

On the spaces of bounded and compact multiplicative Hankel operators

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ABSTRACT. A multiplicative Hankel operator is an operator with matrix representation $M(\alpha) = \{\alpha(nm)\}_{n,m=1}^\infty$, where α is the generating sequence of $M(\alpha)$. Let \mathcal{M} and \mathcal{M}_0 denote the spaces of bounded and compact multiplicative Hankel operators, respectively. In this note it is shown that the distance from an operator $M(\alpha) \in \mathcal{M}$ to the compact operators is minimized by a nonunique compact multiplicative Hankel operator $N(\beta) \in \mathcal{M}_0$. Intimately connected with this result, it is then proven that the bidual of \mathcal{M}_0 is isometrically isomorphic to \mathcal{M} , $\mathcal{M}_0^{**} \simeq \mathcal{M}$. It follows that \mathcal{M}_0 is an M-ideal in \mathcal{M} . The dual space \mathcal{M}_0^* is isometrically isomorphic to a projective tensor product with respect to Dirichlet convolution. The stated results are also valid for small Hankel operators on the Hardy space $H^2(\mathbb{D}^d)$ of a finite polydisk.

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1. Introduction

Given a sequence $\alpha: \mathbb{N} \rightarrow \mathbb{C}$, we consider the corresponding multiplicative Hankel operator $m = M(\alpha): \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$, defined by

$$\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})} = \sum_{n,m=1}^\infty a(n)\overline{b(m)}\alpha(nm), \quad a, b \in \ell^2(\mathbb{N}).$$

Initially, we consider this equality only for finite sequences a and b . It defines a bounded operator $M(\alpha): \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$, with matrix representation $\{\alpha(nm)\}_{n,m=1}^\infty$ in the standard basis of $\ell^2(\mathbb{N})$, if and only if there is a constant $C > 0$ such that

$$|\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})}| \leq C\|a\|_{\ell^2(\mathbb{N})}\|b\|_{\ell^2(\mathbb{N})}, \quad a, b \text{ finite sequences.}$$

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Multiplicative Hankel operators are also known as Helson matrices, having been introduced by Helson in [14, 15].

There are two common alternative interpretations. One is in terms of Dirichlet series. Let \mathcal{H}^2 be the Hardy space of Dirichlet series, the Hilbert space with $(n^{-s})_{n=1}^\infty$ as a basis. Elements $f \in \mathcal{H}^2$ are holomorphic functions in the half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > 1/2\}$. If

$$f(s) = \sum_{n=1}^\infty a(n)n^{-s}, \quad g(s) = \sum_{n=1}^\infty \overline{b(n)}n^{-s}, \quad \rho(s) = \sum_{n=1}^\infty \overline{\alpha(n)}n^{-s},$$

then

$$\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})} = \langle fg, \rho \rangle_{\mathcal{H}^2}.$$

Hence there is an isometric correspondence between Helson matrices and Hankel operators on \mathcal{H}^2 , since the forms associated with the latter are precisely of the type $(f, g) \mapsto \langle fg, \rho \rangle_{\mathcal{H}^2}$.

The second interpretation is in terms of the Hardy space of the infinite polytorus $H^2(\mathbb{T}^\infty)$, the Hilbert space with basis $(z^\kappa)_\kappa$, where $z = (z_1, z_2, \dots)$, and $\kappa = (\kappa_1, \kappa_2, \dots)$ runs through the countably infinite, but finitely supported, multi-indices. Identify each integer n with a multi-index κ of this type through the factorization of n into the primes p_1, p_2, \dots ,

$$n \longleftrightarrow \kappa \text{ if and only if } n = \prod_{j=1}^\infty p_j^{\kappa_j}.$$

Under this equivalence, multiplicative Hankel operators correspond to additive Hankel operators on a countably infinite number of variables,

$$\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})} = \sum_{\kappa, \kappa'} a(\kappa)\overline{b(\kappa')}\alpha(\kappa + \kappa').$$

Hence the multiplicative Hankel operators correspond isometrically to small Hankel operators on $H^2(\mathbb{T}^\infty)$, since the matrix representations of the latter are of the form $\{\alpha(\kappa + \kappa')\}_{\kappa, \kappa'}$. See [14, 15] for details.

In particular, the Helson matrices generalize the small Hankel operators on the Hardy space of any finite polytorus $H^2(\mathbb{T}^d)$, $d < \infty$. In fact, the results in this note have analogous statements for small Hankel operators on $H^2(\mathbb{T}^d)$; every proof given remains valid verbatim after restricting the number of prime factors, that is, the number of variables.

The first result is the following. We denote by $\mathcal{B}(\ell^2(\mathbb{N}))$ and $\mathcal{K}(\ell^2(\mathbb{N}))$, respectively, the spaces of bounded and compact operators on $\ell^2(\mathbb{N})$.

Theorem 1.1. *Let $M(\alpha)$ be a bounded multiplicative Hankel operator. Then there exists a compact multiplicative Hankel operator $N(\beta)$ such that*

$$\|M(\alpha) - N(\beta)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = \inf \{ \|M(\alpha) - K\|_{\mathcal{B}(\ell^2(\mathbb{N}))} : K \in \mathcal{K}(\ell^2(\mathbb{N})) \}. \quad (1)$$

The minimizer $N(\beta)$ is never unique, unless $M(\alpha)$ is compact.

The quantity on the right-hand side of (1) is known as the essential norm of $M(\alpha)$. For classical Hankel operators on $H^2(\mathbb{T})$, this result was proven by Axler, Berg, Jewell, and Shields in [6], and can be viewed as a limiting case of the theory of Adamjan, Arov, and Krein [1]. The demonstration of Theorem 1.1 requires only a minor modification of the arguments in [6], the main point being that a characterization of the class of bounded multiplicative Hankel operators is not necessary for the proof.

On $H^2(\mathbb{T})$, Nehari’s theorem [21] states that the class of bounded Hankel operators can be isometrically identified with $L^\infty(\mathbb{T})/H^\infty(\mathbb{T})$, where $L^\infty(\mathbb{T})$ and $H^\infty(\mathbb{T})$ denote the spaces of bounded and bounded analytic functions on \mathbb{T} , respectively. By Hartman’s theorem [13], the class of compact Hankel operators is isometrically isomorphic to $(H^\infty(\mathbb{T}) + C(\mathbb{T}))/H^\infty(\mathbb{T})$, where $C(\mathbb{T})$ denotes the space of continuous functions on \mathbb{T} . Note that the spaces L^∞ , H^∞ , and $H^\infty + C$ are all algebras, as proven by Sarason [26].

Luecking [20] observed, through a very illustrative argument relying on function algebra techniques, that the compact Hankel operators form an M-ideal in the space of bounded Hankel operators. The concept of an M-ideal will be defined shortly, but let us note for now that M-ideality implies proximality; the distance from a bounded Hankel operator to the compact Hankel operators has a minimizer. Thus Luecking reproved some of the results of [6]. Since

$$((H^\infty + C)/H^\infty)^{**} \simeq L^\infty/H^\infty,$$

it follows that the bidual of the space of compact Hankel operators is isometrically isomorphic to the space of bounded Hankel operators. Spaces which are M-ideals in their biduals are said to be M-embedded.

The multiplicative Hankel operators, on the other hand, have thus far resisted all attempts to characterize their boundedness. It has been shown that a Nehari-type theorem cannot exist [22], and positive results only exist in special cases [14, 24]. In spite of this, the main theorem shows that Luecking’s result holds for multiplicative Hankel operators.

Let

$$\mathcal{M}_0 = \{m = M(\alpha) : M(\alpha) : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N}) \text{ compact}\}$$

and

$$\mathcal{M} = \{m = M(\alpha) : M(\alpha) : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N}) \text{ bounded}\}.$$

Equipped with the operator norm, \mathcal{M}_0 and \mathcal{M} are closed subspaces of $\mathcal{K}(\ell^2(\mathbb{N}))$ and $\mathcal{B}(\ell^2(\mathbb{N}))$, respectively. For a Banach space Y , we denote by ι_Y the canonical embedding $\iota_Y : Y \rightarrow Y^{**}$,

$$\iota_Y y(y^*) = y^*(y), \quad y \in Y, y^* \in Y^*.$$

Theorem 1.2. *There is a unique isometric isomorphism $U : \mathcal{M}_0^{**} \rightarrow \mathcal{M}$ such that $U \iota_{\mathcal{M}_0} m = m$ for every $m \in \mathcal{M}_0$. Furthermore, \mathcal{M}_0 is an M-ideal in \mathcal{M} .*

Remark. As pointed out earlier, Theorem 1.2 is also true when stated for small Hankel operators on $H^2(\mathbb{T}^d)$, $d < \infty$. The biduality has in this case been demonstrated isomorphically in [18], with an argument based on the non-isometric Nehari-type theorems proven in [10, 17].

The M-ideal property means the following: there is an (onto) projection $L: \mathcal{M}^* \rightarrow \mathcal{M}_0^\perp$ such that

$$\|m^*\|_{\mathcal{M}^*} = \|Lm^*\|_{\mathcal{M}^*} + \|m^* - Lm^*\|_{\mathcal{M}^*}, \quad m^* \in \mathcal{M}^*,$$

where \mathcal{M}_0^\perp denotes the space of functionals $m^* \in \mathcal{M}^*$ which annihilate \mathcal{M}_0 . M-ideals were introduced by Alfsen and Effros [3] as a Banach space analogue of closed two-sided ideals in C^* -algebras. Very loosely speaking, the fact that \mathcal{M}_0 is an M-ideal in \mathcal{M} implies that the norm of \mathcal{M} resembles a maximum norm and, in this analogy, that \mathcal{M}_0 is the subspace of elements vanishing at infinity. The book [12] comprehensively treats M-structure theory and its applications.

We will make use of the following consequences of Theorem 1.2. Proximality of \mathcal{M}_0 in \mathcal{M} was already mentioned, but the M-ideal property also implies that the minimizer is never unique [16]. It also ensures that \mathcal{M}_0^* is a strongly unique predual of \mathcal{M} [12, Proposition III.2.10]. This means that every isometric isomorphism of \mathcal{M} onto Y^* , Y a Banach space, is weak*-weak* continuous, that is, arises as the adjoint of an isometric isomorphism of Y onto \mathcal{M}_0^* . On the other hand, \mathcal{M}_0^* has infinitely many different preduals [11, Theoreme 27].

The predual of \mathcal{M} is well known to have an almost tautological characterization as a projective tensor product with respect to Dirichlet convolution,

$$\mathcal{X} = \ell^2(\mathbb{N}) \hat{\star} \ell^2(\mathbb{N}).$$

The space \mathcal{X} is also referred to as a weak product space. We defer the precise definition to the next section – after establishing the main theorems, we essentially show, following [25], that all reasonable definitions of \mathcal{X} coincide.

Theorem 1.3. *There is an isometric isomorphism $L: \mathcal{X} \rightarrow \mathcal{M}_0^*$ such that $L^*U^{-1}: \mathcal{M} \rightarrow \mathcal{X}^*$ is the canonical isometric isomorphism of \mathcal{M} onto \mathcal{X}^* , where $U: \mathcal{M}_0^{**} \rightarrow \mathcal{M}$ is the isometric isomorphism of Theorem 1.2.*

Informally stated, $\mathcal{M}_0^* \simeq \mathcal{X}$ and $\mathcal{X}^* \simeq \mathcal{M}$. Theorem 1.3 follows at once from Theorem 1.2 and the uniqueness of the predual of \mathcal{M} , but we also supply a direct proof. While the duality $\mathcal{X}^* \simeq \mathcal{M}$ is a rephrasing of the definition of \mathcal{M} , it is difficult to identify a common approach to dualities of the type $\mathcal{M}_0^* \simeq \mathcal{X}$ in the existing literature. Often, the latter duality is deduced (isomorphically) via a concrete description of \mathcal{M} . For a small selection of relevant examples, see [4, 8, 12, 18, 19, 23, 28].

The idea behind this note is that the direct view of \mathcal{M} as a subspace of $\mathcal{B}(\ell^2(\mathbb{N}))$ already provides sufficient information to prove Theorems 1.1, 1.2, and 1.3. In this direction, Wu [28] worked with an embedding into the space

of bounded operators to deduce duality results for certain Hankel-type forms on Dirichlet spaces.

The proofs of the results only have two main ingredients. The first is a device to approximate elements of \mathcal{M} by elements of \mathcal{M}_0 (Lemma 2.1). Such an approximation property is necessary, because if $\mathcal{M}_0^{**} \simeq \mathcal{M}$, then the unit ball of \mathcal{M}_0 is weak* dense in the unit ball of \mathcal{M} . The second ingredient is an inclusion of \mathcal{M} into a reflexive space; in our case, $\ell^2(\mathbb{N})$. Analogous theorems could be proven for many other linear spaces of bounded and compact operators using the same technique.

2. Results

For a sequence a and $0 < r < 1$, let

$$D_r a(n) = r^{\sum_{j=1}^{\infty} j\kappa_j} a(n), \text{ where } n = \prod_{j=1}^{\infty} p_j^{\kappa_j}.$$

Note that

$$\sum_{\kappa} r^{2\sum_{j=1}^{\infty} j\kappa_j} = \prod_{j=1}^{\infty} \frac{1}{1 - r^{2j}} < \infty.$$

Hence it follows by the dominated convergence theorem that $D_r: \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ is a compact operator. Furthermore, D_r is self-adjoint and contractive, $\|D_r\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq 1$. The dominated convergence theorem also implies that $D_r \rightarrow \text{id}_{\ell^2(\mathbb{N})}$ in the strong operator topology (SOT) as $r \rightarrow 1$, that is, $\lim_{r \rightarrow 1} D_r a = a$ in $\ell^2(\mathbb{N})$, for every $a \in \ell^2(\mathbb{N})$. A study of the operators D_r in the context of Hardy spaces of the infinite polytorus can be found in [2].

The Dirichlet convolution of two sequences a and b is the new sequence $a \star b$ given by

$$(a \star b)(n) = \sum_{k|n} a(k) \overline{b(n/k)}, \quad n \in \mathbb{N}.$$

If a and b are two finite sequences, then

$$\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})} = \langle \alpha, a \star b \rangle, \tag{2}$$

where $(a, b) = \sum_{n=1}^{\infty} a(n)b(n)$ denotes the bilinear pairing between $a, b \in \ell^2(\mathbb{N})$. Note also that, for $0 < r < 1$,

$$D_r(a \star b) = D_r a \star D_r b. \tag{3}$$

The following simple lemma is key.

Lemma 2.1. *Let $M(\alpha)$ be a bounded multiplicative Hankel operator, $M(\alpha) \in \mathcal{M}$. For $0 < r < 1$, let $\alpha_r = D_r \alpha$. Then $M_{\alpha_r} \in \mathcal{M}_0$,*

$$\|M_{\alpha_r}\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq \|M_{\alpha}\|_{\mathcal{B}(\ell^2(\mathbb{N}))},$$

and $M_{\alpha_r} \rightarrow M_{\alpha}$ and $M_{\alpha_r}^ \rightarrow M_{\alpha}^*$ SOT as $r \rightarrow 1$.*

Proof. By (2) and (3), it holds for finite sequences a and b that

$$\langle M(\alpha_r)a, b \rangle_{\ell^2(\mathbb{N})} = \langle M_\alpha D_r a, D_r b \rangle_{\ell^2(\mathbb{N})}.$$

Hence $M_{\alpha_r} = D_r M_\alpha D_r$. We conclude that M_{α_r} is compact, $\|M_{\alpha_r}\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq \|M_\alpha\|_{\mathcal{B}(\ell^2(\mathbb{N}))}$, and $M_{\alpha_r} \rightarrow M_\alpha$ SOT as $r \rightarrow 1$. Similarly, $M_{\alpha_r}^* = M_{\bar{\alpha}_r} \rightarrow M_{\bar{\alpha}} = M_\alpha^*$ SOT as $r \rightarrow 1$. \square

The following is a recognizable consequence, cf. [27, Theorem 1]. Note that if S_n and T_n are operators such that $S_n \rightarrow S$ and $T_n \rightarrow T$ SOT, and if C is a compact operator, then $S_n C T_n^* \rightarrow S C T^*$ in operator norm.

Proposition 2.2. *Let $M(\alpha) \in \mathcal{M}$. Then $M(\alpha) \in \mathcal{M}_0$ if and only if*

$$\lim_{r \rightarrow 1} \|M(\alpha_r) - M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = 0. \tag{4}$$

Proof. If (4) holds, then $M(\alpha) \in \mathcal{M}_0$, since $M(\alpha_r)$ is compact for every $0 < r < 1$. If $M(\alpha) \in \mathcal{M}_0$, then (4) holds, since $M(\alpha_r) = D_r M(\alpha) D_r = D_r M(\alpha) D_r^*$ and $D_r \rightarrow \text{id}_{\ell^2(\mathbb{N})}$ SOT as $r \rightarrow 1$. \square

Recall next the main tool from [6].

Theorem 2.3 ([6]). *Let $T: \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ be a non-compact operator and (T_n) a sequence of compact operators such that $T_n \rightarrow T$ SOT and $T_n^* \rightarrow T^*$ SOT. Then there exists a sequence (c_n) of non-negative real numbers such that $\sum_n c_n = 1$ for which the compact operator*

$$J = \sum_n c_n T_n$$

satisfies

$$\|T - J\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = \inf \{ \|T - K\|_{\mathcal{B}(\ell^2(\mathbb{N}))} : K \in \mathcal{K}(\ell^2(\mathbb{N})) \}.$$

Lemma 2.1 and Theorem 2.3 immediately yield the existence part of Theorem 1.1.

Proof of Theorem 1.1. Let $M(\alpha)$ be a bounded multiplicative Hankel operator and let (r_k) be a sequence such that $0 < r_k < 1$ and $r_k \rightarrow 1$. Then $M(\alpha)$ has a best compact approximant of the form

$$N = \sum_k c_k M(\alpha_{r_k}).$$

But then $N = N(\beta)$ is a multiplicative Hankel operator, $\beta = \sum_k c_k \alpha_{r_k}$.

The non-uniqueness of $N(\beta)$ follows immediately once we have established Theorem 1.2, by general M-ideal results [16]. In fact, if $M(\alpha) \notin \mathcal{M}_0$, then the set of minimizers $N(\beta)$ is so large that it spans \mathcal{M}_0 . \square

Note that

$$\|M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \geq \overline{\lim}_{N \rightarrow \infty} \frac{1}{\|(\alpha(n))_{n=1}^N\|_{\ell^2(\mathbb{N})}} \sum_{n=1}^N |\alpha(n)|^2 = \|\alpha\|_{\ell^2(\mathbb{N})}.$$

Therefore the inclusion $I: \mathcal{M}_0 \rightarrow \ell^2(\mathbb{N})$ is a contractive operator, $Im = I(M(\alpha)) = \alpha$. We can state Theorem 1.2 slightly more precisely in terms of I .

Theorem 1.2. *Consider the bitranspose $U = I^{**}: \mathcal{M}_0^{**} \rightarrow \ell^2(\mathbb{N})$. Then $U\mathcal{M}_0^{**} = \mathcal{M}$, viewing \mathcal{M} as a (non-closed) subspace of $\ell^2(\mathbb{N})$. Furthermore,*

$$U\iota_{\mathcal{M}_0}m = m, \quad m \in \mathcal{M}_0,$$

and

$$\|Um^{**}\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = \|m^{**}\|_{\mathcal{M}_0^{**}}, \quad m^{**} \in \mathcal{M}_0^{**}.$$

If $V: \mathcal{M}_0^{**} \rightarrow \mathcal{M}$ is another isometric isomorphism such that $V\iota_{\mathcal{M}_0}m = m$ for all $m \in \mathcal{M}_0$, then $V = U$. Furthermore, \mathcal{M}_0 is an M -ideal in \mathcal{M} .

Proof. We identify $(\ell^2(\mathbb{N}))^* \simeq \ell^2(\mathbb{N})$ linearly through the pairing $(a, b) = \sum_{n=1}^\infty a(n)b(n)$ between $a, b \in \ell^2(\mathbb{N})$. With this convention, $I^*: \ell^2(\mathbb{N}) \rightarrow \mathcal{M}_0^*$ is also contractive, and

$$I^*a(m) = (\alpha, a), \quad a \in \ell^2(\mathbb{N}), \quad m = M(\alpha) \in \mathcal{M}_0.$$

Since I is injective, I^* has dense range. In particular, \mathcal{M}_0^* is separable. Furthermore, $I^{**}: \mathcal{M}_0^{**} \rightarrow \ell^2(\mathbb{N})$ is injective. By the reflexivity of $\ell^2(\mathbb{N})$, we have that $I^{**}\iota_{\mathcal{M}_0} = I$, since

$$(I^{**}\iota_{\mathcal{M}_0}m, a) = \iota_{\mathcal{M}_0}m(I^*a) = (\alpha, a) = (Im, a)$$

for every $m = M(\alpha) \in \mathcal{M}_0$ and $a \in \ell^2(\mathbb{N})$. The interpretation, viewing \mathcal{M} as a non-closed subspace of $\ell^2(\mathbb{N})$, is that $I^{**}\iota_{\mathcal{M}_0}m = m$, for all $m \in \mathcal{M}_0$.

Consider any $m^{**} \in \mathcal{M}_0^{**}$, and let $\alpha = I^{**}m^{**} \in \ell^2(\mathbb{N})$. Since \mathcal{M}_0^* is separable, the weak* topology of the unit ball $B_{\mathcal{M}_0^{**}}$ of \mathcal{M}_0^{**} is metrizable. As is the case for every Banach space, $\iota_{\mathcal{M}_0}(B_{\mathcal{M}_0})$ is weak* dense in $B_{\mathcal{M}_0^{**}}$. Hence there is a sequence $(m_n)_{n=1}^\infty$ in \mathcal{M}_0 such that $\iota_{\mathcal{M}_0}m_n \rightarrow m^{**}$ weak* and $\|m_n\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq \|m^{**}\|_{\mathcal{M}_0^{**}}$. Suppose that $m_n = M(\alpha_n)$ and let $a, b \in \ell^2(\mathbb{N})$ be two finite sequences. Then, since $\iota_{\mathcal{M}_0}m_n \rightarrow m^{**}$ weak*,

$$\langle M(\alpha_n)a, b \rangle_{\ell^2(\mathbb{N})} = (\alpha_n, a \star b) = I^*(a \star b)(m_n) \rightarrow m^{**}(I^*(a \star b)) = (\alpha, a \star b),$$

as $n \rightarrow \infty$. It follows that

$$\begin{aligned} |\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})}| &= |(\alpha, a \star b)| \leq \overline{\lim}_{n \rightarrow \infty} \|m_n\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \|a\|_{\ell^2(\mathbb{N})} \|b\|_{\ell^2(\mathbb{N})} \\ &\leq \|m^{**}\|_{\mathcal{M}_0^{**}} \|a\|_{\ell^2(\mathbb{N})} \|b\|_{\ell^2(\mathbb{N})}. \end{aligned}$$

Since a, b were arbitrary finite sequences, it follows that $M(\alpha) \in \mathcal{M}$ and

$$\|M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq \|m^{**}\|_{\mathcal{M}_0^{**}}.$$

Since $\alpha = I^{**}m^{**}$ this proves that I^{**} maps \mathcal{M}_0^{**} contractively into \mathcal{M} .

Conversely, suppose that $m = M(\alpha) \in \mathcal{M}$. By Lemma 2.1, for $0 < r < 1$, $M(\alpha_r) \in \mathcal{M}_0$, $\|M(\alpha_r)\| \leq \|M(\alpha)\|$, and $\alpha_r \rightarrow \alpha$ in $\ell^2(\mathbb{N})$ as $r \rightarrow 1$. Define $m^{**} \in \mathcal{M}_0^{**}$ by

$$m^{**}(I^*a) := (\alpha, a) = \lim_{r \rightarrow 1} (\alpha_r, a) = \lim_{r \rightarrow 1} I^*a(M(\alpha_r)), \quad a \in \ell^2(\mathbb{N}). \quad (5)$$

This specifies an element $m^{**} \in \mathcal{M}_0^{**}$ since I^* has dense range in \mathcal{M}_0^* and

$$|m^{**}(I^*a)| \leq \overline{\lim}_{r \rightarrow 1} \|M(\alpha_r)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \|I^*a\|_{\mathcal{M}_0^*} \leq \|M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \|I^*a\|_{\mathcal{M}_0^*}.$$

From this inequality we also see that

$$\|m^{**}\|_{\mathcal{M}_0^{**}} \leq \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))}. \tag{6}$$

Furthermore, since

$$(I^{**}m^{**}, a) = m^{**}(I^*a) = (\alpha, a), \quad a \in \ell^2(\mathbb{N}),$$

we have that $I^{**}m^{**} = \alpha$. Hence I^{**} maps \mathcal{M}_0^{**} bijectively and contractively onto \mathcal{M} . By (6), $I^{**}: \mathcal{M}_0^{**} \rightarrow \mathcal{M}$ is also expansive, and hence it is an isometric isomorphism.

Recall that $\mathcal{K}(\ell^2(\mathbb{N}))$ is an M-ideal in $\mathcal{B}(\ell^2(\mathbb{N}))$ [9] – indeed, $\mathcal{K}(\ell^2(\mathbb{N}))$ is a two-sided closed ideal in $\mathcal{B}(\ell^2(\mathbb{N}))$. It is well known that there is an isometric isomorphism $E: \mathcal{K}(\ell^2(\mathbb{N}))^{**} \rightarrow \mathcal{B}(\ell^2(\mathbb{N}))$ such that $E\iota_{\mathcal{K}(\ell^2(\mathbb{N}))}K = K$ for all $K \in \mathcal{K}(\ell^2(\mathbb{N}))$. Thus $\mathcal{K}(\ell^2(\mathbb{N}))$ is M-embedded. Since \mathcal{M}_0 is a closed subspace of $\mathcal{K}(\ell^2(\mathbb{N}))$, \mathcal{M}_0 is also M-embedded [12, Theorem III.1.6]. Hence, since we have shown that $I^{**}: \mathcal{M}_0^{**} \rightarrow \mathcal{M}$ is an isometric isomorphism for which $I^{**}\iota_{\mathcal{M}_0}m = m$ for all $m \in \mathcal{M}_0$, it follows that \mathcal{M}_0 is an M-ideal in \mathcal{M} .

Finally, if $V: \mathcal{M}_0^{**} \rightarrow \mathcal{M}$ is another isometric isomorphism such that $V\iota_{\mathcal{M}_0}m = m$, $m \in \mathcal{M}_0$, then $F = V^{-1}I^{**}: \mathcal{M}_0^{**} \rightarrow \mathcal{M}_0^{**}$ is an isometric isomorphism such that $F\iota_{\mathcal{M}_0} = \iota_{\mathcal{M}_0}$. However, since \mathcal{M}_0 is M-embedded, F must be obtained as the bitranspose, $F = G^{**}$, of an isometric isomorphism $G: \mathcal{M}_0 \rightarrow \mathcal{M}_0$ [12, Proposition III.2.2]. But then $G = \text{id}_{\mathcal{M}_0}$, since

$$m^*(Gm) = G^*m^*(m) = F\iota_{\mathcal{M}_0}m(m^*) = m^*(m), \quad m \in \mathcal{M}_0, m^* \in \mathcal{M}_0^*.$$

Hence $F = \text{id}_{\mathcal{M}_0^{**}}$ and so $V = I^{**}$. □

The predual of a space of Hankel operators usually has an abstract description as a projective tensor product [5, 7, 10]. In the present context, let

$$X = \left\{ c : c = \sum_{\text{finite}} a_k \star b_k, \ a_k, b_k \text{ finite sequences} \right\},$$

and equip X with the norm

$$\|c\|_X = \inf \sum_{\text{finite}} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})},$$

where the infimum is taken over all **finite** representations of c . By writing $c = c \star (1, 0, 0, \dots)$ it is clear that $\|c\|_X \leq \|c\|_{\ell^2(\mathbb{N})}$ for $c \in X$.

We define the projective tensor product space $\mathcal{X} = \ell^2(\mathbb{N}) \hat{\star} \ell^2(\mathbb{N})$ with respect to Dirichlet convolution as the Banach space completion of X . It is essentially definition that $\mathcal{X}^* \simeq \mathcal{M}$.

Lemma 2.4. For $m = M(\alpha) \in \mathcal{M}$, let

$$Jm(c) = (\alpha, c), \quad c \in X.$$

Then Jm extends to a bounded functional on \mathcal{X} for every $m \in \mathcal{M}$, and $J: \mathcal{M} \rightarrow \mathcal{X}^*$ is an isometric isomorphism.

Proof. Let $m \in \mathcal{M}$. If $c \in X$ and $\varepsilon > 0$, choose a representation $c = \sum_{k=1}^N a_k \star b_k$, where a_k and b_k are finite sequences for every k , and

$$\sum_{k=1}^N \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} < \|c\|_X + \varepsilon.$$

Then

$$|Jm(c)| = \left| \sum_{k=1}^N \langle M(\alpha)a_k, b_k \rangle_{\ell^2(\mathbb{N})} \right| \leq \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))} (\|c\|_X + \varepsilon).$$

Hence $\|Jm\|_{\mathcal{X}^*} \leq \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))}$. Choosing finite sequences a and b such that $\|a\|_{\ell^2(\mathbb{N})} = \|b\|_{\ell^2(\mathbb{N})} = 1$ and $\langle M(\alpha)a, b \rangle_{\ell^2(\mathbb{N})} > \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))} - \varepsilon$, and letting $c = a \star b$ gives that

$$\|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))} - \varepsilon < \|Jm\|_{\mathcal{X}^*} \|c\|_X \leq \|Jm\|_{\mathcal{X}^*}.$$

Hence J is an isometry.

The inclusion of finite sequences into X extends to a contractive map $E: \ell^2(\mathbb{N}) \rightarrow \mathcal{X}$. Let $\ell \in \mathcal{X}^*$ and let $c \in X$. Then $\ell(c) = (\alpha, c)$, where $\alpha = E^*\ell \in \ell^2(\mathbb{N})$. Then $m = M(\alpha) \in \mathcal{M}$, since $\ell \in \mathcal{X}^*$. Clearly $Jm = \ell$ and thus J is onto. \square

Theorem 1.3. For every $c \in X$, let

$$Lc(m) = (\alpha, c), \quad m = M(\alpha) \in \mathcal{M}_0.$$

Then L extends to an isometric isomorphism $L: \mathcal{X} \rightarrow \mathcal{M}_0^*$, and

$$L^*U^{-1} = J: \mathcal{M} \rightarrow \mathcal{X}^*$$

is the isometric isomorphism of Lemma 2.4. Here $U: \mathcal{M}_0^{**} \rightarrow \mathcal{M}$ is the isometric isomorphism of Theorem 1.2.

Proof. The quickest proof proceeds by noting that \mathcal{M}_0^* is a strongly unique predual of \mathcal{M}_0^{**} , since \mathcal{M}_0 is M-embedded. This implies that the isometric isomorphism $JU: \mathcal{M}_0^{**} \rightarrow \mathcal{X}^*$ is the adjoint of an isometric isomorphism $E: \mathcal{X} \rightarrow \mathcal{M}_0^*$, $E^* = JU$. But then, for $c \in X$ and $m = M(\alpha) \in \mathcal{M}_0$,

$$\begin{aligned} Ec(m) &= \iota_{\mathcal{M}_0} m(Ec) = E^* \iota_{\mathcal{M}_0} m(c) = JU \iota_{\mathcal{M}_0} m(c) \\ &= Jm(c) = (\alpha, c) = Lc(m). \end{aligned} \tag{7}$$

Hence $L = E$, and thus L is an isometric isomorphism.

Alternatively, the weak*-weak* continuity of JU can be proven by hand. L clearly extends to a contractive operator $L: \mathcal{X} \rightarrow \mathcal{M}_0^*$. The computation (7) shows that $JU \iota_{\mathcal{M}_0} = L^* \iota_{\mathcal{M}_0}$. Let $m^{**} \in \mathcal{M}_0^{**}$ and let $M(\alpha) = Um^{**}$.

From (5) we deduce that $m_r^{**} = \iota_{\mathcal{M}_0} M(\alpha_r) \rightarrow m^{**}$ weak* in \mathcal{M}_0^{**} . Hence $L^* m_r^{**} \rightarrow L^* m^{**}$ weak* in \mathcal{X}^* . On the other hand, for $c \in X$,

$$\begin{aligned} JU m^{**}(c) &= (\alpha, c) = \lim_{r \rightarrow 1} (\alpha_r, c) = \lim_{r \rightarrow 1} JU m_r^{**}(c) \\ &= \lim_{r \rightarrow 1} L^* m_r^{**}(c) = L^* m^{**}(c). \end{aligned}$$

This shows that $JU = L^*$, and hence L is an isometric isomorphism. □

Remark. In the notation of Theorem 1.2, $I^*c = Lc$ for $c \in X$. Theorem 1.3 hence completes the picture of Theorem 1.2 by giving an interpretation of the operator I^* .

Suppose that we had instead defined the projective tensor product space $\ell^2(\mathbb{N}) \hat{\star} \ell^2(\mathbb{N})$ as the sequence space

$$\mathcal{Y} = \left\{ c : c = \sum_{k=1}^{\infty} a_k \star b_k, a_k, b_k \in \ell^2(\mathbb{N}), \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} < \infty \right\},$$

normed by

$$\|c\|_{\mathcal{Y}} = \inf \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})},$$

where the infimum is taken over all representations of c . One would like to know that $\mathcal{Y} = \mathcal{X}$. Indeed, it is not a priori clear that \mathcal{X} is a sequence space; or if \mathcal{X} is identifiable with a space of Dirichlet series, if considering multiplicative Hankel operators in that context. For \mathcal{Y} these properties are immediate.

Lemma 2.5. \mathcal{Y} is a Banach space.

Proof. Since $|(a \star b)(n)| \leq \|a\|_{\ell^2(\mathbb{N})} \|b\|_{\ell^2(\mathbb{N})}$ it is clear that

$$e_n(c) = c(n), \quad c \in \mathcal{Y},$$

defines an element $e_n \in \mathcal{Y}^*$, for every $n \in \mathbb{N}$. It follows that $\|c\|_{\mathcal{Y}} = 0$ if and only if $c = 0$.

Suppose that $\sum_{k=1}^{\infty} c_k$ is an absolutely convergent series in \mathcal{Y} . Then there are double sequences $(a_{k,j})$ and $(b_{k,j})$ such that $c_k = \sum_{j=1}^{\infty} a_{k,j} \star b_{k,j}$ for every k and

$$\sum_{k,j=1}^{\infty} \|a_{k,j}\|_{\ell^2(\mathbb{N})} \|b_{k,j}\|_{\ell^2(\mathbb{N})} < \infty.$$

Then $c = \sum_{k,j=1}^{\infty} a_{k,j} b_{k,j}$ is an element of \mathcal{Y} and

$$\|c - \sum_{k=1}^N c_k\|_{\mathcal{Y}} \leq \sum_{k=N+1}^{\infty} \sum_{j=1}^{\infty} \|a_{k,j}\|_{\ell^2(\mathbb{N})} \|b_{k,j}\|_{\ell^2(\mathbb{N})} \rightarrow 0, \quad N \rightarrow \infty.$$

Hence $\sum_{k=1}^{\infty} c_k$ converges in \mathcal{Y} to c . Thus \mathcal{Y} is complete. □

We now prove that $\mathcal{Y} = \mathcal{X}$. The details are similar to those of [25], where projective tensor products of spaces of holomorphic functions were considered. Note that X is contractively contained in \mathcal{Y} .

Proposition 2.6. *The inclusion $V: X \rightarrow \mathcal{Y}$ extends to an isometric isomorphism $V: \mathcal{X} \rightarrow \mathcal{Y}$.*

Proof. We make the following preliminary observation. Since for every $0 < r < 1$,

$$D_r(a \star b) = D_r a \star D_r b, \quad \|D_r\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq 1,$$

D_r defines a bounded operator $D_r: \mathcal{X} \rightarrow \mathcal{X}$,

$$\|D_r\|_{\mathcal{B}(\mathcal{X})} \leq 1.$$

Furthermore, since $D_r \rightarrow \text{id}_{\ell^2(\mathbb{N})}$ SOT on $\ell^2(\mathbb{N})$ as $r \rightarrow 1$, it follows that $\|D_r c - c\|_{\mathcal{X}} \leq \|D_r c - c\|_{\ell^2(\mathbb{N})} \rightarrow 0$ as $r \rightarrow 1$ for every $c \in X$. Hence $D_r \rightarrow \text{id}_{\mathcal{X}}$ SOT on \mathcal{X} as $r \rightarrow 1$.

As in Lemma 2.5, for each $n \in \mathbb{N}$,

$$e_n(c) = c(n), \quad c \in X,$$

extends to a functional $e_n \in \mathcal{X}^*$ with $\|e_n\|_{\mathcal{X}^*} \leq 1$. We show now that (e_n) is a complete sequence in \mathcal{X}^* with respect to the weak* topology. Suppose that $c \in \mathcal{X}$ and that $e_n(c) = 0$ for all n . Pick a sequence (c_k) in X such that $c_k \rightarrow c$ in \mathcal{X} . Then for fixed $r < 1$,

$$\begin{aligned} \|D_r c\|_{\mathcal{X}} &\leq \overline{\lim}_{k \rightarrow \infty} (\|D_r(c - c_k)\|_{\mathcal{X}} + \|D_r c_k\|_{\mathcal{X}}) \\ &= \overline{\lim}_{k \rightarrow \infty} \|D_r c_k\|_{\mathcal{X}} \leq \overline{\lim}_{k \rightarrow \infty} \|D_r c_k\|_{\ell^2(\mathbb{N})}. \end{aligned}$$

Since $c_k \rightarrow c$ in \mathcal{X} and $e_n \in \mathcal{X}^*$, we have that $\lim_{k \rightarrow \infty} c_k(n) = e_n(c) = 0$ for every n . Furthermore, $|c_k(n)| \leq \|e_n\|_{\mathcal{X}^*} \|c_k\|_{\mathcal{X}} \leq \|c_k\|_{\mathcal{X}}$ is uniformly bounded in k and n . Hence it follows by the dominated convergence theorem that $\overline{\lim}_{k \rightarrow \infty} \|D_r c_k\|_{\ell^2(\mathbb{N})} = 0$ and thus that $D_r c = 0$. Since $D_r c \rightarrow c$ in \mathcal{X} as $r \rightarrow 1$ we conclude that $c = 0$. Therefore (e_n) is complete.

Hence \mathcal{X} is a space of sequences. More precisely, since every evaluation e_n is a bounded functional on \mathcal{Y} as well, the extension $V: \mathcal{X} \rightarrow \mathcal{Y}$ of the inclusion map is given by

$$Vc = (e_n(c))_{n=1}^{\infty}, \quad c \in \mathcal{X}. \tag{8}$$

The completeness of (e_n) implies that V is injective.

We next prove that V is onto. The argument is precisely as in [25], but we include it for completeness. For a sequence a and $m \in \mathbb{N}$, let $a^m = (a(1), \dots, a(m), 0, \dots)$. Given $a \in \ell^2(\mathbb{N})$ and $\delta > 0$, choose a sequence (m_1, m_2, \dots) such that $\|a - a^{m_k}\|_{\ell^2(\mathbb{N})} \leq 2^{-k}$. Let $a_k = a^{m_{k+1}} - a^{m_k}$. Then, for sufficiently large K ,

$$a = a^{m_K} + \sum_{k=K}^{\infty} (a^{m_{k+1}} - a^{m_k}), \quad \sum_{k=K}^{\infty} \|a^{m_{k+1}} - a^{m_k}\|_{\ell^2(\mathbb{N})} < \delta.$$

Hence we can write $a = \sum_{j=1}^{\infty} a_j$, where each a_j is a finite sequence and $\sum_j \|a_j\|_{\ell^2(\mathbb{N})} < \|a\|_{\ell^2(\mathbb{N})} + \delta$.

Given $c \in \mathcal{Y}$ and $\varepsilon > 0$, choose $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=1}^{\infty}$ such that

$$c = \sum_{k=1}^{\infty} a_k \star b_k, \quad \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} < \|c\|_{\mathcal{Y}} + \varepsilon.$$

For each k , write, as in the preceding paragraph, $a_k = \sum_{j=1}^{\infty} a_{k,j}$, $b_k = \sum_{j=1}^{\infty} b_{k,j}$, where each $a_{k,j}$ and $b_{k,j}$ is a finite sequence and

$$\sum_{j=1}^{\infty} \|a_{k,j}\|_{\ell^2(\mathbb{N})} < \|a_k\|_{\ell^2(\mathbb{N})} + \delta_k, \quad \sum_{j=1}^{\infty} \|b_{k,j}\|_{\ell^2(\mathbb{N})} < \|b_k\|_{\ell^2(\mathbb{N})} + \delta_k.$$

Here the δ_k are chosen so that

$$\sum_{k=1}^{\infty} (\|a_k\|_{\ell^2(\mathbb{N})} + \delta_k)(\|b_k\|_{\ell^2(\mathbb{N})} + \delta_k) < \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} + \varepsilon.$$

Then $c = \sum_{k,j,l=1}^{\infty} a_{k,j} \star b_{k,l}$, and

$$\sum_{k,j,l=1}^{\infty} \|a_{k,j}\|_{\ell^2(\mathbb{N})} \|b_{k,l}\|_{\ell^2(\mathbb{N})} < \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} + \varepsilon < \|c\|_{\mathcal{Y}} + 2\varepsilon.$$

Relabeling, we have a representation $c = \sum_{n=1}^{\infty} a_n \star b_n$ where a_n and b_n are finite sequences and $\sum_n \|a_n\|_{\ell^2(\mathbb{N})} \|b_n\|_{\ell^2(\mathbb{N})} < \|c\|_{\mathcal{Y}} + 2\varepsilon$. Let $c_N = \sum_{n=1}^N a_n \star b_n$. Then $c_N \rightarrow c$ in \mathcal{Y} , and furthermore (c_N) is a Cauchy sequence in X , hence has a limit \tilde{c} in \mathcal{X} . By continuity of the functionals e_n on both \mathcal{Y} and \mathcal{X} , we find in view of (8) that $V\tilde{c} = c$. Hence V is onto.

Furthermore, since V is contractive,

$$\|c\|_{\mathcal{Y}} \leq \|\tilde{c}\|_{\mathcal{X}} = \lim_{N \rightarrow \infty} \|c_N\|_X < \|c\|_{\mathcal{Y}} + 2\varepsilon.$$

We already showed that V is injective, so that \tilde{c} is uniquely defined by c . On the other hand, ε is arbitrary. We conclude that $\|c\|_{\mathcal{Y}} = \|\tilde{c}\|_{\mathcal{X}}$. It follows that V is an isometric isomorphism. \square

References

- [1] ADAMJAN, VADYM M.; AROV, DAMIR Z.; KREĀN, MARK G. Analytic properties of the Schmidt pairs of a Hankel operator and the generalized Schur–Takagi problem. *Mat. Sb. (N.S.)* **86(128)** (1971), 34–75. MR0298453, Zbl 0248.47019, doi:10.1070/sm1971v015n01abeh001531. 591
- [2] ALEMAN, ALEXANDRU; OLSEN, JAN-FREDRIK; SAKSMAN, EERO. Fatou and brothers Riesz theorems in the infinite-dimensional polydisc. *J. Anal. Math.* **137** (2019), no. 1, 429–447. MR3938009, Zbl 07074794, arXiv:1512.01509, doi:10.1007/s11854-019-0006-x. 593
- [3] ALFSEN, ERIK M.; EFFROS, EDWARD G. Structure in real Banach spaces. I, II. *Ann. of Math. (2)* **96** (1972), 98–128; *ibid.* (2) **96** (1972), 129–173. MR0352946, Zbl 0248.46019, doi:10.2307/1970895. 592

- [4] ARAZY, JONATHAN; FISHER, STEPHEN D.; PEETRE, JAAK. Möbius invariant function spaces. *J. Reine Angew. Math.* **363** (1985), 110–145. MR0814017, Zbl 0566.30042. 592
- [5] ARCOZZI, NICOLA; ROCHBERG, RICHARD; SAWYER, ERIC; WICK, BRETT D. Bilinear forms on the Dirichlet space. *Anal. PDE* **3** (2010), no. 1, 21–47. MR2663410, Zbl 1262.30066, arXiv:0811.4107, doi:10.2140/apde.2010.3.21. 596
- [6] AXLER, SHELDON; BERG, I. DAVID; JEWELL, NICHOLAS; SHIELDS, ALLEN. Approximation by compact operators and the space $H^\infty + C$. *Ann. of Math. (2)* **109** (1979), no. 3, 601–612. MR0534765, Zbl 0399.47024, doi:10.2307/1971228. 591, 594
- [7] COIFMAN, RONALD R.; ROCHBERG, RICHARD; WEISS, GUIDO. Factorization theorems for Hardy spaces in several variables. *Ann. of Math. (2)* **103** (1976), no. 3, 611–635. MR0412721, Zbl 0326.32011, doi:10.2307/1970954. 596
- [8] COIFMAN, RONALD R.; WEISS, GUIDO. Extensions of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.* **83** (1977), no. 4, 569–645. MR0447954, Zbl 0358.30023, doi:10.1090/s0002-9904-1977-14325-5. 592
- [9] DIXMIER, J. Les fonctionnelles linéaires sur l'ensemble des opérateurs bornés d'un espace de Hilbert. *Ann. of Math. (2)* **51** (1950), 387–408. MR0033445, Zbl 0036.35801, doi:10.2307/1969331. 596
- [10] FERGUSON, SARAH H.; LACEY, MICHAEL T. A characterization of product BMO by commutators. *Acta Math.* **189** (2002), no. 2, 143–160. MR1961195, Zbl 1039.47022, arXiv:math/0104144, doi:10.1007/bf02392840. 592, 596
- [11] GODEFROY, GILLES. Sous-espaces bien disposés de L^1 -applications. *Trans. Amer. Math. Soc.* **286** (1984), no. 1, 227–249. MR0756037, Zbl 0521.46012, doi:10.2307/1999403. 592
- [12] HARMAND, PETER; WERNER, DIRK; WERNER, WEND. M -ideals in Banach spaces and Banach algebras. Lecture Notes in Mathematics, 1547. Springer-Verlag, Berlin, 1993. viii+387 pp. ISBN: 3-540-56814-X. MR1238713, Zbl 0789.46011, doi:10.1007/bfb0084355. 592, 596
- [13] HARTMAN, PHILIP. On completely continuous Hankel matrices. *Proc. Amer. Math. Soc.* **9** (1958), 862–866. MR0108684, Zbl 0090.09203, doi:10.2307/2033318. 591
- [14] HELSON, HENRY. Hankel forms and sums of random variables. *Studia Math.* **176** (2006), no. 1, 85–92. MR2263964, Zbl 1108.43003, doi:10.4064/sm176-1-6. 590, 591
- [15] HELSON, HENRY. Hankel forms. *Studia Math.* **198** (2010), no. 1, 79–84. MR2640082, Zbl 1229.47042, doi:10.4064/sm198-1-5. 590
- [16] HOLMES, RICHARD; SCRANTON, BRUCE; WARD, JOSEPH. Approximation from the space of compact operators and other M -ideals. *Duke Math. J.* **42** (1975), 259–269. MR0394301, Zbl 0332.47024, doi:10.1215/s0012-7094-75-04224-6. 592, 594
- [17] LACEY, MICHAEL; TERWILLEGER, ERIN. Hankel operators in several complex variables and product BMO. *Houston J. Math.* **35** (2009), no. 1, 159–183. MR2491875, Zbl 1163.47024, arXiv:math/0310348. 592
- [18] LACEY, MICHAEL T.; TERWILLEGER, ERIN; WICK, BRETT D. Remarks on product VMO. *Proc. Amer. Math. Soc.* **134** (2006), no. 2, 465–474. MR2176015, Zbl 1130.42022, arXiv:math/0405097, doi:10.1090/s0002-9939-05-07974-8. 592
- [19] LI, FENGYING. Essential norm of Toeplitz operators and Hankel operators on the weighted Bergman space. *Integral Equations Operator Theory* **75** (2013), no. 4, 517–525. MR3032666, Zbl 1287.47026, arXiv:1210.4793, doi:10.1007/s00020-012-2024-2. 592
- [20] LUECKING, DANIEL H. The compact Hankel operators form an M -ideal in the space of Hankel operators. *Proc. Amer. Math. Soc.* **79** (1980), no. 2, 222–224. MR0565343, Zbl 0437.46043, doi:10.2307/2043239. 591

- [21] NEHARI, ZEEV. On bounded bilinear forms. *Ann. of Math. (2)* **65** (1957), 153–162. MR0082945, Zbl 0077.10605, doi: 10.2307/1969670. 591
- [22] ORTEGA-CERDÀ, JOAQUIM; SEIP, KRISTIAN. A lower bound in Nehari’s theorem on the polydisc. *J. Anal. Math.* **118** (2012), no. 1, 339–342. MR2993031, Zbl 1276.32005, arXiv:1107.0175, doi: 10.1007/s11854-012-0038-y. 591
- [23] PERFEKT, KARL-MIKAEL. Duality and distance formulas in spaces defined by means of oscillation. *Ark. Mat.* **51** (2013), no. 2, 345–361. MR3090201, Zbl 1283.46011, arXiv:1110.6766, doi: 10.1007/s11512-012-0175-7. 592
- [24] PERFEKT, KARL-MIKAEL; PUSHNITSKI, ALEXANDER. On Helson matrices: moment problems, non-negativity, boundedness, and finite rank. *Proc. Lond. Math. Soc. (3)* **116** (2018), no. 1, 101–134. MR3747045, Zbl 06836089, arXiv:1611.03772, doi: 10.1112/plms.12068. 591
- [25] RICHTER, STEFAN; SUNDBERG, CARL. Weak products of Dirichlet functions. *J. Funct. Anal.* **266** (2014), 5270–5299. MR3177336, Zbl 1303.30050, doi: 10.1016/j.jfa.2014.02.005. 592, 599
- [26] SARASON, DONALD. Algebras of functions on the unit circle. *Bull. Amer. Math. Soc.* **79** (1973), 286–299. MR0324425, Zbl 0257.46079, doi: 10.1090/s0002-9904-1973-13144-1. 591
- [27] SARASON, DONALD. Functions of vanishing mean oscillation. *Trans. Amer. Math. Soc.* **207** (1975), 391–405. MR0377518, Zbl 0319.42006, doi: 10.2307/1997184. 594
- [28] WU, ZHI JIAN. The predual and second predual of W_α . *J. Funct. Anal.* **116** (1993), no. 2, 314–334. MR1239074, Zbl 0809.47024, doi: 10.1006/jfan.1993.1115. 592

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