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Boundary-value problems and associated eigenvalue problems for systems describing vibrations of a rotation shell

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Abstract. In this paper, we consider boundary-value and eigenvalue problems for a system of differential equations with variable coefficients, some of which fail to be integrable in any neighborhood of zero. These problems describe vibrations of an inhomogeneous elastic shell.

We introduce certain functional weighted spaces which make it possible to study and consequently to prove the existence and uniqueness of solutions of the boundary-value problem and the existence of solutions of the eigenvalue problem. The properties of the functions of these weighted spaces are also studied, and some embedding theorems are proved.

Contents

1. Introduction

Weighted spaces and their embedding theorems play a prominent role in the theory of degenerate elliptic equations. For such equations it is possible to give an exact formulation of boundary-value problems and to obtain necessary and sufficient conditions for the solvability of several boundary-value problems [9, 10, 11]. Functional weighted spaces play an important role in

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proving the existence and uniqueness of solutions of boundary-value problems and the existence of solutions of the eigenvalue problem, as well as in establishing the stability of the solution in the sense of the energy integral as the boundary values are varied [1, 3, 6, 9, 10, 11, 12, 13, 14, 15].

With this paper we try to establish the advantages of using the weighted Sobolev and weighted Lebesgue spaces in the boundary-value and eigenvalue problems. Here we consider boundary-value and eigenvalue problems for a system of differential equations with variable coefficients, some of which fail to be integrable in any neighborhood of zero. The motivation for the study of the shell vibration problem arises primarily from the association of this problem with different optimal control problems. The innovation of this paper is the introduction of certain functional weighted spaces and in studying their properties in order to prove existence and uniqueness results for the above mentioned boundary-value and eigenvalue problems.

It is worth mentioning that the main result of the paper is the key to derive the continuous dependence of the eigenvalues and eigenfunctions on the control perturbations as well as to prove the existence of solutions of the optimal control problems (see, e.g., [2]).

2. Basic notions and problem formulation

Let us start with an eigenvalue problem of the thin shell theory in the following formulation. Under certain assumptions, the transverse vibrations of the rotation shell (see Fig. 1) are described by the following system of differential equations [7, 8, 16, 17]

$$(Lu)_1 \equiv \big(rDW''\big)'' + \left(\left(\nu D' - \frac{D}{r}\right) W'\right)' + (f'\varphi)' = \lambda r h \rho W, \tag{2.1}$$

$$(Lu)_2 \equiv \big(ar\varphi'\big)' - \left(\frac{a}{r} + \nu a'\right)\varphi - f'W' = 0, \tag{2.2}$$

where W and φ are the unknowns, and $u = (W, \varphi)$.

Below we will verify that each equation here contains a coefficient which is nonintegrable in any neighborhood of 0.

Let us mention that in the case when the coefficients of the system of differential equations (2.1)–(2.2) are integrable then the existence and uniqueness of the solution of the system are solved in the general theory (see, e.g. [5]).

Now let us describe the quantities in the system of differential equations (2.1)–(2.2):

The variable $r \in [0, b]$ indicates the current radius;

$W(r)$ is the amplitude of the middle surface point displacements;

$\varphi(r)$ characterizes the tangential displacement;

$f(r)$ defines the shape of the middle surface of the rotation shell;

$\rho(r) > 0$ is the specific weight of the shell material;

$h(r)$ is the given thickness of the shell satisfying the condition

$$0 < h_0 \le h(r) \le h_1. \tag{2.3}$$

Finally, $D(r)$ is the bending stiffness, $D(r) \ge D_0 > 0$, determined by

$$D(r) = Eh^3(r)/\left(12\left(1-\nu^2\right)\right), \quad a(r) = 1/(Eh(r)), \tag{2.4}$$

where $b = const$, $b > 0$, $E > 0$ is Young's modulus and ν, $-1 < \nu < 0.5$, is Poisson's ratio of the shell material.

Notice that in the differential equation (2.2) the coefficient of the unknown φ is nonintegrable in $[0, b]$ due to the term $\frac{a}{r}$ there. Here we take into account also the relations (2.3) and (2.4).

Notice also that in the same way in the differential equation (2.1) the coefficient of W', i.e., the expression $\left(\nu D' - \frac{D}{r}\right)'$, as well as the coefficient of W'', i.e., the expression $\nu D' - \frac{D}{r}$ is nonintegrable. Here we use the well known fact that $\frac{1}{r^\alpha}$ is nonintegrable in $[0, b]$ if and only if $\alpha \ge 1$.

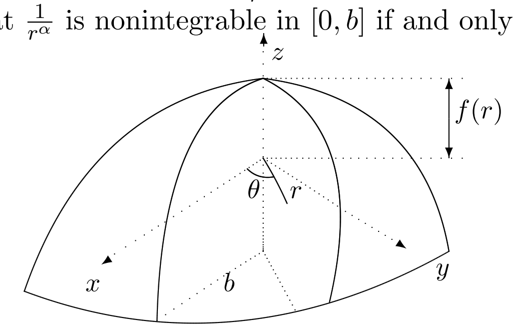

Fig. 1. The rotation shell

The eigenvalue problem is the system of equations (2.1), (2.2) together with the following boundary conditions for a rigidly supported shell:

$$\begin{aligned} W'\big|_{r=0} &= \frac{1}{r}\left[\left(rDW''\right)' + \left(\nu D' - \frac{D}{r}\right)W'\right]\Bigg|_{r=0} = 0, \\ W\big|_{r=b} &= -D\left(W'' + \nu\frac{W'}{r}\right)\Bigg|_{r=b} = 0, \\ a(\nu\varphi - r\varphi')\big|_{r=0} &= a(\nu\varphi - r\varphi')\big|_{r=b} = 0. \end{aligned} \tag{2.5}$$

The eigenvalue problem (2.1)–(2.5) can be associated with different optimal control problems [2, 4, 18].

In order to study the eigenvalue problem, we need to study the following problem:

$$(Lu)_1 = f_1 \tag{2.6}$$

$$(Lu)_2 = f_2 \tag{2.7}$$

and boundary conditions (2.5).

The first question that arises in the study of the problems (2.1)-(2.5) and (2.5)-(2.7) is how we can treat their solution. In fact, some of the system coefficients fail to be integrable in any neighborhood of zero. Therefore, a thorough analysis is required. For this purpose, we introduce specific weighted spaces in the next section.

3. Weighted spaces and embedding theorems

As is customary, we denote by $L_{1,loc}[0,b]$ the space of functions that are Lebesgue integrable on any segment contained strictly inside the segment $[0,b]$, and by $L_2[0,b]$ the space of functions, the square of which is integrable on $[0,b]$. We denote by v' the generalized derivative of v. Let us introduce the following weighted Hilbert spaces with the indicated inner products:

$$\langle u,v\rangle_{H_r^2} = \int\limits_0^b \left(ru''v'' + \frac{u'v'}{r} + uv \right) dr,$$

$$\langle u,v\rangle_{H_r^1} = \int\limits_0^b \left(ru'v' + \frac{uv}{r} \right) dr$$

and associated norms:

$$\|u\|_{H_r^2} = \left(\langle u,u\rangle_{H_r^2}\right)^{1/2}, \quad \|u\|_{H_r^1} = \left(\langle u,u\rangle_{H_r^1}\right)^{1/2}.$$

It should be noted that $H_r^2[0,b]$ consists of functions for which $\sqrt{r}\,v''$, $\frac{v'}{\sqrt{r}}$, $v \in L_2[0,b], v, v', v'' \in L_{1,loc}[0,b]$, while $H_r^1[0,b]$ consists of functions for which $\frac{v}{\sqrt{r}}, \sqrt{r}\,v' \in L_2[0,b]$, $v, v' \in L_{1,loc}[0,b]$.

Lemma 3.1. *The spaces $H_r^2[0,b]$ and $H_r^1[0,b]$ are Hilbert spaces.*

Proof. First, we prove the completeness of $H_r^2[0,b]$. Consider a Cauchy sequence $\{v_n\}_{n=1}^{\infty} \subset H_r^2[0,b]$.

From this we have

$$\int\limits_0^b (v_n - v_m)^2\, dr \to 0 \quad \text{and} \int\limits_0^b \left(\frac{v_n'}{\sqrt{r}} - \frac{v_m'}{\sqrt{r}} \right)^2 dr \to 0 \quad \text{as } n,m \in \infty. \tag{3.1}$$

Furthermore, from the completeness of the space $L_2[0,b]$ it follows that there exist functions $v,u \in L_2[0,b]$ such that

$$\int\limits_0^b (v - v_n)^2\, dr \to 0 \quad \text{and} \int\limits_0^b \left(u - \frac{v_n'}{\sqrt{r}} \right)^2 dr \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

Next, let us set $v_* = \sqrt{r}u$. We have that

$$\int_0^b \frac{v_*^2}{r}\,dr = \int_0^b u^2 dr < +\infty,$$

i.e., $\frac{v_*}{\sqrt{r}} \in L_2[0,b]$. But then from (3.1) it follows that

$$\int_0^b \frac{(v_* - v_n')^2}{r}\,dr \to 0 \quad \text{for } \; n \to \infty. \tag{3.3}$$

Now, let us show that $v_* = v'$. Since $v_n \in H_r^2[0,b], v_n, v_n' \in L_{1,loc}[0,b]$. Therefore, we have

$$\int_0^b v_n \varphi'\,dr = -\int_0^b v_n' \varphi\,dr \quad \text{for all } \varphi \in C_0^\infty[0,b]. \tag{3.4}$$

It is easy to see that

$$\left|\int_0^b \left(v_n' - v_*\right)\varphi\,dr\right| = \left|\int_\varepsilon^b \left(v_n' - v_*\right)\varphi\,dr\right| \leqslant$$
$$\leqslant b^{1/2}\left(\int_\varepsilon^b \frac{\left(v_n' - v_*\right)^2}{r}\,dr\right)^{1/2}\left(\int_\varepsilon^b \varphi^2\,dr\right)^{1/2} \to 0 \quad \text{for } \; n \to \infty$$
$$\text{and all } \varphi \in C_0^\infty[0,b]. \tag{3.5}$$

Similarly, from (3.1) we obtain that

$$\left|\int_0^b \left(v_n - v\right)\varphi'\,dr\right| \to 0 \quad \text{for } \; n \to \infty \text{ for all } \varphi \in C_0^\infty[0,b]. \tag{3.6}$$

By using the relations (3.5), (3.6), and by taking into account the equality (3.4) we get

$$\int_0^b v\varphi'\,dr = -\int_0^b v_*\varphi\,dr.$$

Hence, $v_* = v'$.

Next, by taking into account the relations (3.2) and (3.3) we conclude that

$$\int_0^b \left(\left(v - v_n\right)^2 + \frac{\left(v' - v_n'\right)^2}{r}\right)dr \to 0 \quad \text{as } \; n \to \infty. \tag{3.7}$$

It can be proved in a similar fashion that

$$\int_0^b r\left(v'' - v''_n\right)^2 dr \to 0 \quad \text{for } \ n \to \infty.$$

This, together with the relation (3.7), establishes the completeness of the space $H_r^2[0,b]$.

In the same manner we can prove that $H_r^1[0,b]$ is a Hilbert space. □

It should be noted that in addition to being complete these weighted spaces have a number of other remarkable properties.

Theorem 3.1. *We have the following embeddings*

$$H_r^2[0,b] \subset H^2[\delta,b], \qquad H_r^1[0,b] \subset H^1[\delta,b]$$

and estimates

$$\|u\|_{H^2[\delta,b]} \leqslant C_1 \|u\|_{H_r^2[0,b]}\,, \tag{3.8}$$

$$\|v\|_{H^1[\delta,b]} \leqslant C_1 \|v\|_{H_r^1[0,b]}\,, \tag{3.9}$$

where $u \in H_r^2[0,b]$, $v \in H_r^1[0,b]$ *and* $C_1 = \left(\max\{1/\delta, b, 1\}\right)^{1/2}$, $\delta > 0$.

Proof. Let us establish the estimate (3.8) which yields the embedding

$$H_r^2[0,b] \subset H^2[\delta,b].$$

It is easy to see that

$$\frac{(u')^2}{r} \leqslant \frac{1}{\delta} u'^2 \quad \text{and} \quad \left(ru''\right)^2 \leqslant bu''^2 \qquad \text{for all } r \in [\delta,b].$$

From this and the finiteness of

$$\int_\delta^b \left(u^2 + \frac{1}{\delta}u'^2 + bu''^2\right) dr$$

we find that $u \in H^2[\delta,b]$ and that the estimate (3.8) holds. The estimate (3.9) and therefore, the embedding $H_r^1[0,b] \subset H^1[\delta,b]$ can be proved similarly. □

Theorem 3.2. *Each function* $u \in H_r^2[0,b]$ *can be associated with a function on* $[0,b]$ *that is continuously differentiable. Likewise each function* $v \in H_r^1[0,b]$ *determines a continuous function on* $[0,b]$. *Further the estimates*

$$\max_{[0,b]} \left(|u'(r)| + |u(r)|\right) \leqslant C_2 \|u\|_{H_r^2}\,, \qquad u'(0) = 0 \quad \textit{and} \tag{3.10}$$

$$\max_{[0,b]} |v(r)| \leqslant \sqrt{5}\, \|v\|_{H_r^1}\,, \qquad v(0) = 0, \tag{3.11}$$

are satisfied, where $C_2 = \sqrt{5} + \left(\max(b + \frac{4}{b}, b^2+1)\right)^{1/2}$.

Proof. Consider an arbitrary function $u \in H_r^2[0,b]$. Theorem 3.1 implies that $u \in H^2[\delta,b]$. Hence, the function $u(r)$ can be associated with a continuously differentiable function on $[\delta,b]$.

Evidently, we have that

$$\begin{gathered}\left|u'^2(r)-u'^2(b)\right| = \left|\int\limits_r^b \left(u'^2\right)' \, d\xi\right| = \left|\int\limits_r^b 2u'u''\, d\xi\right| \leqslant \\ \leqslant \int\limits_r^b \left(\frac{u'^2}{\xi}+\xi u''^2\right) d\xi \leqslant \|u\|^2_{H_r^2[0,b]} \,. \end{gathered} \tag{3.12}$$

Further, note that $2u'u'' \in L_{1,loc}[0,b]$, which implies that $\int\limits_r^b 2u'u''\,d\xi$ is absolutely continuous. Hence, the limit $\lim\limits_{r\to 0}\int\limits_r^b 2u'u''\,d\xi = \lim\limits_{r\to 0}(u'^2(r)-u'^2(b))$ exists.

Since $u \in C^1[\delta,b]$, we conclude that the limit $\lim\limits_{r\to 0} u'^2(r)$ exists. This implies that the function $u'^2(r)$ is continuous on $[0,b]$. Furthermore, for any $\tau, r \in \left[\frac{b}{2},b\right]$ we get

$$u'^2(r) \leqslant 2u'^2(\tau) + 2\left(\int\limits_\tau^r u''d\xi\right)^2 \leqslant 2b\frac{u'^2(\tau)}{\tau} + 2\int\limits_{\frac{b}{2}}^b \xi u''^2 d\xi.$$

This implies that

$$u'^2(r) \leqslant 4\int\limits_{\frac{b}{2}}^b \left(\frac{u'^2}{\xi}+\xi u''^2\right) d\xi$$

and therefore from the estimate (3.12) we obtain

$$\max_{[0,b]}\left|u'^2(r)\right| \leqslant \|u\|^2_{H_r^2[0,b]} + u'^2(b) \leqslant 5\,\|u\|^2_{H_r^2[0,b]}\,.$$

In the same way, we derive

$$\max_{[0,b]}\left|u^2(r)\right| \leqslant \max(b+\frac{4}{b}, b^2+1)\int\limits_0^b \left(u^2+\frac{u'^2}{\xi}\right) d\xi.$$

Thus the inequality in the relation (3.10) holds with

$$C_2 = \sqrt{5} + \left(\max\{b+\frac{4}{b}, b^2+1\}\right)^{1/2}.$$

Next, we prove that $u'(0)=0$. Assume to the contrary that $u'(0)\neq 0$. In view of continuity of u'^2 we have $\lim_{r\to 0} u'^2(r) = a^2 \neq 0$. This implies that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $r, 0 \leqslant r \leqslant \delta$, $u'^2(r) > a^2 - \varepsilon$.

Set $\varepsilon = \frac{a^2}{2}$, then

$$\int_0^\delta \frac{u'^2(r)}{r}\, dr > \frac{a^2}{2} \int_0^\varepsilon \frac{dr}{r} = +\infty,$$

which is a contradiction. Thus proves that $u'(0) = 0$.

The second part of Theorem 3.2 can be proved similarly. □

We denote by $K[0,b]$ the space of functions in $C^\infty[0,b]$ whose derivatives vanish in a neighborhood of 0, and we denote by $M[0,b]$ the space of functions in $C^\infty[0,b]$ that vanish in a neighborhood of 0.

Lemma 3.2. *The space $K[0,b]$ is dense in $H_r^2[0,b]$ and the space $M[0,b]$ is dense in $H_r^1[0,b]$.*

Proof. If $v \in H_r^2[0,b]$, then $v' \in H_r^1[0,b]$. By Theorem 3.2, $v'(0) = 0$ and by Theorem 3.1 for any $\delta > 0$ we have that $u = v' \in H^1[\delta/2, b]$. But then, in view of the density of the space $C^\infty[\delta/2, b]$ in $H^1[\delta/2, b]$, there exists a sequence $\{u_n\}_{n=1}^\infty \subset C^\infty[\delta/2, b]$ such that

$$\|u - u_n\|_{H^1[\delta/2,b]} \to 0 \quad \text{for } n \to \infty. \tag{3.13}$$

The functions u_n are defined on $[\delta/2, b]$. We extend them to functions $w_n(r)$ defined on all of $[0,b]$ as follows (see Fig. 2):

$$w_n(r) = \begin{cases} 0, & 0 \leqslant r \leqslant \delta \\ u_n(2\delta)\dfrac{r-\delta}{\delta}, & \delta \leqslant r \leqslant 2\delta \\ u_n(r), & 2\delta \leqslant r \leqslant b \end{cases}$$

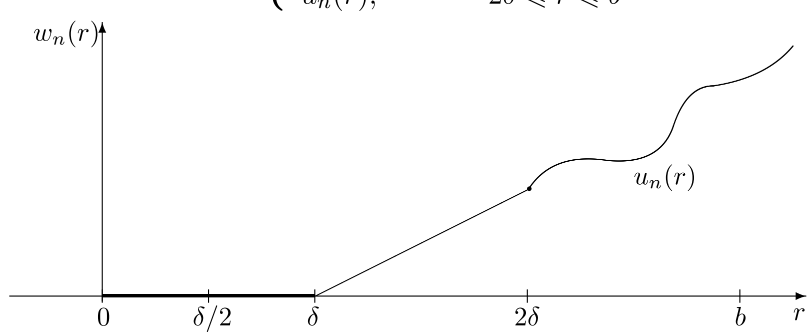

Fig. 2. Graph of w_n

Note that $w_n \in H^1[0,b]$, but $w_n \notin C^\infty[0,b]$. For the validity of the first statement of Lemma 3.2 we only need to replace the functions w_n with functions belonging to $C^\infty[0,b]$. In view of what was proved above, there exists a sequence $\{z_{n_m}\}_{m=1}^\infty \subset C^\infty[0,b]$ such that

$$\|w_n - z_{n_m}\|_{H^1[0,b]} \to 0 \quad \text{for } m \to \infty \tag{3.14}$$

for a fixed n, where

$$z_{n_m}(r) = 0 \qquad \forall r : 0 \leqslant r \leqslant \delta/2. \tag{3.15}$$

It is easy to see that $z_{n_m} \in H_r^1[0,b]$ and the following inequality holds:

$$\begin{aligned}\|u - z_{n_m}\|^2_{H_r^1[0,b]} &= \|u - z_{n_m}\|^2_{H_r^1[0,\delta/2]} + \|u - z_{n_m}\|^2_{H_r^1[\delta/2,b]} \leqslant \\ &\leqslant \|u - z_{n_m}\|^2_{H_r^1[0,\delta/2]} + 3\,\|u - u_n\|^2_{H_r^1[\delta/2,b]} + \\ &\quad + 3\,\|u_n - w_n\|^2_{H_r^1[\delta/2,b]} + 3\,\|w_n - z_{n_m}\|^2_{H_r^1[\delta/2,b]}\,. \end{aligned} \tag{3.16}$$

We have that

$$\begin{aligned}\|u_n - w_n\|^2_{H_r^1[\delta/2,b]} &\leqslant 2\,\|u_n\|^2_{H_r^1[\delta/2,2\delta]} + 2\left\|u_n(2\delta)\frac{r-\delta}{\delta}\right\|^2_{H_r^1[\delta,2\delta]} \leqslant \\ &\leqslant 4\,\|u\|^2_{H_r^1[\delta/2,2\delta]} + 4\,\|u - u_n\|^2_{H_r^1[\delta/2,\delta]} + \\ &\quad + 10\left(u^2(2\delta) + (u(2\delta) - u_n(2\delta))^2\right). \end{aligned} \tag{3.17}$$

By the well-known embedding theorem ($H^1[0,b] \subset C[0,b]$), we have

$$|u(2\delta) - u_n(2\delta)| \leqslant \max_{[\delta,2\delta]} |u(r) - u_n(r)| \leqslant$$

$$\leqslant \left(\max\left(\delta, \frac{1}{\delta}\right)\right)^{1/2} \|u - u_n\|_{H^1[\delta,2\delta]} = \frac{1}{\delta^{1/2}}\,\|u - u_n\|_{H^1[\delta,2\delta]}\,,$$

for sufficiently samll δ.

By taking into account the relations (3.15)-(3.17), this yields

$$\begin{aligned}\|u - z_{n_m}\|^2_{H_r^1[0,b]} &\leqslant 12\,\|u\|^2_{H_r^1[0,2\delta]} + 15\,\|u - u_n\|^2_{H_r^1[\delta/2,b]} + \\ &+ 30\left(u^2(2\delta) + \frac{1}{\delta}\,\|u - u_n\|^2_{H^1[\delta,2\delta]}\right) + 3\,\|w_n - z_{n_m}\|^2_{H_r^1[\delta/2,b]}\,. \end{aligned} \tag{3.18}$$

Next we prove that for any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that for any $\delta, 0 < \delta \leqslant \delta_0$, we have

$$\|u\|^2_{H_r^1[0,2\delta]} < \varepsilon \quad \text{and} \quad u^2(2\delta) < \varepsilon. \tag{3.19}$$

Indeed, the first inequality follows from the absolute continuity of the Lebesgue integral and the second inequality follows from the continuity of u and the equality $u(0) = 0$ proved above.

Let us fix a $\delta_1, 0 < \delta_1 \leqslant \delta_0$. By using the relation (3.13), let us choose $N_0(\delta_1, \varepsilon)$ such that for any $n, n \geqslant N_0$, the following estimate holds:

$$\|u - u_n\|^2_{H^1[\delta_1/2,b]} < \varepsilon\delta_1. \tag{3.20}$$

Next we fix an $n_1, n_1 \geqslant N_0$. By the relation (3.14) we can choose $M_0 = M_0(n_1, \delta_1, \varepsilon)$ such that for any $m, m \geqslant M_0$, we have

$$\|w_n - z_{n_m}\|^2_{H^1[0,b]} < \varepsilon. \tag{3.21}$$

By using this and the relations (3.18)-(3.20) we get

$$\|u - z_{n_m}\|_{H_r^1[0,b]} \to 0 \quad \text{for} \;\; m \to \infty.$$

Since $v' \in H_r^1[0,b]$, therefore, in view of what was proved above we have that there exists a sequence $z_m \in C^\infty[0,b]$ such that,

$$\left\|v' - z_m\right\|_{H_r^1[0,b]} \to 0 \quad \text{for} \;\; m \to \infty. \tag{3.22}$$

We define

$$y_m(r) = \int\limits_0^r z_m(\xi)\, d\xi + v(0) \tag{3.23}$$

and prove that they are the desired ones.

Obviously,

$$y_m'(r) = z_m(r) \qquad \forall r \in [0,b]. \tag{3.24}$$

This means that $y_m \in C^\infty[0,b]$. From the relation (3.23) we obtain

$$|v(r) - y_m(r)| = \left| \int\limits_0^r \left(v' - z_m\right)(\xi)\, d\xi \right| \leqslant b \left(\int\limits_0^b \frac{\left(v' - z_m\right)^2}{r}\, dr \right)^{1/2}.$$

This, in view of (3.22) and (3.24), implies

$$\|v - y_m\|_{H_r^2[0,b]} \to 0 \quad \text{for} \;\; m \to \infty.$$

The first statement of Lemma 3.2 is proved. The second statement can be proved similarly. □

In order to investigate the smoothness properties of a generalized solution of boundary-value or spectral problems, we need to introduce the following weighted spaces $\tilde{H}_r^2[0,b]$, $H_r^3[0,b]$ with the inner products:

$$\langle u, v\rangle_{\tilde{H}_r^2} = \int\limits_0^b \left(r u'' v'' + u'v' + \frac{uv}{r^2} \right) dr,$$

$$\langle u, v\rangle_{H_r^3} = \int\limits_0^b \left(r u''' v''' + u''v'' + \frac{u'v'}{r^2} + uv \right) dr,$$

respectively. Set $\|v\|_{\tilde{H}_r^2} = \left(\langle v, v\rangle_{\tilde{H}_r^2}\right)^{1/2}$, $\|v\|_{H_r^3} = \left(\langle v, v\rangle_{H_r^3}\right)^{1/2}$.

Thus we have

$$\tilde{H}_r^2[0,b] = \left\{ v : v, v', v'' \in L_{1,loc}[0,b], \|v\|^2_{\tilde{H}_r^2} < +\infty \right\},$$

$$H_r^3[0,b] = \left\{ v : v, v', v'', v''' \in L_{1,loc}[0,b], \|v\|^2_{H_r^3} < +\infty \right\}.$$

In the same way as in Lemma 3.1, we can prove the following

Lemma 3.3. *The spaces $\tilde{H}_r^2[0,b]$, $H_r^3[0,b]$ are complete.*

Below, in the next three lemmas we present some embedding properties of the functions from the weighted spaces $H_r^2[0,b], \tilde{H}_r^2[0,b], H_r^3[0,b]$.

Lemma 3.4. *The ball* $S = \Big\{u(r) : u(r) \in H_r^2[0,b], \|u\|_{H_r^2} \le R\Big\}$ *is precompact in the space* $L_2[0,b]$.

Proof. It is easily seen that for any function $u \in S$ we have that

$$I(h) := \int\limits_0^b (u(r+h) - u(r))^2 \, dr = \int\limits_0^b \left(\int\limits_r^{r+h} u'(\xi)\, d\xi \right)^2 dr \le$$
$$\le \int\limits_0^b \left(\int\limits_r^{r+h} \xi \, d\xi \cdot \int\limits_r^{r+h} \frac{(u'(\xi))^2}{\xi} \, d\xi \right) dr \le hb^2R^2.$$

Therefore $I(h) \to 0$ as $h \to 0$. Notice that we have $\int\limits_0^b u^2 \, dr \le R^2$ for any $u \in S$. Hence, equicontinuity and uniform boundedness of the set S are established. Therefore, by the Riesz-Frechet-Kolmogorov theorem, the ball S is precompact. $\square$

Similarly we can prove the following lemmas.

Lemma 3.5. *The ball* $S = \Big\{u(r) : u(r) \in H_r^1[0,b], \|u\|_{H_r^1} \le R\Big\}$ *is precompact in the space* $L_2[0,b]$.

Lemma 3.6. *The ball* $\tilde{S} = \Big\{u(r) : u(r) \in \tilde{H}_r^2[0,b], \|u\|_{\tilde{H}_r^2} \le R\Big\}$ *is precompact in the space* $H_r^1[0,b]$ *and the ball* $S = \Big\{u(r) : u(r) \in H_r^3[0,b], \|u\|_{H_r^3} \le R\Big\}$ *is precompact in the space* $H_r^2[0,b]$, *respectively.*

4. On the existence and uniqueness of generalized solution of the boundary-value problem

Next, let us turn to the problem (2.6)–(2.7) with boundary conditions (2.5).

Let $p(r) = f'(r)$ and V be the following linear subspace of the product space $H_r^2 \times H_r^1$:

$$V = \left\{ v : \, v = (v_1, v_2) \in H_r^2 \times H_r^1, v_1(b) = 0 \right\}.$$

For the norm in V we define:

$$\|v\|_V = \left(\|v_1\|_{H_r^2}^2 + \|v_2\|_{H_r^1}^2 \right)^{1/2}.$$

Let us consider the following bilinear form in the space V:

$$B(u,v)=\int\limits_0^b\Big[rDu_1''v_1''+(D-\nu D'r)\frac{u_1'v_1'}{r}-pu_2v_1'+$$
$$+aru_2'v_2'+(a+\nu a'r)\frac{u_2v_2}{r}+pu_1'v_2\Big]\,dr-a\nu u_2v_2|_0^b+\nu Du_1'v_1'\big|_0^b,$$

generated by the differential operator (2.6)–(2.7) and boundary conditions (2.5).

Now we are in a position to present the first of one main results:

Theorem 4.1. *Given functions $D(r)$, $D(r)-\nu D'(r)r$, $a'(r),h(r),\rho(r)\in L_\infty[0,b]$, $D(r)\in L_1[0,b]$ such that $D(r)\geqslant D_0>0$, $D(r)-\nu D'(r)r\geqslant D_{10}>0$, $a(r)\geqslant a_0>0$, $h(r)\geqslant h_0>0$, $\rho(r)\geqslant\rho_0>0$. Then for any function $f=(f_1,f_2)$, $f_1,f_2\in L_2[0,b]$ the problem*

$$B(u,v)=(f_1,v_1)_{L_2}-(f_2,v_2)_{L_2}\qquad\forall v\in V\tag{4.1}$$

has a unique solution $u\in V$. Moreover $\|u\|_V$ satisfies

$$\|u\|_V\leqslant\alpha^{-1}\left(\|f_1\|_{L_2}^2+\|f_2\|_{L_2}^2\right)^{1/2}.\tag{4.2}$$

Furthermore, the solution u satisfies the following boundary conditions

$$u_1(b)=u_1'(0)=u_2(0)=0,\tag{4.3}$$

in the classical sense.

Proof. In view of the completeness of the spaces H_r^2, H_r^1 and the estimate (3.10) we obtain the closedness of V.

Now, let us prove that the bilinear form $B(u,v)$ is V-elliptic, i.e.,

$$B(v,v)\geqslant\alpha\,\|v\|_V^2\quad\forall v\in V,\tag{4.4}$$

where $\alpha=\min\left(D_0,\frac{D_{10}}{2},\frac{D_{10}}{b^3},(1-\varepsilon)a_0,1-\frac{\nu^2}{\varepsilon}\right)$ and $\nu^2<\varepsilon<1$.

It is easy to see that

$$B(v,v)=\int\limits_0^b\left[rDv_1''^2+(D-\nu D'r)\frac{v_1'^2}{r}+arv_2'^2+(a+\nu a'r)\frac{v_2^2}{r}\right]dr-$$
$$-a\nu v_2^2\big|_0^b+\nu Dv_1'^2\big|_0^b.\tag{4.5}$$

By simplifying some members of $B(v,v)$ we obtain

$$\int_0^b \left[arv_2'^2 + (a+\nu a' r)\frac{v_2^2}{r} \right] dr - a\nu v_2^2\big|_0^b =$$

$$= \int_0^b \left[arv_2'^2 + (a+\nu a' r)\frac{v_2^2}{r} - a'\nu v_2^2 - 2a\nu v_2 v_2' \right] dr \geqslant$$

$$\geqslant \int_0^b \left[(1-\varepsilon)arv_2'^2 + a\left(1-\frac{\nu^2}{\varepsilon}\right)\frac{v_2^2}{r} \right] dr. \quad (4.6)$$

Since $v_1(r) = -\int_r^b v_1'(\xi)\,d\xi$ we have

$$\int_0^b v_1^2 dr \leqslant \int_0^b \left(\int_0^b \xi\,d\xi \int_0^b \frac{v_1'^2}{\xi}\,d\xi \right) dr \leqslant \frac{b^3}{2}\int_0^b \frac{v_1'^2}{r}\,dr. \quad (4.7)$$

By taking ν such that $\nu^2 < \varepsilon < 1$ in relations (4.5)–(4.7) we obtain(4.4).

It is easy to prove the boundedness of $B(u,v)$ and of the linear functional $(f_1,v_1)_{L_2} - (f_2,v_2)_{L_2}$ on V, i.e.,

$$|B(u,v)| \leqslant N\,\|u\|_V\,\|v\|_V\,, \qquad |(f_1,v_1)_{L_2} - (f_2,v_2)_{L_2}| \leqslant L\,\|v\|_V\,,$$

where N and L are constants.
Therefore, by the well-known Lax-Milgram Lemma [5] there exists a unique solution of the problem (4.1). Thus the estimate (4.2) is true.

The last statement (4.3) of Theorem 4.1 follows from the statements (3.10) and (3.11) of Theorem 3.2.

Thus Theorem 4.1 is proved. □

5. On the existence of solutions of the eigenvalue problem

Let us consider the following eigenvalue problem

$$B(u,v) = \lambda(rh\rho u_1, v_1)_{L_2} \qquad \forall v \in V. \quad (5.1)$$

Consider an arbitrary function $\psi \in L_2[0,b]$. Then on setting $f = \binom{\sqrt{rh\rho}\,\psi}{0}$ in (4.1) we get the following relation

$$B(u,v) = (\sqrt{rh\rho}\psi, v_1)_{L_2} \qquad \forall v \in V. \quad (5.2)$$

From this and Theorem 4.1 we have that $u \in V$ is determined uniquely. Thus, the operator $G : \psi \in L_2[0,b] \to u = G\psi \in V$ is well defined and in view of (4.2), the following estimate holds:

$$\|u\|_{H_r^2\times H_r^1} = \|G\psi\|_{H_r^2\times H_r^1} \leq \alpha^{-1}\left\|\sqrt{rh\rho}\,\psi\right\|_{L_2}. \quad (5.3)$$

Next, we prove that the operator $F_1\psi = \sqrt{rh\rho}(G\psi)_1$, as an operator mapping $L_2[0,b] \to L_2[0,b]$, is compact and self-adjoint. Let us substitute $u = \binom{v_1}{0}$ and $v = \binom{0}{-v_2}$ in (5.2). By adding the resulting equation we get

$$B_1(G\psi, v) = (\sqrt{rh\rho}\psi, v_1)_{L_2} \qquad \forall v \in V, \tag{5.4}$$

where $B_1(u,v)$ is a symmetric, bilinear form already. Thus, the solution of the problem (5.2) is the solution of the equation (5.4). The converse is also true: the solution of (5.4) is the solution of (5.2).

The operator F_1 defined on $L_2[0,b]$ can be represented as

$$F_1\psi = \sqrt{rh\rho} \cdot I \cdot (G\psi)_1,$$

where I is an embedding operator from $H^2_r[0,b]$ to $L_2[0,b]$. With the help of Lemma 3.4 we obtain that the operator I is compact. Next, from the estimate (5.3) we get that the operator $(G)_1$ is bounded. Thus, the operator $I \cdot (G)_1$ is compact. From the symmetry and boundedness of the operator F_1, we get that it is self-adjoint.

Let us consider the following eigenvalue problem:

$$F_1\psi = \mu\psi. \tag{5.5}$$

We have the following

Lemma 5.1. *There exist eigenvalues $\mu_1, \mu_2, ..., \mu_k, ...,\quad \mu_k \to 0, \mu_k > 0$, of the operator F_1. Moreover, each eigenvalue has a finite multiplicity and the corresponding sequence of eigenfunctions ψ_k makes up a complete orthonormal system in $L_2[0,b]$.*

Proof. Since the operator F_1 is compact and self-adjoint we need to prove only that $F_1\psi = 0$ implies $\psi = 0$. Indeed, from $\sqrt{rh\rho}(G\psi)_1 = 0$ we get that $(G\psi)_1 = 0$. Then, by substituting in (5.2) $v = u = G\psi$, we obtain

$$\int\limits_0^b \left(aru_2'^2 + \left(a + \nu a' r\right) \frac{u_2^2}{r} \right) dr - a\nu u_2^2|_0^b = 0.$$

Then by using the estimate (4.4) for $v = (0, u_2)$, we get $\alpha \left\| u_2 \right\|^2_{H^1_r} \le 0$, i.e., $u_2 = 0$. Since, we have that $u_1 = (G\psi)_1 = 0$, we get $u = (u_1, u_2) \equiv 0$. Therefore, from (5.3) we obtain that $\left(\sqrt{rh\rho}\psi, v_1\right)_{L_2} = 0$ for any $v \in V$. Now, let us prove that $\psi = 0$. Indeed, by taking into account that $v_1'(0) = v_1(b) = 0$, we get

$$\begin{aligned} 0 = \int\limits_0^b \sqrt{rh\rho}\psi v_1\, dr = \int\limits_0^b \left(\int\limits_0^r \sqrt{\xi h\rho}\psi\, d\xi \right)' v_1\, dr = \\ = \int\limits_0^b \left(\int\limits_b^r \int\limits_0^z \sqrt{\xi h\rho}\psi\, d\xi\, dz \right) v_1''\, dr. \end{aligned} \tag{5.6}$$

It is easily seen that $\binom{v_1}{0} \in V$, where $v_1'' = \int\limits_0^r \int\limits_0^z \sqrt{\xi h\rho}\psi \, d\xi \, dz$. By substituting this in (5.6), we obtain the following $\int\limits_0^b \left(\int\limits_0^r \int\limits_0^z \sqrt{\xi h\rho}\psi \, d\xi \, dz \right)^2 dr = 0$, i.e., $\sqrt{rh\rho}\psi = 0$ for any $r \in [0, b]$. Therefore, $\psi = 0$. ☐

Denote by $L_{2,rh\rho}$ the weighted space with the following norm $\|u\| = \left(\int\limits_0^b u^2 rh\rho \, dr \right)^{\frac{1}{2}}$. Clearly $L_2 \subset L_{2,rh\rho}$.

Now we are in a position to present the second main result of this paper.

Theorem 5.1. *Given the framework of Theorem 4.1, the following statements hold*

(a) *There exists a sequence of positive eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_k, ...\}$ of the problem (5.1), with $\lambda_k \to +\infty \quad for \quad k \to \infty$.*
(b) *To each eigenvalue λ_k there corresponds only finite number of linear independent eigenfunctions from V.*
(c) *The system $\{u_{1k}\}_{k=1}^{\infty}$ of the first components of the eigenfunctions forms a complete orthonormal system with the weight $rh\rho$.*
(d) *For any function $u \in L_{2,rh\rho}$ the following Fourier decomposition takes place in the $L_{2,rh\rho}$ norm:*

$$u = \sum_{k=1}^{\infty} a_k u_{1k}$$

Proof. Suppose that μ is an eigenvalue of F_1 and ψ is a corresponding eigenfunction, i.e., $F_1\psi = \mu\psi$. Then $\lambda = \frac{1}{\mu}$ is an eigenvalue and $u = G\psi$ is a corresponding eigenfunction of the problem (5.1).

Indeed, we have that

$$\begin{aligned} B(u, v) = B(G\psi, v) = (\sqrt{rh\rho}\,\psi, v_1)_{L_2} = \frac{1}{\mu}(rh\rho(G\psi)_1, v_1)_{L_2} = \\ = \lambda(rh\rho u_1, v_1)_{L_2} \quad \forall v \in V. \end{aligned}$$

In the same way, we can prove that if λ is an eigenvalue and u is a corresponding eigenfunction of the problem (5.1), then $\mu = \frac{1}{\lambda}$ is an eigenvalue and $\psi = \sqrt{rh\rho}u_1$ is a corresponding eigenfunction of F_1.

Thus, by taking into account Lemma 5.1, to complete the proof it remains to verify the positiveness of eigenvalues λ_k. Indeed,

$$\alpha \|u\|_V^2 \leq B(u, u) = \lambda \left(rh\rho u_1, u_1 \right)_{L_2}.$$

☐

6. Conclusion

We introduced certain functional weighted spaces generated by an eigenvalue problem describing vibrations of an elastic shell. We studied the properties of these spaces and proved a series of embedding results. As an application the existence and uniqueness of the generalized solution of the boundary-value problem (2.5)–(2.7) and the existence of generalized solutions of the eigenvalue problem (2.1)–(2.5) have been established.

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