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# Computations of de Rham cohomology rings of classifying stacks at torsion primes

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Abstract. We compute the de Rham cohomology rings of $BG_2$ and $B\mathrm{Spin}(n)$ for $7 \leq n \leq 11$ over base fields of characteristic 2.

Contents



## Introduction

Let $G$ be a smooth affine algebraic group over a commutative ring $R$. In [17], Totaro defines the Hodge cohomology group $H^i(BG, \Omega^j)$ for $i, j \geq 0$ to be the $i$th étale cohomology group of the sheaf of differential forms $\Omega^j$ over $R$ on the big étale site of the classifying stack $BG$. For $n \geq 0$, let $H^n_{\mathrm{H}}(BG/R) := \oplus_j H^j(BG, \Omega^{n-j})$ denote the total Hodge cohomology group of degree $n$. De Rham cohomology groups $H^n_{\mathrm{dR}}(BG/R)$ are defined to be the étale cohomology groups of the de Rham complex of $BG$. Let $\mathfrak{g}$ denote the Lie algebra associated to $G$ and let $O(\mathfrak{g}) = S(\mathfrak{g}^*)$ denote the ring of polynomial functions on $\mathfrak{g}$. In [17, Corollary 2.2], Totaro showed that the Hodge cohomology of $BG$ is related to the representation theory of $G$:

$$H^i(BG, \Omega^j) \cong H^{i-j}(G, S^j(\mathfrak{g}^*)).$$

Let $G$ be a split reductive group defined over $\mathbb{Z}$. From the work of Bhatt-Morrow-Scholze in p-adic Hodge theory [1, Theorem 1.1], one might expect that

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$$\dim_{\mathbb{F}_p} H^i_{\mathrm{dR}}(BG_{\mathbb{F}_p}/\mathbb{F}_p) \geq \dim_{\mathbb{F}_p} H^i(BG_{\mathbb{C}}, \mathbb{F}_p) \tag{1}$$

for all primes $p$ and $i \geq 0$. The results from [1] do not immediately apply to $BG$ since $BG$ is not proper as a stack over $\mathbb{Z}$. For $p$ a non-torsion prime of a split reductive group $G$ defined over $\mathbb{Z}$, Totaro showed that

$$H^*_{\mathrm{dR}}(BG_{\mathbb{F}_p}/\mathbb{F}_p) \cong H^*(BG_{\mathbb{C}}, \mathbb{F}_p) \tag{2}$$

[17, Theorem 9.2]. It remains to compare $H^*_{\mathrm{dR}}(BG_{\mathbb{F}_p}/\mathbb{F}_p)$ with $H^*(BG_{\mathbb{C}}, \mathbb{F}_p)$ for $p$ a torsion prime of $G$. For $n \geq 3$, 2 is a torsion prime for the split group $SO(n)$. Totaro showed that

$$H^*_{\mathrm{dR}}(BSO(n)_{\mathbb{F}_2}/\mathbb{F}_2) \cong H^*(BSO(n)_{\mathbb{C}}, \mathbb{F}_2) \cong \mathbb{F}_2[w_2, \ldots, w_n]$$

as graded rings where $w_2, \ldots, w_n$ are the Stiefel-Whitney classes [17, Theorem 11.1]. In general, the rings $H^*_{\mathrm{dR}}(BG_{\mathbb{F}_p}/\mathbb{F}_p)$ and $H^*(BG_{\mathbb{C}}, \mathbb{F}_p)$ are different though. For example,

$$\dim_{\mathbb{F}_2} H^{32}_{\mathrm{dR}}(B\mathrm{Spin}(11)_{\mathbb{F}_2}/\mathbb{F}_2) > \dim_{\mathbb{F}_2} H^{32}(B\mathrm{Spin}(11)_{\mathbb{C}}, \mathbb{F}_2)$$

[17, Theorem 12.1].

In this paper, we verify inequality (1) for more examples. For the torsion prime 2 of the split reductive group $G_2$ over $\mathbb{Z}$, we show that

$$H^*_{\mathrm{dR}}(B(G_2)_{\mathbb{F}_2}/\mathbb{F}_2) \cong H^*(B(G_2)_{\mathbb{C}}, \mathbb{F}_2) \cong \mathbb{F}_2[y_4, y_6, y_7]$$

as graded rings where $|y_i| = i$ for $i = 4, 6, 7$. For the spin groups, we show that

$$H^*_{\mathrm{dR}}(B\mathrm{Spin}(n)_{\mathbb{F}_2}/\mathbb{F}_2) \cong H^*(B\mathrm{Spin}(n)_{\mathbb{C}}, \mathbb{F}_2) \tag{3}$$

for $7 \leq n \leq 10$. Note that 2 is a torsion prime for $\mathrm{Spin}(n)$ for $n \geq 7$. The isomorphism (3) holds for $1 \leq n \leq 6$ by the "accidental" isomorphisms for spin groups along with (2).

For $n = 11$, we make a full computation of the de Rham cohomology ring of $B\mathrm{Spin}(n)_{\mathbb{F}_2}$ :

$$H^*_{\mathrm{dR}}(B\mathrm{Spin}(11)_{\mathbb{F}_2}/\mathbb{F}_2) \cong \mathbb{F}_2[y_4, y_6, y_7, y_8, y_{10}, y_{11}, y_{32}]/(y_7 y_{10} + y_6 y_{11})$$

where $|y_i| = i$ for all $i$. We can compare this result with the computation of the singular cohomology of $B\mathrm{Spin}(11)_{\mathbb{C}}$ given by Quillen [14]:

$$H^*(B\mathrm{Spin}(11)_{\mathbb{C}}, \mathbb{F}_2) \cong \mathbb{F}_2[w_4, w_6, w_7, w_8, w_{10}, w_{11}, w_{64}]/(w_7 w_{10} + w_6 w_{11}, \\ w_{11}^3 + w_{11}^2 w_7 w_4 + w_{11} w_8 w_7^2)$$

where $|w_i| = i$ for all $i$. Equivalently,

$$H^*(B\mathrm{Spin}(11)_{\mathbb{C}}, \mathbb{F}_2) \cong H^*(BSO(11)_{\mathbb{C}}, \mathbb{F}_2)/J \otimes \mathbb{F}_2[w_{64}]$$

where $J$ is the ideal generated by the regular sequence

$$w_2, Sq^1(w_2), Sq^2 Sq^1(w_2), \ldots, Sq^{16} Sq^8 \cdots Sq^1 w_2.$$

Thus, the rings $H^*_{\mathrm{dR}}(B\mathrm{Spin}(n)_{\mathbb{F}_2}/\mathbb{F}_2)$ and $H^*(B\mathrm{Spin}(n)_{\mathbb{C}}, \mathbb{F}_2)$ are not isomorphic in general even though $H^*_{\mathrm{dR}}(BSO(n)_{\mathbb{F}_2}/\mathbb{F}_2) \cong H^*(BSO(n)_{\mathbb{C}}, \mathbb{F}_2)$ for all $n$. Steenrod squares on de Rham cohomology over a base field of characteristic 2 have not yet been constructed. If they exist, our calculation suggests that their action on $H^*_{\mathrm{dR}}(BSO(n)_{\mathbb{F}_2}/\mathbb{F}_2) \cong H^*(BSO(n)_{\mathbb{C}}, \mathbb{F}_2)$ would have to be different from the action of the topological Steenrod operations.

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## 1. Preliminaries

In this section, we recall results from [17] that will be used in our computations. These results were also used by Totaro in [17, Theorem 11.1] to compute the de Rham cohomology of $BSO(n)_k$ for $k$ a field of characteristic 2.

The first result we mention [17, Proposition 9.3] is an analogue of the Leray-Serre spectral sequence from topology.

**Proposition 1.1.** *Let $G$ be a split reductive group defined over a field $F$ and let $P$ be a parabolic subgroup of $G$ with Levi quotient $L$ (this means that $P \cong R_u(P) \rtimes L$ where $R_u(P)$ is the unipotent radical of $P$* [2, 14.19]*). There exists a spectral sequence of algebras*

$$E_2^{i,j} = H^i_{\mathrm{H}}(BG/F) \otimes H^j_{\mathrm{H}}((G/P)/F) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/F).$$

Proposition 1.1 is the main tool that we will use to compute Hodge cohomology rings of classifying stacks. To apply Proposition 1.1, we will choose a parabolic subgroup $P$ for which $H^*_{\mathrm{H}}(BL/F)$ is a polynomial ring.

To fill in the 0th column of the $E_2$ page in Proposition 1.1, we use a result of Srinivas [15].

**Proposition 1.2.** *Let $G$ be split reductive over a field $F$ and let $P$ be a parabolic subgroup of $G$. The cycle class map*

$$CH^*(G/P) \otimes_{\mathbb{Z}} F \to H^*_{\mathrm{H}}((G/P)/F)$$

*is an isomorphism.*

Under the cycle class map, $CH^i(G/P) \otimes_{\mathbb{Z}} F$ maps to $H^i(G/P, \Omega^i)$. From the work of Chevalley [5] and Demazure [6], $CH^*(G/P)$ is independent of the field $F$ and is isomorphic to the singular cohomology ring $H^*(G_{\mathbb{C}}/P_{\mathbb{C}}, \mathbb{Z})$.

The last piece of information we will use to compute $H^*_{\mathrm{H}}(BG/F)$ is the ring of $G$-invariants $O(\mathfrak{g})^G = \oplus_i H^i(BG, \Omega^i)$. Let $T$ be a maximal torus in $G$ with Lie algebra $\mathfrak{t}$ and Weyl group $W$. There is a restriction homomorphism

$$O(\mathfrak{g})^G \to O(\mathfrak{t})^W. \tag{4}$$

We will need the following theorem which is due to Chaput and Romagny [4, Theorem 1.1]. For the following theorem, a split algebraic group $G$ over a field $F$ is simple if every proper smooth normal connected subgroup of $G$ is trivial.

**Theorem 1.3.** *Assume that $G$ is simple over a field $F$. Then the restriction homomorphism (4) is an isomorphism unless* $\operatorname{char}(F) = 2$ *and $G_{\overline{F}}$ is a product of copies of $Sp(2n)$ for some $n \in \mathbb{N}$.*

From the rings $O(\mathfrak{g})^G, CH^*(G/P), H^*_{\mathrm{H}}(BL/F)$, we will be able to determine the $E_\infty$ terms of the spectral sequence in Proposition 1.1. This will allow us to determine $H^*_{\mathrm{H}}(BG/F)$ by using the following version of the Zeeman comparison theorem [12, Theorem VII.2.4].

**Theorem 1.4.** *Fix a field $F$. Let $\{\bar{E}_r^{i,j}\}, \{E_r^{i,j}\}$ be first quadrant (cohomological) spectral sequences of $F$-vector spaces such that $\bar{E}_2^{i,j} = \bar{E}_2^{i,0} \otimes_F \bar{E}_2^{0,j}$ and $E_2^{i,j} = E_2^{i,0} \otimes_F E_2^{0,j}$ for all $i, j$. Let $\{f_r^{i,j} : \bar{E}_r^{i,j} \to E_r^{i,j}\}$ be a morphism of spectral sequences such that $f_2^{i,j} = f_2^{i,0} \otimes f_2^{0,j}$ for all $i, j$. Fix $N, Q \in \mathbb{N}$. Assume that $f_\infty^{i,j}$ is an isomorphism for all $i, j$ with $i + j < N$ and an injection for $i + j = N$. If $f_2^{0,i}$ is an isomorphism for all $i < Q$ and an injection for $i = Q$, then $f_2^{i,0}$ is an isomorphism for all $i < \min(N, Q + 1)$ and an injection for $i = \min(N, Q + 1)$.*

We recall a result from [17, Section 11] on the degeneration of the Hodge spectral sequence for split reductive groups, under some assumptions. The result in [17, Section 11] was proved for the special orthogonal groups but the proof works more generally.

**Proposition 1.5.** *Let $G$ be a split reductive group over a field $F$ and assume that the Hodge cohomology ring of $BG$ is generated as an $F$-algebra by classes in $\oplus_i H^{i+1}(BG, \Omega^i)$ and $\oplus_i H^i(BG, \Omega^i)$. Then the Hodge spectral sequence*

$$E_1^{i,j} = H^j(BG, \Omega^i) \Rightarrow H^{i+j}_{\mathrm{dR}}(BG/F) \tag{5}$$

*for $BG$ degenerates at the $E_1$ page.*

**Proof.** From [17, Lemma 8.2], there are natural maps

$$H^i(BG, \Omega^i) \to H^{2i}_{\mathrm{dR}}(BG/F)$$

and

$$H^{i+1}(BG, \Omega^i) \to H^{2i+1}_{\mathrm{dR}}(BG/F)$$

for all $i \geq 0$. These maps are compatible with products. Let $T$ denote a maximal torus of $G$. From the group homomorphism $T \to G$, we have the commuting square

$$\begin{array}{ccc} \oplus_i H^i(BG,\Omega^i) & \longrightarrow & \oplus_i H^{2i}_{\mathrm{dR}}(BG/F) \\ \big\downarrow & & \big\downarrow \\ \oplus_i H^i(BT,\Omega^i) & \xrightarrow{\cong} & H^{2i}_{\mathrm{dR}}(BT/F). \end{array} \tag{6}$$

The restriction homomorphism (4) induces an injection

$$\oplus_i H^i(BG,\Omega^i) \to \oplus_i H^i(BT,\Omega^i)$$

[17, Lemma 8.2]. Hence, from diagram (6), we get that the natural map

$$\oplus_i H^i(BG,\Omega^i) \to \oplus_i H^{2i}_{\mathrm{dR}}(BG/F)$$

is an injection. Hence, any differentials into the diagonal in the spectral sequence (5) must be 0. Then all classes in $\oplus_i H^{i+1}(BT,\Omega^i)$ must be permanent cycles (an element $x$ in the $E_2$ page of a spectral sequence $E_*$ is called a permanent cycle if $d_i(x) = 0$ for all $i \geq 2$) in (5). Classes in $\oplus_i H^i(BT,\Omega^i)$ must be permanent cycles in the spectral sequence (5) since $H^i(BG,\Omega^j) = 0$ for $i < j$ by [17, Corollary 2.2]. This proves that the Hodge spectral sequence for $BG$ degenerates. □

The following definition will be used later to describe the Hodge cohomology of flag varieties.

*Definition* 1.6. Let $F$ be a field. For variables $x_1,\dots,x_n$ let $\Delta(x_1,\dots,x_n)$ denote the $F$-vector space with basis given by the products $x_{i_1}\cdots x_{i_r}$ for $1 \leq i_1 < i_2 < \cdots < i_r \leq n$.

## 2. $G_2$

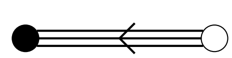
Let $k$ be a field of characteristic 2 and let $G$ denote the split form of $G_2$ over $k$.

**Theorem 2.1.** *The Hodge cohomology ring of $BG$ is freely generated as a commutative $k$-algebra by generators $y_4 \in H^2(BG,\Omega^2)$, $y_6 \in H^3(BG,\Omega^3)$, and $y_7 \in H^4(BG,\Omega^3)$. The Hodge spectral sequence for $BG$ degenerates at $E_1$ and we have*

$$H^*_{\mathrm{dR}}(BG/k) \cong H^*_{\mathrm{H}}(BG/k) = k[y_4,y_6,y_7].$$

From the computation [12, Corollary VII.6.3] of the singular cohomology ring of $B(G_2)_{\mathbb{C}}$ with $\mathbb{F}_2$-coefficients, we then have $H^*(B(G_2)_{\mathbb{C}},k) \cong H^*_{\mathrm{dR}}(BG/k)$.

**Proof.** We first choose a suitable parabolic subgroup of $G$. Let $P$ be the parabolic subgroup of $G$ corresponding to inclusion of the long root.



From Proposition 1.2, $CH^*(G/P)$ is independent of the field $k$ and the characteristic of $k$. As discussed in [9, §23.3], if we consider $(G_2)_{\mathbb{C}}$ over $\mathbb{C}$ along with the corresponding parabolic subgroup $P_{\mathbb{C}}$, $(G_2)_{\mathbb{C}}/P_{\mathbb{C}}$ is isomorphic to a smooth quadric $Q_5$ in $\mathbb{P}^6$. Hence, by [8, Chapter XIII], $H^*_{\mathrm{H}}((G/P)/k)$ is isomorphic to

$$CH^*(Q_5) \otimes_{\mathbb{Z}} k \cong k[v,w]/(v^6, w^2, v^3 - 2w) = k[v,w]/(v^3, w^2)$$

where $|v| = 2$ and $|w| = 6$ in $H^*_{\mathrm{H}}((G/P)/k)$ .

We next show that the Levi quotient $L$ of $P$ is isomorphic to $GL(2)_k$. This can be seen by constructing an isomorphism from the root datum of $GL(2)_k$ to the root datum of the Levi quotient. Let $(X_1, R_1, X_1^\vee, R_1^\vee)$ be the usual root datum of $GL(2)_k$ where $X_1 = \mathbb{Z}\chi_1 + \mathbb{Z}\chi_2$, $R_1 = \mathbb{Z}(\chi_1 - \chi_2)$, and we take our torus to be the set of diagonal matrices in $GL(2)_k$. We take $(X_2, R_2, X_2^\vee, R_2^\vee)$ to be the root datum of $G$ as described in [3, Plate IX]. Here, $X_2 = \{(a,b,c) \in \mathbb{Z}^3 \mid a+b+c = 0\}$. The long root $\alpha$ for $G$ is then $(-2,1,1)$ and the root datum of $P/R_u(P)$ is $(X_2, \pm\alpha, X_2^\vee, \pm\frac{1}{3}\alpha)$. An isomorphism from the root datum of $GL(2)_k$ to the root datum of $G$ can then be obtained from the isomorphism

$$X_1 \to X_2$$
$$\chi_1 \longmapsto (-1,1,0), \chi_2 \longmapsto (1,0,-1).$$

Thus, $L \cong GL(2)_k$.

We now analyze the spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/k) \tag{7}$$

from Proposition 1.1. From [7, Proposition] and [10, II.4.22],

$$H^*_{\mathrm{H}}(BL/k) = S^*(\mathfrak{gl}_2)^{GL(2)_k} \cong S^*(\mathfrak{t})^{S_2} = k[x_1, x_2]$$

where $x_1 \in H^1(BL, \Omega^1)$ and $x_2 \in H^2(BL, \Omega^2)$. Here, $\mathfrak{t}$ is the space of all diagonal matrices in $\mathfrak{gl}_2$ and $S_2$ acts on $\mathfrak{t}$ by permuting the diagonal entries.

In order to compute $H^*_{\mathrm{H}}(BG/k)$ from the spectral sequence above, we must first compute the ring of invariants of $S^*(\mathfrak{g}_2)^G$. From Theorem 1.3, $S^*(\mathfrak{g}_2)^G \cong S^*(\mathfrak{t}_\circ)^W$ where $\mathfrak{t}_\circ$ is the Lie algebra of a maximal torus $T$ in $G$ and $W$ is the corresponding Weyl group of $G$. By [17, Corollary 2.2],

$$H^i(BG, \Omega^i) \cong S^i(\mathfrak{t}_\circ)^W$$

for $i \geq 0$.

**Proposition 2.2.** *The ring of invariants $S^*(\mathfrak{t}_\circ)^W$ is equal to $k[y_4, y_6]$ where $|y_4| = 2$ and $|y_6| = 3$ in $S^*(\mathfrak{t}_\circ)^W$ .*

**Proof.** Following the notation in [3, Plate IX], $W \cong Z_2 \times S_3$ acts on the root lattice $X_2 = \{(a,b,c) \in \mathbb{Z}^3 \mid a+b+c = 0\}$ by multiplication by $-1$ and by permuting the coordinates. Hence, since we are working in characteristic 2, $W$ acts on $S^*(\mathfrak{t}_\circ) = k[t_1, t_2, t_3]/(t_1 + t_2 + t_3)$ by permuting $t_1$, $t_2$, and $t_3$. We then have $S^*(\mathfrak{t}_\circ)^W = k[t_1t_2 + t_1t_3 + t_2t_3, t_1t_2t_3] = k[y_4, y_6]$. $\square$

We can now carry out the computation of $H^*_{\mathrm{H}}(BG/k)$. First, we show that the class $v \in E_2^{0,2}$ is a permanent cycle. Consider the filtration on $H^2_{\mathrm{H}}(BL/k) = k{\cdot}v$ given by (7): $H^2_{\mathrm{H}}(BL/k) \hookleftarrow E_\infty^{2,0}$, where $H^2_{\mathrm{H}}(BL/k)/E_\infty^{2,0} \cong E_\infty^{0,2}$. Here, $E_2^{1,1} = 0$ and

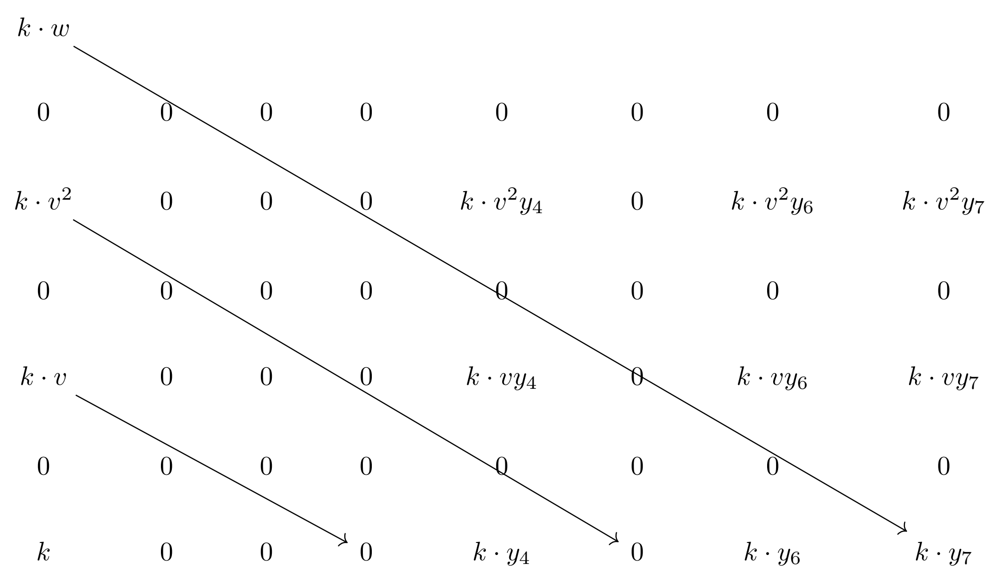
$$E_\infty^{2,0} = E_2^{2,0} = H^2_{\mathrm{H}}(BG/k) = H^1(BG, \Omega^1)$$

(we have $H^2(BG, \mathcal{O}) = 0$ since $H^2(BL, \mathcal{O}) = 0$ and there are no differentials entering $E_2^{2,0}$ ) since $H^*_{\mathrm{H}}((G/P)/k) = \oplus_i H^i(G/P, \Omega^i)$ is concentrated in even degrees. Hence,

$$E_\infty^{2,0} = H^2_{\mathrm{H}}(BG/k) = H^1(BG, \Omega^1) = 0,$$

by Proposition 2.2. It follows that $E_\infty^{0,2} \cong E_2^{0,2} = k \cdot v$ which implies that $d_3(v) = 0$. As (7) is a spectral sequence of algebras, it follows that $v$ and $v^2$ are permanent cycles. Using that $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees, we then get that $H^3_{\mathrm{H}}(BG/k) = E_2^{3,0} = E_\infty^{3,0} = 0$ and $H^5_{\mathrm{H}}(BG/k) = E_2^{5,0} = E_\infty^{5,0} = 0$.

Next, we show that $w \in H^6_{\mathrm{H}}((G/P)/k) = E_2^{0,6}$ is transgressive with $0 \neq d_7(w) \in E_7^{7,0}$. Note that $\dim_k H^6_{\mathrm{H}}(BL/k) = 2$. As $v$ is a permanent cycle in $E_*$, we observe that $E_\infty^{4,2} \cong E_2^{4,2} \cong k{\cdot}y_4 \otimes_k k{\cdot}v \cong k$ and $E_\infty^{6,0} \cong E_2^{6,0} \cong k{\cdot}y_6 \cong k$. Hence, $\dim_k H^6_{\mathrm{H}}(BL/k) = 2 = \dim_k E_\infty^{4,2} + \dim_k E_\infty^{6,0}$. From the filtration on $H^6_{\mathrm{H}}(BL/k)$ given by the spectral sequence (7), it follows that $E_\infty^{0,6} = 0$. As $H^3_{\mathrm{H}}(BG/k) = E_2^{3,0} = E_\infty^{3,0} = 0$ and $H^5_{\mathrm{H}}(BG/k) = E_2^{5,0} = E_\infty^{5,0} = 0$, we then get that $0 \neq d_7(w) \in E_7^{7,0}$ and $d_7(w)$ lifts to a non-zero element $y_7 \in H^4(BG, \Omega^3) \subseteq H^7_{\mathrm{H}}(BG/k)$.



Now, we can determine the $E_\infty$ terms in (7). For $n$ odd, $E_\infty^{i,n-i} = 0$ since $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees. Let $n \in \mathbb{N}$ be even. The $k$-dimension of $H^n_{\mathrm{H}}(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a, b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 2a + 4b = n\}.$$

For $i = 0, 1, 2$, set $V_{i,n} := H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $i = 0, 1, 2$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a, b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 4a + 6b = n - 2i\}$. As $v$ is a permanent cycle in (7), $E_2^{n-2i,2i} \cong E_7^{n-2i,2i}$ for $i = 0, 1, 2$. As $y_7 \in H^4(BG, \Omega^3)$ and $H^i(BG, \Omega^j) = 0$ for $i < j$,

$$y_7 \cdot x \notin \oplus_j H^j(BG, \Omega^j)$$

for all $x \in H^*_{\mathrm{H}}(BG/k)$. Hence,

$$H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2}) \otimes_k k \cdot v^i \subseteq E_2^{n-2i,2i} \cong E_7^{n-2i,2i}$$

injects into $E_\infty^{n-2i,2i}$ for $i = 0, 1, 2$.

Define a bijection $f_n : S_n \to S_{0,n} \cup S_{1,n} \cup S_{2,n}$ by

$$f_n(a, b) = \begin{cases} (b, a/3) \in S_{0,n} \text{ if } a \equiv 0 \mod 3, \\ (b, (a-1)/3) \in S_{1,n} \text{ if } a \equiv 1 \mod 3, \\ (b, (a-2)/3) \in S_{2.n} \text{ if } a \equiv 2 \mod 3. \end{cases}$$

Then

$$\begin{aligned} \dim_k H^n_{\mathrm{H}}(BL/k) &= |S_n| = |S_{0,n}| + |S_{1,n}| + |S_{2,n}| \\ &\leq \dim_k E_\infty^{n,0} + \dim_k E_\infty^{n-2,2} + \dim_k E_\infty^{n-4,4} \end{aligned}$$

where the inequality follows from the fact proved above that

$$H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$$

injects into $E_\infty^{n-2i,2i}$ for $i = 0, 1, 2$. From the filtration on $H^n_{\mathrm{H}}(BL/k)$ defined by the spectral sequence (7), it follows that $H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2}) \cong E_\infty^{n-2i,2i}$ for $i = 0, 1, 2$ and $E_\infty^{n-2i,2i} = 0$ for $i \geq 3$.

We can now finish the computation of the Hodge cohomology of $BG$ by using Zeeman's comparison theorem. Let $F_*$ denote the cohomological spectral sequence of $k$-vector spaces concentrated on the 0th column with $E_2$ page given by

$$F_2^{0,i} = \begin{cases} k \text{ if } i = 0, \\ k \cdot v \text{ if } i = 2, \\ k \cdot v^2 \text{ if } i = 4, \\ 0 \text{ if } i \neq 0,\ 2,\ 4. \end{cases}$$

As $v \in E_2^{0,2}$ in the spectral sequence (7) is transgressive with $d_r(v) = 0$ for all $r \geq 2$, there exists a map of of spectral sequences $F_* \to E_*$ that takes $v \in F_2^{0,2}$ to $v \in E_2^{0,2}$ and $v^2 \in F_2^{0,4}$ to $v^2 \in E_2^{0,4}$.

Fixing a variable $y$, let $H_*$ denote the cohomological spectral sequence with $E_2$ page given by $H_2 = \Delta(w) \otimes k[y]$ where $w$ is of bidegree $(0, 6)$, $y$ is of bidegree $(7,0)$, and $w$ is transgressive with $d_7(wy^i) = y^{i+1}$ for all $i \geq 0$. As $w \in E_2^{0,6}$ is transgressive with $d_7(w) = y_7 \in E_2^{7,0}$, there exists a map of spectral sequence $H_* \to E_*$ such that $w \in H_2^{0,6}$ maps to $w \in E_2^{0,6}$ and $y \in H_2^{7,0}$ maps to $y_7 \in E_2^{7,0}$. Elements of the ring of $G$-invariants $k[y_4, y_6]$ are permanent cycles in the spectral sequence (7) since they are concentrated on the 0th row. Thus, by tensoring the previous maps of spectral sequences, we get a map

$$\alpha : I_* := F_* \otimes H_* \otimes k[y_4, y_6] \to E_*$$

of spectral sequences.

As shown above, the map $\alpha$ induces an isomorphism $I_\infty \cong F_2 \otimes k[y_4, y_6] \to E_\infty$ on $E_\infty$ pages. The 0th columns of the $E_2$ pages of the spectral sequences $I_*$ and $E_*$ are both isomorphic to $k[v,w]/(v^3, w^2)$ and $\alpha$ induces an isomorphism on the 0th columns of the $E_2$ pages. Thus, by Theorem 1.4, $\alpha$ induces an isomorphism on the 0th rows of the $E_2$ pages. Hence,

$$H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7].$$

From Proposition 1.5, the Hodge spectral sequence for $BG$ degenerates. □

**Corollary 2.3.** *Let $G$ be a $k$-form of $G_2$. Then*

$$H^*_{\mathrm{H}}(BG/k) \cong k[x_4, x_6, x_7]$$

*where $|x_i| = i$ for $i = 4, 6, 7$.*

**Proof.** Letting $k_s$ denote the separable closure of $k$, we have $BG \times_k \mathrm{Spec}(k_s) \cong B(G_2)_{k_s}$. From Theorem 2.1, $H^*_{\mathrm{H}}(B(G_2)_{k_s})/k_s) \cong k_s[x_4', x_6', x_7']$ for some $x_4', x_6', x_7' \in H^*_{\mathrm{H}}(B(G_2)_{k_s}/k_s)$ with $|x_i'| = i$ for all $i$. As Hodge cohomology commutes with extensions of the base field,

$$H^*_{\mathrm{H}}((BG \times_k \mathrm{Spec}(k_s))/k_s) \cong H^*_{\mathrm{H}}(BG/k) \otimes_k k_s.$$

It follows that $H^*_{\mathrm{H}}(BG/k) \cong k[x_4, x_6, x_7]$ for some $x_4, x_6, x_7 \in H^*_{\mathrm{H}}(BG/k)$. □

## 3. Spin groups

Let $k$ be a field of characteristic 2 and let $G$ denote the split group $\mathrm{Spin}(n)_k$ over $k$ for $n \geq 7$.

Let $P_0 \subset SO(n)_k$ denote a parabolic subgroup that stabilizes a maximal isotropic subspace. Let $P \subset G$ denote the inverse image of $P_0$ under the double cover map $G \to SO(n)_k$. The Hodge cohomology of $G/P$ is given by Proposition 1.2 and [12, Theorem III.6.11].

**Proposition 3.1.** *There is an isomorphism*

$$H^*_{\mathrm{H}}((G/P)/k) \cong k[e_1, \ldots, e_s]/(e_i^2 = e_{2i}),$$

*where $s = \lfloor (n-1)/2 \rfloor$, $e_m = 0$ for $m > s$, and $|e_i| = 2i$ for all $i$.*

The Levi quotient of $P_0$ is isomorphic to $GL(r)_k$ where $r = \lfloor n/2 \rfloor$. Hence, the Levi quotient $L$ of $P$ is a double cover of $GL(r)_k$.

**Proposition 3.2.** *The torsion index of $L$ is equal to* 1.

**Proof.** We show that the torsion index of the corresponding compact connected Lie group $M$ is equal to 1. As $M$ is a double cover of $U(r)$, $M$ is isomorphic to $(S^1 \times SU(r))/2\mathbb{Z}$ where $k \in \mathbb{Z}$ acts on $S^1 \times SU(r)$ by

$$(z, A) \mapsto (ze^{2\pi ik/r}, e^{-2\pi ik/r}A).$$

Hence, the derived subgroup $[M, M]$ of $M$ is isomorphic to $SU(r)$. As $SU(r)$ has torsion index 1, $M$ has torsion index 1 by [16, Lemma 2.1]. Thus, $L$ has torsion index equal to 1. □

**Corollary 3.3.** *We have*

$$H^*_{\mathrm{H}}(BL/k) = O(\mathfrak{l})^L = k[A, c_2, \ldots, c_r]$$

*where $|c_i| = 2i$ in $H^*_{\mathrm{H}}(BL/k)$ for all $i$ and $|A| = 2$.*

**Proof.** From Proposition 3.2 and [17, Theorem 9.1],

$$H^*_{\mathrm{H}}(BL/k) = O(\mathfrak{l})^L.$$

Let $T$ be a maximal torus in $L$ with Lie algebra $\mathfrak{t}$ and Weyl group $W$. From Theorem 1.3, $O(\mathfrak{l})^L \cong O(\mathfrak{t})^W$. To compute $O(\mathfrak{t})^W$, we use that $L$ is a double cover of $GL(r)_k$. We have

$$\begin{aligned} S(X^*(T) \otimes k) &\cong \mathbb{Z}[x_1, \ldots, x_r, A]/(2A = x_1 + \cdots + x_r) \otimes k \\ &\cong k[x_1, \ldots, x_r, A]/(x_1 + \cdots + x_r). \end{aligned}$$

The Weyl group $W$ of $L$ is isomorphic to the symmetric group $S_r$ and acts on $S(X^*(T) \otimes k)$ by permuting $x_1, \ldots, x_r$. From [13, Proposition 4.1],

$$(k[x_1, \ldots, x_r, A]/(x_1 + \cdots + x_r))^{S_r} = k[A, c_2, \ldots, c_r]$$

where $c_1, \ldots, c_r$ are the elementary symmetric polynomials in the variables

$$x_1, \ldots, x_r.$$

□

For our calculations, we will need to know the Hodge cohomology of $BSO(n)_k$ [17, Theorem 11.1].

**Theorem 3.4.** *The Hodge spectral sequence for $BSO(n)_k$ degenerates and*

$$H^*_{\mathrm{H}}(BSO(n)_k/k) = k[u_2, \ldots, u_n]$$

*where $u_{2i} \in H^i(BSO(n)_k, \Omega^i)$ and $u_{2i+1} \in H^{i+1}(BSO(n)_k, \Omega^i)$ for all relevant $i$.*

We'll also need to know the ring of invariants of $G = \mathrm{Spin}(n)_k$ for all $n \geq 6$. This can be found in [17, Section 12].

**Lemma 3.5.** *For $n \geq 6$,*

$$O(\mathfrak{g})^G = \begin{cases} k[c_2, \ldots, c_r, \eta_{r-1}] \text{ if } n = 2r+1 \\ k[c_2, \ldots, c_r, \mu_{r-1}] \text{ if } n = 2r \text{ and } r \text{ is even} \\ k[c_2, \ldots, c_r, \mu_r] \text{ if } n = 2r \text{ and } r \text{ is odd} \end{cases}$$

*where $|c_i| = i$, $|\eta_j| = 2^j$, and $|\mu_j| = 2^{j-1}$ in $O(\mathfrak{g})^G$ for all $i$ and $j$.*

Note that under the inclusion $O(\mathfrak{g})^G \subset H^*_{\mathrm{H}}(BG/k)$, the degree of an invariant function in $H^*_{\mathrm{H}}(BG/k)$ is twice its degree in $O(\mathfrak{g})^G$.

**Theorem 3.6.** *Let $n = 7$. The Hodge spectral sequence for $BG$ degenerates and*

$$H^*_{\mathrm{dR}}(BG/k) \cong H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8]$$

*where $|y_i| = i$ for $i = 4, 6, 7, 8$.*

**Proof.** From Lemma 3.5,

$$O(\mathfrak{g})^G = k[y_4, y_6, y_8]$$

where $|y_i| = i$ in $H^*_{\mathrm{H}}(BG/k)$, viewing $O(\mathfrak{g})^G$ as a subring of $H^*_{\mathrm{H}}(BG/k)$. Consider the spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/k) \tag{8}$$

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

$$H^*_{\mathrm{H}}((G/P)/k) \cong k[e_1, e_2, e_3]/(e_i^2 = e_{2i}) = k[e_1, e_3]/(e_1^4, e_3^2)$$

and

$$H^*_{\mathrm{H}}(BL/k) \cong k[A, c_2, c_3].$$

First, we show that $e_1 \in E_2^{0,2}$ is a permanent cycle. From the filtration on $H^2_{\mathrm{H}}(BL/k) = k \cdot A$ given by (8), we have

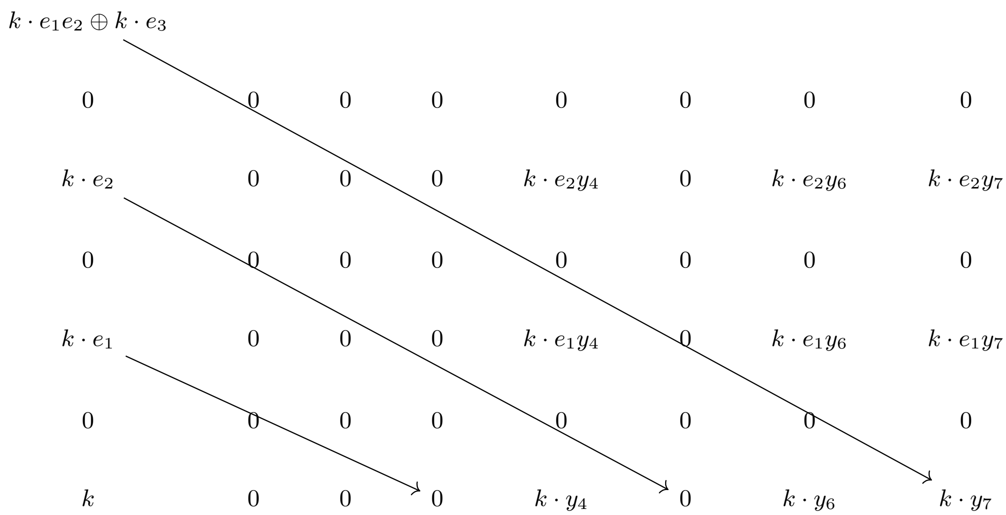
$$1 = \dim_k E_\infty^{0,2} + \dim_k E_\infty^{2,0} = \dim_k E_\infty^{0,2} + \dim_k E_2^{2,0}.$$

As $H^*_{\mathrm{H}}(BL/k) = \oplus_i H^i(BL, \Omega^i)$, $E_2^{2,0} = H^1(BG, \Omega^1) = 0$. Hence, $E_\infty^{0,2} = E_2^{0,2} = k \cdot e_1$ which implies that $e_1$ is a permanent cycle. As $e_2 = e_1^2$, it follows that $e_2$ is a permanent cycle. Hence, $E_\infty^{4,2} \cong E_2^{4,2} \cong k \cdot (y_4 \otimes e_1)$ and $E_\infty^{6,0} \cong E_2^{6,0} \cong k \cdot y_6$.

We next show that $e_3 \in E_2^{0,6}$ is transgressive with $d_7(e_3) \neq 0$. As $e_1$ is a permanent cycle and $H^i_{\mathrm{H}}(BL/k) = 0$ for $i$ odd, the spectral sequence (8) implies that $E_2^{3,0} = E_2^{5,0} = 0$. Consider the filtration of (8) on $H^6_{\mathrm{H}}(BL/k)$. We have

$$\dim_k H^6_{\mathrm{H}}(BL/k) = 3 = \dim_k E_\infty^{6,0} + \dim_k E_\infty^{4,2} + \dim_k E_\infty^{0,6} = 2 + \dim_k E_\infty^{0,6}$$

which implies that $E_\infty^{0,6} \cong k \cdot e_1 e_2$. As $E_2^{3,0} = E_2^{5,0} = 0$, we must then have $e_3 \in E_7^{0,6}$ and $0 \neq d_7(e_3) \in E_7^{7,0}$. The class $d_7(e_3)$ lifts to a non-zero class $y_7 \in H^4(BG, \Omega^3) \subseteq E_2^{7,0} = H^7_{\mathrm{H}}(BG/k)$.




We can now determine the $E_\infty$ page of (8). For $n$ odd, $E_\infty^{i,n-i} = 0$ since $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees. Assume that $n \in \mathbb{N}$ is even. The $k$-dimension of $H^n_{\mathrm{H}}(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a,b,c) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 2a + 4b + 6c = n\}.$$

For $i = 0,1,2,3$, set $V_{i,n} \coloneqq H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $i = 0,1,2,3$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a,b,c) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 4a + 6b + 8c = n - 2i\}$. As $e_1$ is a permanent cycle in (8),

$$V_{i,n} \cong V_{i,n} \otimes k \cdot e_1^i \subseteq E_\infty^{n-2i,2i}$$

for $i = 0,1,2,3$.

Define a bijection $f_n : S_n \to S_{0,n} \cup S_{1,n} \cup S_{2,n} \cup S_{3,n}$ by

$$f_n(a,b,c) = \begin{cases} (b,c,a/4) \in S_{0,n} \text{ if } a \equiv 0 \mod 4, \\ (b,c,(a-1)/4) \in S_{1,n} \text{ if } a \equiv 1 \mod 4, \\ (b,c,(a-2)/4) \in S_{2,n} \text{ if } a \equiv 2 \mod 4, \\ (b,c,(a-3)/4) \in S_{3,n} \text{ if } a \equiv 3 \mod 4. \end{cases}$$

Then

$$\dim_k H^n_{\mathrm{H}}(BL/k) = |S_n| = |S_{0,n}| + |S_{1,n}| + |S_{2,n}| + +|S_{3,n}|.$$

As

$$\dim_k H^n_{\mathrm{H}}(BL/k) \geq E_\infty^{n,0} + E_\infty^{n-2,2} + E_\infty^{n-4,4} + E_\infty^{n-6,6}$$

and $V_{i,n} \subseteq E_\infty^{n-2i,2i}$ for $i = 0,1,2,3$, it follows that $V_{i,n} \cong E_\infty^{n-2i,2i}$ for $i = 0,1,2,3$ and $E_\infty^{n-2i,2i} = 0$ for $i \geq 4$.

We now use Theorem 1.4 to finish the computation of the Hodge cohomology of $BG$. Let $F_*$ denote the cohomological spectral sequence of $k$-vector spaces concentrated on the 0th column given by $F_2 = \Delta(e_1, e_2)$ where $e_i$ is of bidegree $(0, 2i)$ for $i = 1,2$. As $e_1$ is a permanent cycle in (8) , there is a map of spectral sequences $F_* \to E_*$ taking $e_i \in F_2^{0,2i}$ to $e_i \in E_2^{0,2i}$ for $i = 1, 2$. Fix a variable $y$. Let $H_*$ be the spectral sequence with $E_2$ page given by $H_2 = \Delta(e_3) \otimes k[y]$ where $e_3$ is of bidegree $(0, 6)$, $y$ is of bidegree $(7, 0)$, and $e_3$ is transgressive with $d_7(e_3 y^i) = y^{i+1}$ for all $i$. As $e_3 \in E_2^{0,6}$ is transgressive with $d_7(e_3) = y_7$, there exists a map of spectral sequences $H_* \to E_*$ taking $e_3 \in H_2^{0,6}$ to $e_3 \in E_2^{0,6}$ and $y \in H_2^{7,0}$ to $y_7 \in E_2^{7,0}$.

Elements in the ring of $G$-invariants $k[y_4, y_6, y_8]$ are permanent cycles in the spectral sequence (8). Tensoring maps of spectral sequences, we get a map

$$\alpha : I_* := F_* \otimes H_* \otimes k[y_4, y_6, y_8] \to E_*$$

of spectral sequences. As $I_\infty \cong F_2 \otimes k[y_4, y_6, y_8]$, $\alpha$ induces isomorphisms on $E_\infty$ terms and on the 0th columns of the $E_2$ pages. Hence, by Theorem 1.4, $\alpha$ induces an isomorphism on the 0th rows of the $E_2$ pages. Thus,

$$H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8].$$

The Hodge spectral sequence for $BG$ degenerates by Proposition 1.5. □

As Hodge cohomology commutes with extensions of the base field, we have the following result.

**Corollary 3.7.** *Let $k$ be a field of characteristic 2 and let $G$ be a $k$-form of* Spin(7). *Then*

$$H^*_{\mathrm{H}}(BG/k) \cong k[x_4, x_6, x_7, x_8]$$

*where $|x_i| = i$ for all $i$.*

**Theorem 3.8.** *Let $n = 8$. The Hodge spectral sequence for $BG$ degenerates and*

$$H^*_{\mathrm{dR}}(BG/k) \cong H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_8']$$

*where $|y_i| = i$ for $i = 4, 6, 7, 8$ and $|y_8'| = 8$.*

**Proof.** From Lemma 3.5,

$$O(\mathfrak{g})^G = k[y_4, y_6, y_8, y_8']$$

where $|y_i| = i$ and $|y_8'| = 8$ in $H^*_{\mathrm{H}}(BG/k)$, viewing $O(\mathfrak{g})^G$ as a subring of $H^*_{\mathrm{H}}(BG/k)$. Consider the spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/k) \tag{9}$$

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

$$H^*_{\mathrm{H}}((G/P)/k) \cong k[e_1, e_2, e_3]/(e_i^2 = e_{2i}) = k[e_1, e_3]/(e_1^4, e_3^2)$$

and

$$H^*_{\mathrm{H}}(BL/k) \cong k[A, c_2, c_3, c_4].$$

Calculations similar to those performed in the proof of Proposition 3.6 show that $e_1$ is a permanent cycle in (9) and $e_3 \in E_2^{0,6}$ is transgressive with $0 \neq d_7(e_3) = y_7 \in H^4(BG, \Omega^3)$. We have $H^m_{\mathrm{H}}(BG/k) \cong H^m_{\mathrm{H}}(B\mathrm{Spin}(7)_k/k)$ for $m < 8$ and $H^8_{\mathrm{H}}(BG/k) = k \cdot y_8 \oplus k \cdot y_8'$.

We can now determine the $E_\infty$ terms for (9). For $n$ odd, $E_\infty^{i,n-i} = 0$ since $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees. Assume that $n \in \mathbb{N}$ is even. The $k$-dimension of $H^n_{\mathrm{H}}(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a,b,c,d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 2a + 4b + 6c + 8d = n\}.$$

For $i = 0,1,2,3$, set $V_{i,n} \coloneqq H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $i = 0,1,2,3$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a,b,c,d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 4a + 6b + 8c + 8d = n - 2i\}$. As $e_1$ is a permanent cycle in (9),

$$V_{i,n} \cong V_{i,n} \otimes k \cdot e_1^i \subseteq E_\infty^{n-2i,2i}$$

for $i = 0,1,2,3$.

Define a bijection $f_n : S_n \to S_{0,n} \cup S_{1,n} \cup S_{2,n} \cup S_{3,n}$ by

$$f_n(a,b,c,d) = \begin{cases} (b,c,d,a/4) \in S_{0,n} \text{ if } a \equiv 0 \mod 4, \\ (b,c,d,(a-1)/4) \in S_{1,n} \text{ if } a \equiv 1 \mod 4, \\ (b,c,d,(a-2)/4) \in S_{2,n} \text{ if } a \equiv 2 \mod 4, \\ (b,c,d,(a-3)/4) \in S_{3,n} \text{ if } a \equiv 3 \mod 4. \end{cases}$$

Then

$$\dim_k H^n_{\mathrm{H}}(BL/k) = |S_n| = |S_{0,n}| + |S_{1,n}| + |S_{2,n}| + +|S_{3,n}|.$$

As

$$\dim_k H^n_{\mathrm{H}}(BL/k) \geq E_\infty^{n,0} + E_\infty^{n-2,2} + E_\infty^{n-4,4} + E_\infty^{n-6,6}$$

and $V_{i,n} \subseteq E_\infty^{n-2i,2i}$ for $i = 0,1,2,3$, it follows that $V_{i,n} \cong E_\infty^{n-2i,2i}$ for $i = 0,1,2,3$ and $E_\infty^{n-2i,2i} = 0$ for $i \geq 4$.

Let $F_*$ denote the spectral sequence concentrated on the 0th column with $F_2 = \Delta(e_1, e_2, e_4)$ where $e_i$ is of bidegree $(0, 2i)$. There is a map of spectral sequences $F_* \to E_*$ taking $e_i$ to $e_i$ for $i = 1,2,4$. Fix a variable $y$. Let $H_*$ denote the spectral sequence with $E_2$ page $H_2 = \Delta(e_3) \otimes k[y]$ where $e_3$ is of bidegree $(0,6)$, $y$ is of bidegree $(7,0)$, and $e_3$ is transgressive with $d_7(e_3 y^i) = y^{i+1}$ for all $i$. There is an obvious map of spectral sequences $H_* \to E_*$. Classes in the ring of $G$-invariants are permanent cycles in the spectral sequence (9). Tensoring these maps, we get a map of spectral sequences

$$\alpha : I_* \coloneqq F_* \otimes H_* \otimes k[y_4, y_6, y_8, y_8'] \to E_*.$$

The map $\alpha$ induces an isomorphism on $E_\infty$ terms and on the 0th columns of the $E_2$ pages. Theorem 1.4 then implies that $\alpha$ induces an isomorphism on the 0th rows of the $E_2$ pages. Thus,

$$H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_8'].$$

Proposition 1.5 implies that the Hodge spectral sequence for $BG$ degenerates.

□

**Corollary 3.9.** *Let $k$ be a field of characteristic 2 and let $G$ be a $k$-form for* Spin(8). *Then*

$$H^*_{\mathrm{H}}(BG/k) \cong k[y_4, y_6, y_7, y_8, y_8']$$

*where $|y_i| = i$ for $i = 4, 6, 7, 8$ and $|y_8'| = 8$.*

**Theorem 3.10.** *Let $n = 9$. The Hodge spectral sequence for $BG$ degenerates and*

$$H^*_{\mathrm{dR}}(BG/k) \cong H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_{16}]$$

*where $|y_i| = i$ for $i = 4, 6, 7, 8, 16$.*

**Proof.** From Lemma 3.5,

$$O(\mathfrak{g})^G = k[y_4, y_6, y_8, y_{16}]$$

where $|y_i| = i$ in $H^*_{\mathrm{H}}(BG/k)$, viewing $O(\mathfrak{g})^G$ as a subring of $H^*_{\mathrm{H}}(BG/k)$. Consider the spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/k) \tag{10}$$

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

$$H^*_{\mathrm{H}}((G/P)/k) \cong k[e_1, e_2, e_3, e_4]/(e_i^2 = e_{2i}) = k[e_1, e_3]/(e_1^8, e_3^2)$$

and

$$H^*_{\mathrm{H}}(BL/k) \cong k[A, c_2, c_3, c_4].$$

Calculations similar to those performed in the proof of Proposition 3.6 show that $e_1$ is a permanent cycle in (10) and $e_3 \in E_2^{0,6}$ is transgressive with $0 \neq d_7(e_3) = y_7 \in H^4(BG, \Omega^3)$. We have $H^m_{\mathrm{H}}(BG/k) \cong H^m_{\mathrm{H}}(B\mathrm{Spin}(7)_k/k)$ for $m \leq 10$.

We now determine the $E_\infty$ terms for (10). For $n$ odd, $E_\infty^{i,n-i} = 0$ since $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees. Assume that $n \in \mathbb{N}$ is even. The $k$-dimension of $H^n_{\mathrm{H}}(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a, b, c, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 2a + 4b + 6c + 8d = n\}.$$

For $0 \leq i \leq 7$, set $V_{i,n} := H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $0 \leq i \leq 7$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a, b, c, d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : 4a + 6b + 8c + 16d = n - 2i\}$. As $e_1$ is a permanent cycle in (10),

$$V_{i,n} \cong V_{i,n} \otimes k \cdot e_1^i \subseteq E_\infty^{n-2i,2i}$$

for $0 \leq i \leq 7$.

Define a bijection $f_n : S_n \to \bigcup_{i=0}^{7} S_{i,n}$ by $f_n(a, b, c, d) = (b, c, d, (a - i)/8) \in S_{i,n}$ for $a \equiv i \mod (8)$. Then

$$\dim_k H^n_{\mathrm{H}}(BL/k) = |S_n| = \sum_{i=0}^{7} |S_{i,n}|.$$

As

$$\dim_k H^n_{\mathrm{H}}(BL/k) \geq \sum_{i=0}^{7} E_\infty^{n-2i,2i}$$

and $V_{i,n} \subseteq E_\infty^{n-2i,2i}$ for $0 \leq i \leq 7$, it follows that $V_{i,n} \cong E_\infty^{n-2i,2i}$ for $0 \leq i \leq 7$ and $E_\infty^{n-2i,2i} = 0$ for $i \geq 8$.

Let $F_*$ denote the cohomological spectral sequence concentrated on the 0th column with $E_2$ page given by $F_2 = \Delta(e_1, e_2, e_4)$ where $e_i$ has bidegree $(0, 2i)$ for $i = 1, 2, 4$. As $e_1$ is a permanent cycle in the spectral sequence (10), there exists a map $F_* \to E_*$ of spectral sequences taking $e_i$ to $e_i$ for $i = 1, 2, 4$. Let $y$ be a free variable and let $H_*$ denote the spectral sequence with $E_2$ page $H_2 = \Delta(e_3) \otimes k[y]$ where $e_3$ is of bidegree $(0, 6)$, $y$ is of bidegree $(7, 0)$, and $e_3$ is transgressive with $d_7(e_3 y^i) = y^{i+1}$ for all $i$. As $e_3$ is transgressive in the spectral sequence (10) with $d_7(e_3) = y_7$, there exists a map of spectral sequences $H_* \to E_*$ taking $e_3$ to $e_3$ and $y$ to $y_7$.

Elements in the ring of $G$-invariants $k[y_4, y_6, y_8, y_{16}]$ are permanent cycles in the spectral sequence (10). Tensoring maps of spectral sequences, we get a map

$$\alpha : I_* := F_* \otimes H_* \otimes k[y_4, y_6, y_8, y_{16}] \to E_*.$$

The map $\alpha$ induces an isomorphism on $E_\infty$ terms and on the 0th columns of the $E_2$ pages. Hence, Theorem 1.4 implies that $\alpha$ induces an isomorphism on the 0th rows of the $E_2$ pages. Thus,

$$H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_{16}].$$

Proposition 5 implies that the Hodge spectral sequence for $BG$ degenerates. □

**Corollary 3.11.** *Let $k$ be a field of characteristic 2 and let $G$ be a $k$-form for* Spin(9). *Then*

$$H^*_{\mathrm{H}}(BG/k) \cong k[y_4, y_6, y_7, y_8, y_{16}]$$

*where $|y_i| = i$ for $i = 4, 6, 7, 8, 16$.*

*Remark* 3.12. Assume that $k$ is perfect. Let $\mu_2$ denote the group scheme of the 2nd roots of unity over $k$. For $n \geq 10$, the Hodge cohomology of $BG$ is no longer a polynomial ring. To determine the relations that hold in $H^*_{\mathrm{H}}(BG/k)$, we will restrict cohomology classes to the classifying stack of a certain subgroup of $G$ considered in [17, Section 12]. Let $r = \lfloor n/2 \rfloor$ and let $T \cong \mathbb{G}_m^r$ denote a split maximal torus of $G$. Assume that $n \not\equiv 2 \mod 4$ so that the Weyl group $W$ of $G$ contains $-1$, acting by inversion on $T$. Then $-1$ acts by the identity on $T[2] \cong \mu_2^r$ (for $n \in \mathbb{N}$, $T[n] \subset T$ is the kernel of the $n$th power map $T \to T$) and $G$ contains a subgroup $Q \cong \mu_2^r \times \mathbb{Z}/2$. Under the double cover $G \to SO(n)_k$, the image of $Q$ is isomorphic to $K \cong \mu_2^{r-1} \times \mathbb{Z}/2$ and $Q \to K$ is a split surjection. We will need to know the Hodge cohomology rings of the classifying stacks of these groups. For a commutative ring $R$, we let $\mathrm{rad} \subset R$ denote the ideal of nilpotent elements. From [17, Proposition 10.1],

$$H^*_{\mathrm{H}}(B\mu_2/k)/\mathrm{rad} \cong k[t]$$

where $t \in H^1(B\mu_2, \Omega^1)$. From [17, Lemma 10.2],

$$H^*_{\mathrm{H}}((B\mathbb{Z}/2)/k) = k[s]$$

where $s \in H^1(B\mathbb{Z}/2, \Omega^0)$. The Künneth formula [17, Proposition 5.1] then lets us calculate the Hodge cohomology ring of $B\mu_2^i \times B(\mathbb{Z}/2)^j$ for any $i, j \geq 0$. Fix $i, j > 0$. Then

$$H^*_{\mathrm{H}}((B\mu_2^i \times B(\mathbb{Z}/2)^j)/k)/\mathrm{rad} \cong k[t_1, \ldots, t_i, s_1, \ldots, s_j]$$

where $t_l \in H^1(B\mu_2^i \times B(\mathbb{Z}/2)^j, \Omega^1)$ for all $l$ and $s_l \in H^1(B\mu_2^i \times B(\mathbb{Z}/2)^j, \Omega^0)$ for all $l$.

**Theorem 3.13.** *Let $n = 10$. The Hodge spectral sequence for $BG$ degenerates and*

$$H^*_{\mathrm{dR}}(BG/k) \cong H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_{10}, y_{32}]/(y_7 y_{10})$$

*where $|y_i| = i$ for $i = 4, 6, 7, 8, 10, 32$.*

**Proof.** We may assume that $k = \mathbb{F}_2$ so that Remark 3.12 applies. From Lemma 3.5,

$$O(\mathfrak{g})^G = k[y_4, y_6, y_8, y_{10}, y_{32}]$$

where $|y_i| = i$ in $H^*_{\mathrm{H}}(BG/k)$, viewing $O(\mathfrak{g})^G$ as a subring of $H^*_{\mathrm{H}}(BG/k)$. Consider the spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/k) \tag{11}$$

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

$$H^*_{\mathrm{H}}((G/P)/k) \cong k[e_1, e_2, e_3, e_4]/(e_i^2 = e_{2i}) = k[e_1, e_3]/(e_1^8, e_3^2)$$

and

$$H^*_{\mathrm{H}}(BL/k) \cong k[A, c_2, c_3, c_4, c_5].$$

Calculations similar to those performed in the proof of Proposition 3.6 show that $e_1$ is a permanent cycle in (11) and $e_3 \in E_2^{0,6}$ is transgressive with $0 \neq d_7(e_3) = y_7 \in H^4(BG, \Omega^3)$. We have $H^m_{\mathrm{H}}(BG/k) \cong H^m_{\mathrm{H}}(B\mathrm{Spin}(9)_k/k)$ for $m < 10$.

Let $F_*$ be the spectral sequence concentrated on the 0th column with $E_2$ page given by $F_2 = \Delta(e_1, e_2, e_4)$ where $e_i$ has bidegree $(0, 2i)$ for all $i$. As $e_1$ is a permanent cycle in (11), there exists a map of spectral sequence $F_* \to E_*$ taking $e_i$ to $e_i$ for $i = 1, 2, 4$. Fix a variable $y$. Let $H_*$ denote the spectral sequence with $E_2$ page $H_2 = \Delta(e_3) \otimes k[y]$ where $e_3$ has bidegree $(0, 6)$, $y$ has bidegree $(7, 0)$, and $e_3$ is transgressive with $d_7(e_3 y^i) = y^{i+1}$ for all $i$. As $e_3$ is transgessive in (11) with $d_7(e_3) = y_7$, there exists a map of spectral sequences $H_* \to E_*$ taking $e_3$ to $e_3$ and $y$ to $y_7$. Elements in the ring of $G$-invariants $k[y_4, y_6, y_8, y_{10}, y_{32}]$ are permanent cycles in (11). Tensoring maps of spectral sequences, we get a map

$$\alpha : I_* \coloneqq F_* \otimes H_* \otimes k[y_4, y_6, y_8, y_{10}, y_{32}] \to E_* \tag{12}$$

which induces an isomorphism on the 0th columns of the $E_2$ pages.

Let $n$ be even. The $k$-dimension of $H^n_{\mathrm{H}}(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a, b, c, d, e) \in \mathbb{Z}^5_{\geq 0} : 2a + 4b + 6c + 8d + 10e = n\}.$$

For $0 \leq i \leq 15$, set $V_{i,n} \coloneqq H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $0 \leq i \leq 15$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a, b, c, d, e) \in \mathbb{Z}^5_{\geq 0} : 4a + 6b + 8c + 10d + 32e = n - 2i\}$. As $e_1 \in H^2_{\mathrm{H}}((G/P)/k)$ is a permanent cycle in (11),

$$V_{i,n} \cong V_{i,n} \otimes k \cdot e_1^i \subseteq E_\infty^{n-2i,2i}$$

for $0 \leq i \leq 7$. Hence, the map $\alpha$ in (12) induces injections on all $E_\infty$ terms. For $n$ odd, $\alpha$ induces isomorphisms $0 = I_\infty^{n-i,i} \cong E_\infty^{n-i,i} = 0$ for all $i$ since $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees.

Define a bijection $f_n : S_n \to \bigcup_{i=0}^{15} S_{i,n}$ by $f_n(a, b, c, d, e) = (b, c, d, e, (a - i)/16) \in S_{i,n}$ for $a \equiv i \mod (16)$. Then

$$\dim_k H^n_{\mathrm{H}}(BL/k) = |S_n| = \sum_{i=0}^{15} |S_{i,n}| = \sum_{i=0}^{15} \dim_k V_{i,n}. \tag{13}$$

Now assume that $n \leq 14$. Then $f_n$ gives a bijection

$$S_n \to \bigcup_{i=0}^{7} S_{i,n}.$$

As

$$\dim_k H^n_{\mathrm{H}}(BL/k) \geq \sum_{i=0}^{7} E_\infty^{n-2i,2i}$$

and $V_{i,n} \subseteq E_\infty^{n-2i,2i}$ for $0 \leq i \leq 7$, it follows that $V_{i,n} \cong E_\infty^{n-2i,2i}$ for $0 \leq i \leq 7$ and $E_\infty^{n-2i,2i} = 0$ for $i \geq 8$. As $\alpha$ induces injections on all $E_\infty$ terms, Theorem 1.4 implies that $\alpha$ in (12) induces an isomorphism $I_2^{n,0} \to E_2^{n,0}$ for $n < 16$.

Now we consider the filtration on $H^{16}_{\mathrm{H}}(BL/k)$ given by (11). From the bijection $f_{16}$ defined in the previous paragraph, we have

$$\dim_k H^{16}_{\mathrm{H}}(BL/k) = 1 + \sum_{i=0}^{7} |S_{i,n}| = 1 + \sum_{i=0}^{7} \dim_k V_{i,n} \otimes k \cdot e_1^i.$$

As $e_1$ is a permanent cycle and $\alpha$ induces isomorphisms on 0th row terms of the $E_2$ pages in degrees less than 16, we must then have

$$E_\infty^{10,6} \cong (H^{10}_{\mathrm{H}}(BG/k) \otimes k \cdot e_1^3) \oplus (k \cdot z \otimes k \cdot e_3)$$

for some $0 \neq z \in H^{10}_{\mathrm{H}}(BG/k)$. Hence, $y_7 z = 0$ in $H^*_{\mathrm{H}}(BG/k)$. Write $z = ay_4y_6 + by_{10}$ for some $a, b \in k$.

We now show that $a = 0$ by restricting $y_7 z = 0$ to the Hodge cohomology of the classifying stack of the subgroup $\mathrm{Spin}(8)_k$ of $G$. Under the isomorphism

$$H^*_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k) \cong k[y_4, y_6, y_7, y_8, y_{16}]$$

of Theorem 3.10, the pullback from $H^*_{\mathrm{H}}(BG/k)$ to $H^*_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k)$ maps $y_4, y_6, y_{10} \in H^*_{\mathrm{H}}(BG/k)$ to $y_4, y_6$, and 0 respectively in $H^*_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k)$. Hence, to show that $a = 0$, it suffices to show that $y_7 \in H^*_{\mathrm{H}}(BG/k)$ restricts to $y_7 \in H^*_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k)$. From the isomorphism

$$H^*_{\mathrm{H}}(BSO(m)_k/k) \cong k[u_2, \ldots, u_m]$$

of Theorem 3.4 for $m \geq 0$, the class $u_7 \in H^7_{\mathrm{H}}(BSO(10)_k/k)$ restricts to $u_7 \in H^7_{\mathrm{H}}(BSO(8)_k/k)$. Thus, we are reduced to showing that $u_7 \in H^7_{\mathrm{H}}(BSO(8)_k/k)$ pulls back to a non-zero multiple of $y_7 \in H^*_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k)$.

Consider the subgroups $\mu_2^4 \times \mathbb{Z}/2 \cong Q \subseteq \mathrm{Spin}(8)_k$ and $\mu_2^3 \times \mathbb{Z}/2 \cong K \subseteq SO(8)_k$ defined in Remark 3.12. As the morphism $Q \to K$ is split surjective, if we can show that $u_7$ restricts to a nonzero class in $H^*_{\mathrm{H}}(BK/k)$, then $u_7$ would restrict to a nonzero class in $H^7_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k)$. From the inclusion $O(2)^4_k \subset O(8)_k$, $O(8)_k$ contains a subgroup of the form $\mu_2^4 \times (\mathbb{Z}/2)^4$. As $SO(8)_k$ is the kernel of the Dickson determinant (also called the Dickson invariant in some sources [11, §23]) $O(8)_k \to \mathbb{Z}/2$, it follows that $SO(8)_k$ contains a subgroup $H \cong \mu_2^4 \times (\mathbb{Z}/2)^3$. Write

$$H^*_{\mathrm{H}}(BH/k)/\mathrm{rad} \cong k[t_1, \ldots, t_4, s_1, \ldots, s_4]/(s_1 + s_2 + s_3 + s_4)$$

using Remark 3.12. From the proof of [17, Lemma 11.4], the pullback of $u_7$ to $H^*_{\mathrm{H}}(BH/k)/\mathrm{rad}$ followed by pullback to

$$H^*_{\mathrm{H}}(BK/k)/\mathrm{rad} \cong k[t_1, \ldots, t_4, s]/(t_1 + \cdots + t_4)$$

is given by

$$u_7 \mapsto \sum_{j=1}^{3} s_j(t_j + t_4) \sum_{\substack{1 \leq i_1 < i_2 \leq 3 \\ i_1, i_2 \neq j}} t_{i_1} t_{i_2} \mapsto \sum_{j=1}^{3} s(t_j + t_4) \sum_{\substack{1 \leq i_1 < i_2 \leq 3 \\ i_1, i_2 \neq j}} t_{i_1} t_{i_2}$$

$$= s \sum_{1 \leq i_1 < i_2 \leq 3} (t_{i_1} + t_{i_2}) t_{i_1} t_{i_2} \neq 0.$$

Thus, $u_7 \in H^7_{\mathrm{H}}(BSO(8)_k/k)$ pulls back to a nonzero multiple of

$$y_7 \in H^7_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k)$$

which implies that $y_7 y_{10} = 0$ in $H^*_{\mathrm{H}}(BG/k)$.

$$\begin{array}{ccc}
u_7 \in H^7_{\mathrm{H}}(BSO(10)_k/k) & \longrightarrow & y_7 \in H^7_{\mathrm{H}}(BG/k) \\
\downarrow & & \downarrow \\
u_7 \in H^7_{\mathrm{H}}(BSO(8)_k/k) & \longrightarrow & y_7 \in H^7_{\mathrm{H}}(B\mathrm{Spin}(8)_k/k) \\
\downarrow & & \downarrow \\
\sum_{j=1}^3 s(t_j + t_4) \sum_{\substack{1 \le i_1 < i_2 \le 3 \\ i_1, i_2 \neq j}} t_{i_1} t_{i_2} \in H^7_{\mathrm{H}}(BK/k) & \xrightarrow{\neq 0} & H^7_{\mathrm{H}}(BQ/k)
\end{array}$$

Using the relation $y_7 y_{10} = 0$, we now modify the spectral sequence $I_*$ defined above to define a new spectral sequence $J_*$ that better approximates (and will actually be isomorphic to) the spectral sequence (11). Let

$$(yy_{10}) \coloneqq F_2 \otimes (\Delta(e_3) \otimes yk[y]) \otimes y_{10} k[y_4, y_6, y_8, y_{10}, y_{32}].$$

Define the $E_2$ page of $J_*$ by $J_2 = I_2/(yy_{10})$. Define the differentials $d'_m$ of $J_*$ so that $I_2 \to J_2$ induces a map $I_* \to J_*$ of cohomological spectral sequences of $k$-vector spaces and $d'_m = 0$ for $m > 7$. This means that $d'_7(f \otimes e_3 \otimes y_{10} g) = f \otimes y \otimes y_{10} g = 0$ and $d'_m(f \otimes e_3 \otimes y_{10} g) = 0$ for $m > 7$, $f \in F_2$, and $g \in k[y_4, y_6, y_8, y_{10}, y_{32}]$. The $E_\infty$ page of $J_*$ is given by

$$J_\infty \cong (F_2 \otimes k[y_4, y_6, y_8, y_{10}, y_{32}]) \oplus (F_2 \otimes e_3 \otimes y_{10} k[y_4, y_6, y_8, y_{10}, y_{32}]).$$

As $y_7 y_{10} = 0$ in $H^*_{\mathrm{H}}(BG/k)$, $\alpha$ induces a map $\alpha' : J_* \to E_*$ of spectral sequences. To finish the calculation, we will show that $\alpha'$ induces an isomorphism on $E_\infty$ terms so that Theorem 1.4 will apply. For $n$ odd, $E_\infty^{n-i,i} = 0$ for all $i$ since $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees. Now assume that $n$ is even. For $0 \le i \le 7$,

$$V_{i,n} \cong H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2}) \otimes e_1^i \subseteq E_\infty^{n-2i,2i}.$$

For $8 \le i \le 15$,

$$V_{i,n} \cong y_{10} H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2}) \otimes e_1^{i-8} e_3 \subseteq E_\infty^{n-2i+10,2i-10}.$$

Hence, from the description of the $E_\infty$ terms of $J_*$ given above, it follows that $\alpha'$ induces an injection $J_\infty^{n-2i,2i} \to E_\infty^{n-2i,2i}$ for all $i$. Equation (13) then implies that $J_\infty^{n-2i,2i} \cong E_\infty^{n-2i,2i}$ for all $i$.

Thus, $\alpha'$ induces an isomorphism on $E_\infty$ pages and an isomorphism on the 0th columns of the $E_2$ pages of the 2 spectral sequences. Theorem 1.4 then implies that

$$H^*_{\mathrm{H}}(BG/k) \cong k[y_4, y_6, y_7, y_8, y_{10}, y_{32}]/(y_7 y_{10}).$$

From Proposition 5, the Hodge spectral sequence for $BG$ degenerates. □

**Corollary 3.14.** *Let $G$ be a $k$-form of* Spin(10). *Then*

$$H^*_{\mathrm{H}}(BG/k) \cong k[y_4, y_6, y_7, y_8, y_{10}, y_{32}]/(y_7 y_{10})$$

*where* $|y_i| = i$ *for all* $i$.

**Theorem 3.15.** *Let* $n = 11$. *The Hodge spectral sequence for* $BG$ *degenerates and*

$$H^*_{\mathrm{dR}}(BG/k) \cong H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_{10}, y_{11}, y_{32}]/(y_7y_{10} + y_6y_{11})$$

*where* $|y_i| = i$ *for* $i = 4, 6, 7, 8, 10, 11, 32$.

**Proof.** We may assume that $k = \mathbb{F}_2$ so that Remark 3.12 applies. From Lemma 3.5,

$$O(\mathfrak{g})^G \cong k[y_4, y_6, y_8, y_{10}, y_{32}]$$

where $|y_i| = i$ in $H^*_{\mathrm{H}}(BG/k)$, viewing $O(\mathfrak{g})^G$ as a subring of $H^*_{\mathrm{H}}(BG/k)$. Consider the spectral sequence

$$E_2^{i,j} = H^i_{\mathrm{H}}(BG/k) \otimes H^j_{\mathrm{H}}((G/P)/k) \Rightarrow H^{i+j}_{\mathrm{H}}(BL/k) \tag{14}$$

from Proposition 1.1. From Proposition 3.1 and Corollary 3.3,

$$H^*_{\mathrm{H}}((G/P)/k) \cong k[e_1, e_2, e_3, e_4, e_5]/(e_i^2 = e_{2i}) = k[e_1, e_3, e_5]/(e_1^8, e_3^2, e_5^2)$$

and

$$H^*_{\mathrm{H}}(BL/k) \cong k[A, c_2, c_3, c_4, c_5].$$

Using Theorem 3.4, write $H^*_{\mathrm{H}}(BSO(11)_k/k) = k[u_2, \ldots, u_{11}]$. From the inclusions $O(2)_k^5 \subset O(10)_k \subset SO(11)_k$, $SO(11)_k$ contains a subgroup $H \cong \mu_2^5 \times (\mathbb{Z}/2)^5$. Write $H^*_{\mathrm{H}}(BH/k)/\mathrm{rad} \cong k[t_1, \ldots, t_5, s_1, \ldots, s_5]$ as described in Remark 3.12. Under the pullback map $H^*_{\mathrm{H}}(BSO(11)_k/k) \to H^*_{\mathrm{H}}(BH/k)/\mathrm{rad}$, $u_{2m}$ pulls back to the $m$th elementary symmetric polynomial

$$\sum_{1 \le i_1 < \cdots < i_m \le 5} t_{i_1} \cdots t_{i_m} \tag{15}$$

and $u_{2m+1}$ pulls back to

$$\sum_{j=1}^{5} s_j \sum_{\substack{1 \le i_1 < \cdots < i_m \le 5 \\ \text{one equal to j}}} t_{i_1} \cdots t_{i_m}$$

for $1 \le m \le 5$ [17, Lemma 11.4]. To be concise, from now on we will write $u_{2m}$ to denote the image of $u_{2m}$ under pullback maps to $H^*_{\mathrm{H}}(BH/k)/\mathrm{rad}$ or $H^*_{\mathrm{H}}(BK/k)/\mathrm{rad}$ whenever we are dealing with these two rings.

Let $Q \cong (\mu_2^5 \times \mathbb{Z}/2) \subset G$ and $K \cong (\mu_2^4 \times \mathbb{Z}/2) \subset SO(11)_k$ be the subgroups described in Remark 3.12. Write $H^*_{\mathrm{H}}(BK/k)/\mathrm{rad} \cong k[t_1, \ldots, t_5, s]/(t_1 + \cdots + t_5)$. Under the pullback map $H^*_{\mathrm{H}}(BSO(11)_k/k) \to H^*_{\mathrm{H}}(BK/k)/\mathrm{rad}$, $u_7$ maps to $su_6 \neq 0$ and $u_{11}$ maps to $su_{10} \neq 0$. As $Q \to K$ is split, it follows that $u_7, u_{11}$ restrict to nonzero classes $y_7 \in H^7_{\mathrm{H}}(BG/k)$ and $y_{11} \in H^{11}_{\mathrm{H}}(BG/k)$. Also, $y_4y_7$ and $y_{11}$ are linearly independent in $H^{11}_{\mathrm{H}}(BG/k)$.

Returning to the spectral sequence (14), calculations similar to those performed in the proof of Proposition 3.6 show that $e_1$ is a permanent cycle in (14) and $e_3 \in E_2^{0,6}$ is transgressive with $0 \neq d_7(e_3) = y_7 \in H^4(BG, \Omega^3)$. We have $H^m_{\mathrm{H}}(BG/k) \cong H^m_{\mathrm{H}}(B\mathrm{Spin}(10)_k/k)$ for $m \le 10$.

Let $F_*$ be the spectral sequence concentrated on the 0th column with $E_2$ page given by $\Delta(e_1, e_2, e_4)$ with $e_i$ of bidegree $(0, 2i)$ for $i = 1, 2, 4$. Fix a variable $y$ and let $H_*$ be the spectral sequence with $H_2 = \Delta(e_3) \otimes k[y]$ where $e_3$ is of bidegree $(0, 6)$, $y$ is of bidegree $(7, 0)$, and $e_3$ is transgressive with $d_7(e_3 y^i) = y^{i+1}$ for all $i$. There exists a map of spectral sequence

$$\alpha : I_* \coloneqq F_* \otimes H_* \otimes k[y_4, y_6, y_8, y_{10}, y_{32}] \to E_*$$

taking $e_i$ to $e_i$ for $i = 1, 2, 3, 4$ and taking $y$ to $y_7$. The $E_\infty$ page of $I_*$ is given by $I_\infty \cong F_2 \otimes k[y_4, y_6, y_8, y_{10}, y_{32}]$ and $\alpha$ induces an injection $I_\infty^{i,j} \to E_\infty^{i,j}$ for all $i, j$ with $i + j \leq 17$. For $n$ odd, $\alpha$ induces an isomorphism $0 = I_\infty^{n-i,i} \cong E_\infty^{n-i,i} = 0$ for all $i$ since $H^*_{\mathrm{H}}(BL/k)$ is concentrated in even degrees.

Let $n$ be even. The $k$-dimension of $H^n_{\mathrm{H}}(BL/k)$ is equal to the cardinality of the set

$$S_n = \{(a, b, c, d, e) \in \mathbb{Z}^5_{\geq 0} : 2a + 4b + 6c + 8d + 10e = n\}.$$

For $0 \leq i \leq 15$, set $V_{i,n} \coloneqq H^{(n-2i)/2}(BG, \Omega^{(n-2i)/2})$. For $0 \leq i \leq 15$, $\dim_k V_{i,n}$ is equal to the cardinality of the set $S_{i,n} = \{(a, b, c, d, e) \in \mathbb{Z}^5_{\geq 0} : 4a + 6b + 8c + 10d + 32e = n - 2i\}$. As $e_1 \in H^2_{\mathrm{H}}((G/P)/k)$ is a permanent cycle in (14),

$$V_{i,n} \cong V_{i,n} \otimes k \cdot e_1^i \subseteq E_\infty^{n-2i,2i}$$

for $0 \leq i \leq 7$ and $n \leq 16$.

Define a bijection $f_n : S_n \to \bigcup_{i=0}^{15} S_{i,n}$ by $f_n(a, b, c, d, e) = (b, c, d, e, (a - i)/16) \in S_{i,n}$ for $a \equiv i \mod (16)$. Then

$$\dim_k H^n_{\mathrm{H}}(BL/k) = |S_n| = \sum_{i=0}^{15} |S_{i,n}| = \sum_{i=0}^{15} \dim_k V_{i,n}. \tag{16}$$

Now assume that $n \leq 14$. Then $f_n$ gives a bijection

$$S_n \to \bigcup_{i=0}^{7} S_{i,n}.$$

As

$$\dim_k H^n_{\mathrm{H}}(BL/k) \geq \sum_{i=0}^{7} E_\infty^{n-2i,2i}$$

and $V_{i,n} \subseteq E_\infty^{n-2i,2i}$ for $0 \leq i \leq 7$, it follows that $V_{i,n} \cong E_\infty^{n-2i,2i}$ for $0 \leq i \leq 7$ and $E_\infty^{n-2i,2i} = 0$ for $i \geq 8$. In particular, $E_\infty^{0,10} \cong k \cdot e_1^5$. As mentioned above, we have $H^m_{\mathrm{H}}(BG/k) = 0$ for $m = 3, 5, 9$. After adding a $k$-multiple of $e_3 e_1^2$ to $e_5$, we can assume that $d_7(e_5) = 0$. Then the isomorphism $E_\infty^{0,10} \cong k \cdot e_1^5$ implies that $d_{11}(e_5) \neq 0$. Hence, $e_5$ is transgressive in (14) and $y_{11} \in H^6(BG, \Omega^5)$ is a lifting of $d_{11}(e_5)$ to $E_2^{11,0}$.

Fix a variable $x$. Let $J_*$ denote the spectral sequence with $E_2$ page $J_2 = \Delta(e_5) \otimes k[x]$ where $e_5$ has bidegree $(0, 10)$, $x$ has bidegree $(11, 0)$, and $e_5$ is transgressive with $d_{11}(e_5 x^i) = x^{i+1}$ for all $i$.

$$\begin{array}{ccccc} k \cdot e_5 & & k \cdot e_5 x & & \cdots \\ & \searrow^{d_{11}} & & \searrow^{d_{11}} & \\ 0 & & k \cdot x & & k \cdot x^2 \qquad \cdots \end{array}$$

As $e_5$ is transgressive in (14), there exists a map of spectral sequences $J_* \to E_*$ taking $e_5$ to $e_5$ and $x$ to $y_{11}$. Tensoring with the map $\alpha$ defined above, we get a map

$$\alpha' : K_* := I_* \otimes J_* \to E_*$$

which induces an isomorphism on the 0th columns of the $E_2$ pages. The $E_\infty$ page of $K_*$ is given by

$$K_\infty \cong I_\infty \cong F_2 \otimes k[y_4, y_6, y_8, y_{10}, y_{32}].$$

As mentioned above, $\alpha$ and hence $\alpha'$ induce isomorphisms on $E_\infty^{n-i,i}$ terms for $n < 16$ and injections on all $E_\infty$ terms on or below the line $i + j = 17$. Theorem 1.4 implies that $\alpha'$ induces an isomorphism $K_2^{n,0} \to E_2^{n,0}$ for $n < 16$.

Next, we consider the filtration on $H^{16}_{\mathrm{H}}(BL/k)$ given by (14). From (16),

$$\dim_k H^{16}_{\mathrm{H}}(BL/k) = 1 + \sum_{i=0}^{7} |S_{i,16}| = 1 + \sum_{i=0}^{7} \dim_k V_{i,16} \otimes k \cdot e_1^i.$$

We must then have either $d_7(e_3 f) = y_7 f = 0 \in H^{17}_{\mathrm{H}}(BG/k)$ for some $0 \neq f \in H^{10}_{\mathrm{H}}(BG/k)$ or $d_{11}(e_5 g) = y_{11} g = d_7(e_3)h = y_7 h \in H^{17}_{\mathrm{H}}(BG/k)$ for some $0 \neq g \in H^{6}_{\mathrm{H}}(BG/k)$ and $h \in H^{10}_{\mathrm{H}}(BG/k)$. Let $a, b, c \in k$, not all zero, such that $ay_{11}y_6 + by_7y_{10} + cy_7y_4y_6 = 0 \in H^{17}_{\mathrm{H}}(BG/k)$.

The class $au_{11}u_6 + bu_7u_{10} + cu_7u_4u_6 \in H^{17}_{\mathrm{H}}(BSO(11)_k/k)$ pulls back to $ay_{11}y_6 + by_7y_{10} + cy_7y_4y_6 = 0 \in H^{17}_{\mathrm{H}}(BG/k)$. Under the pullback map

$$H^*_{\mathrm{H}}(BSO(11)_k/k) \to H^*_{\mathrm{H}}(BK/k)/\mathrm{rad} \cong k[t_1, \ldots, t_5, s]/(t_1 + \cdots + t_5),$$

$au_{11}u_6 + bu_7u_{10} + cu_7u_4u_6$ maps to $asu_{10}u_6 + bsu_6u_{10} + csu_6u_4u_6$, which equals 0 since $Q \to K$ is split. Then $c = 0$ and $a = b$ since the elementary symmetric polynomials (15) in

$$k[t_1, \ldots, t_5]/(t_1 + \cdots + t_5)$$

generate a polynomial subring.

$$\begin{array}{ccc} au_{11}u_6 + bu_7u_{10} + cu_7u_4u_6 \in H^{17}_{\mathrm{H}}(BSO(11)_k/k) & \longrightarrow & 0 \in H^{17}_{\mathrm{H}}(BG/k) \\ \downarrow & & \downarrow \\ asu_{10}u_6 + bsu_6u_{10} + csu_6u_4u_6 \in H^{17}_{\mathrm{H}}(BK/k)/\mathrm{rad} & \longrightarrow & 0 \in H^{17}_{\mathrm{H}}(BQ/k)/\mathrm{rad} \end{array}$$

Thus, the relation $y_7y_{10} + y_6y_{11} = 0$ holds in $H^*_{\mathrm{H}}(BG/k)$ and $E_\infty^{6,10} \cong (k \cdot y_6 \otimes e_5) \oplus (k \cdot y_6 \otimes e_1^5)$. We now use the relation $y_7y_{10} + y_6y_{11} = 0$ to define a new spectral sequence $L_*$ from $K_*$. Let $(y_6x + yy_{10}) \subset K_2$ denote the ideal generated by $y_6x + yy_{10}$ and let $L_2 \coloneqq K_2/(y_6x + yy_{10})$. Define the differentials $d'_m$ of $L_*$ so that $K_2 \to L_2$ induces a map of spectral sequences $K_* \to L_*$ and $d'_m = 0$ for $m > 11$. Then $\alpha' : K_* \to E_*$ induces a map of spectral sequences $\alpha'' : L_* \to E_*$. The $E_\infty$ page of $L_*$ is given by

$$L_\infty \cong (F_2 \otimes k[y_4, y_6, y_8, y_{10}, y_{32}]) \oplus (F_2 \otimes y_6k[y_4, y_6, y_8, y_{10}, y_{32}] \otimes e_5).$$

We now show by induction that $\alpha''$ induces an isomorphism $L_2^{n,0} \to E_2^{n,0}$ for all $n$. For $n < 16$, we have shown that $L_2^{n,0} \cong E_2^{n,0}$. Now let $n \geq 16$ and assume that $\alpha''$ induces an isomorphism $L_2^{m,0} \to E_2^{m,0}$ for all $m < n$. First, suppose that $n$ is even. As $L_2^{m,0} \cong E_2^{m,0}$ for $m < n$, $y_7g \neq 0 \in H^*_{\mathrm{H}}(BG/k)$ for all $0 \neq g \in H^*_{\mathrm{H}}(BG/k)$ with $|g| < n - 7$. Hence, for any $0 \neq g \in H^m_{\mathrm{H}}(BG/k)$ with $|g| = m < n - 7$, $g \otimes e_3e_5 \in E_2^{m,16} \cong E_7^{m,16}$ is not in the kernel of the differential $d_7 : E_7^{m,16} \to E_7^{m+7,10} \cong E_2^{m+7,10}$. As $y_7 \in H^4(BG, \Omega^3)$ and $y_{11} \in H^6(BG, \Omega^5)$, $y_7z, y_{11}z \notin \oplus_i H^i(BG, \Omega^i)$ for all $z \in H^*_{\mathrm{H}}(BG/k)$. It follows that $\alpha''$ induces an injection $L_\infty^{i,j} \to E_\infty^{i,j}$ for all $i, j$ with $m = i + j \leq n$ :

$$L_\infty^{m-2i,2i} \cong V_{i,m} \otimes e_1^i \subseteq E_\infty^{m-2i,2i}$$

for $0 \leq i \leq 4$,

$$L_\infty^{m-2i,2i} \cong (V_{i,m} \otimes e_1^i) \oplus (y_6V_{i+3,m} \otimes e_1^{i-5}e_5) \subseteq E_\infty^{m-2i,2i}$$

for $5 \leq i \leq 7$, and

$$L_\infty^{m-2i,2i} \cong y_6V_{i+3,m} \otimes e_1^{i-5}e_5 \subseteq E_\infty^{m-2i,2i}$$

for $8 \leq i \leq 12$. The equality in (16) then implies that $\alpha''$ induces isomorphisms $L_\infty^{i,j} \to E_\infty^{i,j}$ for all $i, j$ with $i + j \leq n$. As mentioned above, $\alpha''$ induces isomorphisms $0 = L_\infty^{n+1-i,i} \to E_\infty^{n+1-i,i} = 0$ for all $i$ since $n + 1$ is odd. Theorem 1.4 then implies that $\alpha''$ induces an isomorphism $L_2^{n,0} \cong E_2^{n,0} = H^n_{\mathrm{H}}(BG/k)$.

Now assume that $n$ is odd. We have $0 = L_\infty^{i,j} \cong E_\infty^{i,j} = 0$ for all $i, j$ with $i + j = n$. An argument similar to the one used above for when $n$ is even shows that $\alpha''$ induces injections $L_\infty^{i,j} \to E_\infty^{i,j}$ for all $i, j$ with $i + j \leq n + 1$. Equation (16) then implies that $\alpha''$ induces isomorphisms $L_\infty^{i,j} \to E_\infty^{i,j}$ for all $i, j$ with $i + j \leq n + 1$. It follows that $\alpha''$ induces an isomorphism $L_2^{n,0} \cong E_2^{n,0} = H^n_{\mathrm{H}}(BG/k)$ by an application of Theorem 1.4. Thus, by induction, we have obtained that the 0th row of $L_2$ is isomorphic to the 0th row of $E_2$ :

$$H^*_{\mathrm{H}}(BG/k) = k[y_4, y_6, y_7, y_8, y_{10}, y_{11}, y_{32}]/(y_7y_{10} + y_6y_{11}).$$

The Hodge spectral sequence for $BG$ degenerates by Proposition 5.

□

**Corollary 3.16.** *Let $G$ be a $k$-form of* $\mathrm{Spin}(11)$. *Then*

$$H^*_{\mathrm{H}}(BG/k) \cong k[y_4, y_6, y_7, y_8, y_{10}, y_{11}, y_{32}]/(y_7y_{10} + y_6y_{11})$$

*where* $|y_i| = i$ *for* $i = 4, 6, 7, 8, 10, 11, 32$.

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