$ is equal to D : there are D terms with constant coefficient 1 in the sum defining $F_{k,p}$. In addition, the constant coefficient of F_q is 1 for all $q\geq 1$. As a consequence, the constant coefficient of $\prod_{q|p} G_{k,q}$ is D for all $p \geq 1$. It follows by induction on $p \geq 1$ that the constant coefficient of $G_{k,p}$ is equal to D if $p = 1$ and to 1 if $p \geq 2$. $\square$

4. On a question of Baker and DeMarco

We conclude the article with the proof of Proposition 6 : if 0 and 1 are simultaneously (pre)periodic for $q_a : w \mapsto w^2 + a$, then $a \in \{0, -1, -2\}$. In fact, this is an equivalence since

- for q_0, 0 and 1 are fixed;
- for q_{-1}, 0 is periodic of period 2 and 1 is preperiodic with preperiod 1 and period 2 (with orbit $1 \mapsto 0 \mapsto -1 \mapsto 0$);
- for q_{-2}, 0 is preperiodic with preperiod 2 and period 1 (with orbit $0 \mapsto -2 \mapsto 2 \mapsto 2$) and 1 is preperiodic with preperiod 1 and period 1 (with orbit $1 \mapsto -1 \mapsto -1$).

The proof relies on Lemma 11 below which asserts that if 0 and 1 have a bounded orbit for q_a, then a is contained in

$$\Delta(-1, 1) \cup \Delta(-1/4, 1/2) \cup \{-2\},$$

where $\Delta(z, r)$ is the open Euclidean disk centered at z with radius r.

Proof of Proposition 6 assuming Lemma 11. Denote by A the set of parameters $a \in \mathbb{C}$ such that 0 and 1 are simultaneously (pre)periodic for q_a. If $a \in A$, the orbits of 0 and 1 are finite for q_a, thus bounded. According to Lemma 11,

$$A \subset \Delta(-1, 1) \cup \Delta(-1/4, 1/2) \cup \{-2\}.$$

The disk $\Delta(-1/4, 1/2)$ is contained in the main cardioid of the Mandelbrot set $\mathcal{M}_2$. It follows that the only parameter $a \in \Delta(-1/4, 1/2)$ for which 0 is (pre)periodic is $a = 0$. So, $A \subset \Delta(-1, 1) \cup \{0, -2\}$.

Assume $a \in A \cap \Delta(-1, 1)$. Then, $q_a^{\circ k_0}(0) = q_a^{\circ(k_0+p_0)}(0)$ for some integers $k_0 \geq 0$ and $p_0 \geq 1$ and $q_a^{\circ k_1}(1) = q_a^{\circ(k_1+p_1)}(1)$ for some integers $k_1 \geq 0$ and $p_1 \geq 1$. Note that

$$Q_0(a) := q_a^{\circ(k_0+p_0)}(0) - q_a^{\circ k_0}(0) \quad \text{and} \quad Q_1(a) := q_a^{\circ(k_1+p_1)}(1) - q_a^{\circ k_1}(1)$$

are polynomials in $\mathbb{Z}[a]$. In particular, a is an algebraic integer. Moreover, if a' is a Galois conjugate of a, then $Q_0(a') = Q_1(a') = 0$, so that $a' \in A$. Thus, a and its Galois conjugates are contained in $\Delta(-1, 1)$. It follows that $a + 1$ and all its Galois conjugates are contained in the unit disk. In particular, their product is an integer contained in the unit disk, i.e., is equal to 0. Thus, $a + 1 = 0$ and $a = -1$. $\square$

Denote by $\mathcal{N}$ the set of parameters $a \in \mathbb{C}$ such that 0 and 1 have a bounded orbit for q_a.

Lemma 11. *The set $\mathcal{N}$ is contained in* $\Delta(-1, 1) \cup \Delta(-1/4, 1/2) \cup \{-2\}$.

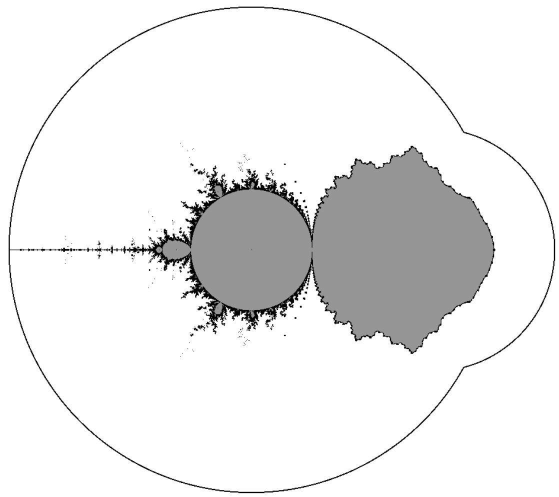

FIGURE 2. The set $\mathcal{N}$ and the boundary of $\Delta(-1,1)\cup\Delta(-1/4,1/2)$.

Proof. For $a\in\mathbb{C}$ and for $n\geq 0$, set

$$P_n(a) := q_a^{\circ n}(0) = af_a^{\circ n}(0) \quad \text{and} \quad Q_n(a) := q_a^{\circ n}(1) = af_a^{\circ n}(1/a).$$

According to Lemma 7 with $D=2$,

$$a\in\mathcal{N} \quad \Longrightarrow \quad \bigl|P_n(a)\bigr|\leq 2 \text{ and } \bigl|Q_n(a)\bigr|\leq 2 \text{ for all } n\geq 0.$$

Let us subdivide

$$U := \mathbb{C}\setminus\Delta(-1,1)\cup\Delta(-1/4,1/2)$$

into two pieces:

$$U_0 := \bigl\{a\in U\ ;\ \mathrm{Re}(a)\leq -1\bigr\},\quad U_1 := \bigl\{a\in U\ ;\ \mathrm{Re}(a)\geq -1\bigr\}.$$

It is enough to prove that

- $\bigl|P_3(a)\bigr|>2$ for $a\in U_0\setminus\{-2\}$ and
- $\bigl|Q_3(a)\bigr|>2$ for $a\in U_1$.

Let us begin with

$$P_3(a) = a\cdot(a^3+2a^2+a+1).$$

The roots of P_3 are 0, $-1.7548\ldots$, $-0.1225\ldots\mathrm{i}0.7448\ldots$ which do not belong to U_0. Thus,

$$\min_{U_0}|P_3| = \min_{\partial U_0}|P_3|.$$

An elementary computation yields

$$\bigl|P_3(-1+\mathrm{i}y)\bigr|^2 = y^8+2y^6+3y^4+3y^2+1.$$

Note that $y^2 \geq 1$ when $-1+\mathrm{i}y \in \partial U_0$. In this case

$$\left|P_3(-1+\mathrm{i}y)\right|^2 \geq 10 > 4.$$

In addition,

$$\left|P_3(-1+\mathrm{e}^{\mathrm{i}\theta})\right|^2 = h\big(\cos(\theta)\big)$$

with

$$A(x) := -16x^4 + 24x^3 + 8x^2 - 26x + 10.$$

Note that $\cos(\theta) \in (-1,0)$ when

$$-1+\mathrm{e}^{\mathrm{i}\theta} \in \partial U_0 \smallsetminus \{-2\}.$$

Thus, it is enough to prove that $A > 4$ on $(-1,0)$. Observe that

$$A(x) - 4 = -2(x+1)B(x)$$

with

$$B(x) := 8x^3 - 20x^2 + 16x - 3.$$

We have to prove that $B < 0$ on $(-1,0)$. Note that

$$B'(x) = 24(x-1)(x-2/3),$$

so that B is increasing on $(-1,0)$ and $B < -3$ on $(-1,0)$.

Let us now consider

$$Q_3(a) = (a+1)(a^3+5a^2+6a+1).$$

The roots of Q_3 are -1, $-0.198\ldots$, $-3.24\ldots$ and $-1.55\ldots$ which do not belong to U_1. Thus,

$$\min_{U_1} |Q_3| = \min_{\partial U_1} |Q_3|.$$

An elementary computation yields

$$\left|Q_3(-1+\mathrm{i}y)\right|^2 = y^8 + 6y^6 + 5y^4 + y^2.$$

Note that $y^2 \geq 1$ when $-1+\mathrm{i}y \in \partial U_1$. In this case

$$\left|Q_3(-1+\mathrm{i}y)\right|^2 \geq 13 > 4.$$

The circles $C(-1,1)$ and $C(-1/4,1/2)$ intersect at the points $-1/8\mathrm{i}\sqrt{15}/8$. We have that

$$\left|Q_3(-1+\mathrm{e}^{\mathrm{i}\theta})\right|^2 = C\big(\cos(\theta)\big)$$

with

$$C(x) := -8x^3 - 12x^2 + 8x + 13.$$

Note that $\cos(\theta) \in (0,7/8)$ when $-1+\mathrm{e}^{\mathrm{i}\theta} \in \partial U_1$. The derivative $C'(x)$ vanishes at

$$x_0 = (-3-\sqrt{21})/6 < 0$$

and

$$x_1 = (-3+\sqrt{21})/6 \in (0,7/8).$$

So, on $(0,7/8)$,

$$C(x) \geq \min\big(C(0), C(7/8)\big) > 4.$$

Finally,

$$\left|Q_3(-1/4+\mathrm{e}^{\mathrm{i}\theta}/2)\right|^2 = \delta\big(\cos(\theta)\big)$$

with

$$\delta(x) := -\frac{39}{256}x^4 - \frac{31}{256}x^3 + \frac{5607}{2048}x^2 + \frac{25933}{4096}x + \frac{242593}{65536}.$$

Note that $\cos(\theta) \in (1/4, 1)$ when $-1/4 + \mathrm{e}^{\mathrm{i}\theta}/2 \in \partial U_1$. In this case,

$$\delta\big(\cos(\theta)\big) \geq -\frac{39}{256} - \frac{31}{256} + \frac{5607}{2048}\cdot\frac{1}{4^2} + \frac{25933}{4096}\cdot\frac{1}{4} + \frac{242593}{65536} > 4. \qquad \square$$

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