

On the motive of Quot schemes of zero-dimensional quotients on a curve

Massimo Bagnarol, Barbara Fantechi
and Fabio Perroni

ABSTRACT. For any locally free coherent sheaf on a fixed smooth projective curve, we study the class, in the Grothendieck ring of varieties, of the Quot scheme that parametrizes zero-dimensional quotients of the sheaf. We prove that this class depends only on the rank of the sheaf and on the length of the quotients. As an application, we obtain an explicit formula that expresses it in terms of the symmetric products of the curve.

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1. Introduction

Let C be a smooth projective curve over an algebraically closed ground field k , and let \mathcal{E} be a locally free sheaf of rank r on C . For any $n \in \mathbb{Z}_{>0}$, let $\text{Quot}_{C/k}^n(\mathcal{E})$ be the Quot scheme which parametrizes coherent quotients of \mathcal{E} with finite support and n -dimensional space of global sections. This Quot scheme is a smooth projective k -variety, and it follows easily from its definition that for any invertible sheaf \mathcal{L} on C there is a natural isomorphism $\text{Quot}_{C/k}^n(\mathcal{E}) \cong \text{Quot}_{C/k}^n(\mathcal{E} \otimes \mathcal{L})$; in particular, if $r = 1$, $\text{Quot}_{C/k}^n(\mathcal{E})$ is isomorphic to $\text{Quot}_{C/k}^n(\mathcal{O}_C)$. In the case where $r > 1$, the isomorphism class

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of $\text{Quot}_{C/k}^n(\mathcal{E})$ depends on \mathcal{E} , as one already sees when $n = 1$, in which case $\text{Quot}_{C/k}^1(\mathcal{E})$ is isomorphic to $\mathbb{P}(\mathcal{E})$, the projective space bundle associated to \mathcal{E} (see Section 2.3). Our aims are to study the class of $\text{Quot}_{C/k}^n(\mathcal{E})$ (where $r \geq 2$) in the Grothendieck ring of k -varieties and to compute it in terms of the classes $[\text{Sym}^m(C)]$, $m \geq 0$. In particular, we prove the following result.

Main Theorem. *Under the previous hypotheses, the equality*

$$[\text{Quot}_{C/k}^n(\mathcal{E})] = [\text{Quot}_{C/k}^n(\mathcal{O}_C^{\oplus r})]$$

holds true in the Grothendieck ring $K_0(\text{Var})$ of k -varieties.

Over the field of complex numbers \mathbb{C} , this theorem has been recently extended by Ricolfi to quasi-projective smooth varieties of any dimension [Ric19]. His proof relies on the theory of power structures, while ours is more elementary and self-contained.

The results of this paper were applied by the first author to the study of the Betti and Hodge numbers of moduli spaces of genus 0 stable maps to Grassmannians [Bag19].

Organization of the paper. In Section 2 we present some well-known facts about Quot schemes on smooth projective curves. In particular, in §2.1 we recall the existence of a natural morphism $\sigma : \text{Quot}_{C/k}^n(\mathcal{E}) \rightarrow \text{Sym}^n(C)$, which we describe explicitly in §2.2. Finally, in §2.3 we consider $\text{Quot}_{C/k}^1(\mathcal{E})$, and we show that, in general, it is not isomorphic to $\text{Quot}_{C/k}^1(\mathcal{O}_C^{\oplus r})$.

A more detailed study of the morphism σ is the subject of Section 3, where we show that the fibers of σ only depend (up to isomorphism) on the rank of the locally free sheaf \mathcal{E} .

Section 4 contains the proof of our main theorem. As an application, we explicitly compute $[\text{Quot}_{C/k}^n(\mathcal{E})] \in K_0(\text{Var})$ and we prove that $\text{Quot}_{C/k}^n(\mathcal{E})$ is irreducible.

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2. Notations and basic results

In this section, we recall some basic results that are relevant for us and we fix the notation. For the proofs and for more details we refer to [Gro61]. Throughout the paper, we work over an algebraically closed ground field k . By a variety we mean a reduced separated scheme of finite type over k , not necessarily irreducible.

Let X be a projective scheme. Let $\mathcal{O}_X(1)$ be a very ample line bundle on X , let $P \in \mathbb{Q}[t]$ be a polynomial with rational coefficients, and let \mathcal{F} be a coherent sheaf on X . We denote by $\text{Quot}_{X/k}^P(\mathcal{F})$ the Quot scheme that parametrizes coherent quotients of \mathcal{F} with Hilbert polynomial P . Let us

recall that $\mathrm{Quot}_{X/k}^P(\mathcal{F})$ is a projective scheme, which represents the contravariant functor that associates to any locally noetherian scheme S the set of isomorphism classes of S -flat coherent quotients $q : \mathcal{F}_S \rightarrow \mathcal{H}$, such that the Hilbert polynomial of \mathcal{H}_s is equal to P , for all $s \in S$. Here, \mathcal{F}_S (respectively \mathcal{H}_s) is the pullback of \mathcal{F} to $S \times_k X$ under the projection onto the second factor (respectively the pullback of \mathcal{H} to X_s). In particular, the identity morphism of $\mathrm{Quot}_{X/k}^P(\mathcal{F})$ corresponds to the universal quotient $u : \mathcal{F}_{\mathrm{Quot}_{X/k}^P(\mathcal{F})} \rightarrow \mathcal{Q}$.

A similar result holds true if X is replaced by a quasi-projective scheme U . In this case, one defines a functor as before, with the additional requirement that \mathcal{H} has proper support over S . Then, this functor is representable by a quasi-projective scheme $\mathrm{Quot}_{U/k}^P(\mathcal{F})$. The relation between the two constructions is given by the following result (see also [Nit05]).

Theorem 2.1. *Let X and \mathcal{F} be as before, and let $U \subseteq X$ be an open subscheme. Then $\mathrm{Quot}_{U/k}^P(\mathcal{F}|_U)$ is naturally an open subscheme of $\mathrm{Quot}_{X/k}^P(\mathcal{F})$.*

Now, let us consider the case where P is a constant polynomial equal to $n \in \mathbb{Z}_{>0}$. Then $\mathrm{Quot}_{X/k}^n(\mathcal{F})$ parametrizes coherent quotients of \mathcal{F} with finite support and such that the dimension of the space of sections is equal to n . Therefore, $\mathrm{Quot}_{X/k}^n(\mathcal{F})$ is independent of $\mathcal{O}_X(1)$.

In this paper, the main object of study is $\mathrm{Quot}_{C/k}^n(\mathcal{E})$, where C is a smooth projective curve and \mathcal{E} is locally free.

Notation. Throughout the article, $\mathrm{Quot}_{C/k}^n(\mathcal{E})$ will be denoted by $\mathbb{Q}_C^n(\mathcal{E})$. Accordingly, the corresponding universal quotient on $\mathbb{Q}_C^n(\mathcal{E}) \times C$ will be denoted by $u : \mathcal{E}_{\mathbb{Q}_C^n(\mathcal{E})} \rightarrow \mathcal{Q}$. Whenever $U \subseteq X$ is an open subscheme, we will write $\mathbb{Q}_U^n(\mathcal{E})$ for the Quot scheme $\mathrm{Quot}_{U/k}^n(\mathcal{E}|_U)$, and $u_0 : \mathcal{E}_{\mathbb{Q}_U^n(\mathcal{E})} \rightarrow \mathcal{Q}_0$ for its universal quotient.

Let us first recall the following fact.

Lemma 2.2. *Let C be a smooth projective curve, and let \mathcal{E} be a locally free coherent sheaf of rank r on C . Then $\mathbb{Q}_C^n(\mathcal{E})$ is a smooth variety of dimension nr .*

Proof. Let $[q : \mathcal{E} \rightarrow \mathcal{H}]$ be a k -rational point of $\mathbb{Q}_C^n(\mathcal{E})$. Since the support of \mathcal{H} is 0-dimensional, we have that

$$\mathrm{Ext}^1(\ker(q), \mathcal{H}) \cong \mathrm{H}^1(C, \ker(q)^\vee \otimes \mathcal{H}) = 0.$$

The smoothness now follows from [HL10, Prop. 2.2.8]. Moreover, the dimension coincides with that of the Zariski tangent space at the point $[q : \mathcal{E} \rightarrow \mathcal{H}]$, which is equal to $\dim \mathrm{H}^0(C, \ker(q)^\vee \otimes \mathcal{H}) = nr$. \square

As a consequence of Theorem 2.1 and Lemma 2.2, $\mathbb{Q}_U^n(\mathcal{E})$ is a smooth quasi-projective variety, for any $U \subseteq C$ open.

Remark 2.3. We will prove in Corollary 4.7 that, under the above hypotheses, $\mathbb{Q}_C^n(\mathcal{E})$ is irreducible.

2.1. The morphism σ . In the proof of the Main Theorem we will use the morphism $\mathfrak{N}_{X/k}$ defined in [Gro61, §6], which will be denoted by σ in this article. The following result is a special case of Grothendieck's construction.

Proposition 2.4. *Let \mathcal{F} be a coherent sheaf on C . Then there exists a canonical morphism $\sigma : \mathbb{Q}_C^n(\mathcal{F}) \rightarrow \mathrm{Sym}^n(C)$ that maps any k -rational point $[q : \mathcal{F} \rightarrow \mathcal{H}]$ to the effective divisor*

$$\mathrm{div}(\mathcal{H}) := \sum_{p \in C} \dim_k(\mathcal{H}_p) p,$$

where \mathcal{H}_p is the stalk of \mathcal{H} at p .

Remark 2.5. If $U \subseteq C$ is open, then $\sigma^{-1}(\mathrm{Sym}^n(U))$ can be naturally identified with $\mathbb{Q}_U^n(\mathcal{E})$. Hereafter, the morphism induced by σ will be denoted by $\sigma_U : \mathbb{Q}_U^n(\mathcal{E}) \rightarrow \mathrm{Sym}^n(U)$.

When $\mathcal{E} = \mathcal{O}_C$, $\mathbb{Q}_U^n(\mathcal{O}_C) = \mathrm{Hilb}_{U/k}^n$ and this morphism is the Hilbert-Chow morphism $\rho_U : \mathrm{Hilb}_{U/k}^n \rightarrow \mathrm{Sym}^n(U)$ of [FG05]. Notice that ρ_U is an isomorphism. Therefore, for any \mathcal{E} , σ_U factors through ρ_U via the morphism $\tau_U := \rho_U^{-1} \circ \sigma_U : \mathbb{Q}_U^n(\mathcal{E}) \rightarrow \mathrm{Hilb}_{U/k}^n$.

2.2. Explicit construction of τ_U . For later use, we provide here an explicit construction of τ_U .

Let us consider the universal quotient $u_0 : \mathcal{E}_{\mathbb{Q}_U^n(\mathcal{E})} \rightarrow \mathcal{Q}_0$ associated to $\mathbb{Q}_U^n(\mathcal{E})$. If $\mathcal{K} = \ker(u_0)$ and $\iota : \mathcal{K} \rightarrow \mathcal{E}_{\mathbb{Q}_U^n(\mathcal{E})}$ is the inclusion, then we have the short exact sequence

$$0 \rightarrow \mathcal{K} \xrightarrow{\iota} \mathcal{E}_{\mathbb{Q}_U^n(\mathcal{E})} \xrightarrow{u_0} \mathcal{Q}_0 \rightarrow 0.$$

Since both $\mathcal{E}_{\mathbb{Q}_U^n(\mathcal{E})}$ and \mathcal{Q}_0 are flat over $\mathbb{Q}_U^n(\mathcal{E})$, \mathcal{K} is flat over $\mathbb{Q}_U^n(\mathcal{E})$ too. Moreover, the restriction of \mathcal{K} to U_q is locally free, for all $q \in \mathbb{Q}_U^n(\mathcal{E})$. It follows that \mathcal{K} is a locally free sheaf of rank $r = \mathrm{rk}(\mathcal{E})$.

Let $\wedge^r(\iota) : \wedge^r(\mathcal{K}) \rightarrow \wedge^r(\mathcal{E}_{\mathbb{Q}_U^n(\mathcal{E})})$ be the r -th exterior power of ι . By tensoring it with $\wedge^r(\mathcal{E}_{\mathbb{Q}_U^n(\mathcal{E})})^\vee$, we get a short exact sequence

$$0 \rightarrow \wedge^r(\mathcal{K}) \otimes \wedge^r(\mathcal{E}_{\mathbb{Q}_U^n(\mathcal{E})})^\vee \rightarrow \mathcal{O}_{\mathbb{Q}_U^n(\mathcal{E}) \times U} \rightarrow \mathcal{G} \rightarrow 0.$$

Notice that \mathcal{G} is flat over $\mathbb{Q}_U^n(\mathcal{E})$, since $\mathcal{O}_{\mathbb{Q}_U^n(\mathcal{E}) \times U}$ is $\mathbb{Q}_U^n(\mathcal{E})$ -flat and $\wedge^r(\iota)$ remains injective when restricted to every fiber (see [Mat80], Thm. 49 and its corollaries). Moreover, the Hilbert polynomial of the restriction of \mathcal{G} to every fiber is equal to n ; indeed, the elementary divisor theorem for PIDs implies that the restriction of \mathcal{G} to every fiber is isomorphic to the restriction of \mathcal{Q}_0 to the same fiber. Therefore, the quotient $\mathcal{O}_{\mathbb{Q}_U^n(\mathcal{E}) \times U} \rightarrow \mathcal{G}$ corresponds to a morphism $\mathbb{Q}_U^n(\mathcal{E}) \rightarrow \mathrm{Hilb}_{U/k}^n$, which is exactly the morphism τ_U defined in Section 2.1.

2.3. The case $n = 1$. The following result should be well known, but we include it here for lack of a suitable reference.

Proposition 2.6. *The Quot scheme $Q_C^1(\mathcal{E})$ is isomorphic to the projective space bundle $\mathbb{P}(\mathcal{E})$.*

Proof. In order to simplify the notation, let us denote $Q_C^1(\mathcal{E})$ by Q , and the universal quotient over $Q \times C$ by $u : \mathcal{E}_Q \rightarrow \mathcal{Q}$.

Let $\sigma : Q \rightarrow C$ be as in Section 2.1, and let $\varphi = (\text{id}_Q, \sigma) : Q \rightarrow Q \times C$ be the morphism with components the identity of Q and σ , respectively. Then the pullback of u via φ gives a quotient $\varphi^*u : \varphi^*\mathcal{E}_Q \rightarrow \varphi^*\mathcal{Q}$. Notice that $\varphi^*\mathcal{Q} \otimes k(q) \cong k(q)$ for any $q \in Q$ (where $k(q)$ is the residue field of q), therefore $\varphi^*\mathcal{Q}$ is locally free of rank 1. Since $\varphi^*\mathcal{E}_Q = \sigma^*\mathcal{E}$, by [Har77, Prop. 7.12] we obtain a morphism $\psi : Q \rightarrow \mathbb{P}(\mathcal{E})$.

On the other hand, let $\pi : \mathbb{P}(\mathcal{E}) \rightarrow C$ be the projection associated to \mathcal{E} , and let $\pi^*\mathcal{E} \rightarrow \mathcal{L}$ be the universal quotient over $\mathbb{P}(\mathcal{E})$. Let us consider the subscheme $\tilde{\Delta} \subset \mathbb{P}(\mathcal{E}) \times C$ whose structure sheaf is $(\pi \times \text{id}_C)^*\mathcal{O}_\Delta$, where $\pi \times \text{id}_C : \mathbb{P}(\mathcal{E}) \times C \rightarrow C \times C$ is the morphism with components π and the identity of C respectively, and $\Delta \subset C \times C$ is the diagonal. The tensor product of the quotient $\mathcal{O}_{\mathbb{P}(\mathcal{E}) \times C} \rightarrow \mathcal{O}_{\tilde{\Delta}}$ with the sheaf $\mathcal{E}_{\mathbb{P}(\mathcal{E})}$ gives a surjection $q : \mathcal{E}_{\mathbb{P}(\mathcal{E})} \rightarrow \mathcal{E}_{\mathbb{P}(\mathcal{E})} \otimes \mathcal{O}_{\tilde{\Delta}}$.

If $\text{pr}_1 : \mathbb{P}(\mathcal{E}) \times C \rightarrow \mathbb{P}(\mathcal{E})$ is the projection, then there is an isomorphism $\mathcal{E}_{\mathbb{P}(\mathcal{E})} \otimes \mathcal{O}_{\tilde{\Delta}} \cong (\text{pr}_1)^*\pi^*\mathcal{E} \otimes \mathcal{O}_{\tilde{\Delta}}$. Therefore, we can compose q with the morphism $(\text{pr}_1)^*\pi^*\mathcal{E} \otimes \mathcal{O}_{\tilde{\Delta}} \rightarrow (\text{pr}_1)^*\mathcal{L} \otimes \mathcal{O}_{\tilde{\Delta}}$, and we obtain a surjective morphism $\mathcal{E}_{\mathbb{P}(\mathcal{E})} \rightarrow (\text{pr}_1)^*\mathcal{L} \otimes \mathcal{O}_{\tilde{\Delta}}$. Since $(\text{pr}_1)^*\mathcal{L} \otimes \mathcal{O}_{\tilde{\Delta}}$ is flat over $\mathbb{P}(\mathcal{E})$ and has constant Hilbert polynomial 1, this surjection corresponds to a morphism $\mathbb{P}(\mathcal{E}) \rightarrow Q$, which is the inverse of ψ by construction. \square

We conclude this section with the following result, from which we deduce that in general $Q_C^n(\mathcal{E})$ depends on \mathcal{E} , if $\text{rk}(\mathcal{E}) \geq 2$.

Proposition 2.7. *Let \mathcal{E} and \mathcal{E}' be two locally free coherent \mathcal{O}_C -modules of the same rank $r \geq 2$. Assume that one of the following conditions holds true:*

- (i) *the genus of C is greater than or equal to 1;*
- (ii) *$r > 2$.*

Then $Q_C^1(\mathcal{E}) \cong Q_C^1(\mathcal{E}')$ if and only if there exists an automorphism ψ of C and an invertible sheaf \mathcal{L} on C , such that $\mathcal{E}' \cong \psi^\mathcal{E} \otimes \mathcal{L}$.*

Proof. First, assume that $Q_C^1(\mathcal{E}) \cong Q_C^1(\mathcal{E}')$. By Proposition 2.6, we thus have an isomorphism $\varphi : \mathbb{P}(\mathcal{E}') \xrightarrow{\sim} \mathbb{P}(\mathcal{E})$. Let us denote the projections of these bundles by $\pi : \mathbb{P}(\mathcal{E}) \rightarrow C$ and $\pi' : \mathbb{P}(\mathcal{E}') \rightarrow C$. Under the hypothesis (i) or (ii), the morphism $\pi \circ \varphi : \mathbb{P}(\mathcal{E}') \rightarrow C$ is constant on the fibers of π' . Therefore, there exists an automorphism ψ of C such that the diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}') & \xrightarrow{\varphi} & \mathbb{P}(\mathcal{E}) \\ \downarrow \pi' & & \downarrow \pi \\ C & \xrightarrow{\psi} & C \end{array}$$

commutes. Since there is also a cartesian diagram

$$\begin{array}{ccc}
 \mathbb{P}(\psi^*\mathcal{E}) & \xrightarrow{\cong} & \mathbb{P}(\mathcal{E}) \\
 \downarrow p & & \downarrow \pi \\
 C & \xrightarrow{\psi} & C
 \end{array} \tag{1}$$

where p is the canonical projection, it follows that $\pi' : \mathbb{P}(\mathcal{E}') \rightarrow C$ and $p : \mathbb{P}(\psi^*\mathcal{E}) \rightarrow C$ are isomorphic over C . By [Har77, §2.7, Exer. 7.9], we deduce that $\mathcal{E}' \cong \psi^*\mathcal{E} \otimes \mathcal{L}$ for some invertible sheaf \mathcal{L} on C .

The inverse implication directly follows from [Har77, §2.7, Exer. 7.9] and the diagram (1). □

3. The fibers of σ

In this section we describe the fibers of the morphism σ introduced in Section 2.1 (analogous results in higher dimension have been obtained in [BF08], [BR18] and [Ric18]). Throughout the section, C denotes a smooth projective curve over k and \mathcal{E} is a coherent locally free \mathcal{O}_C -module of rank r .

Proposition 3.1. *The fiber of the morphism $\sigma : \mathbb{Q}_C^n(\mathcal{E}) \rightarrow \text{Sym}^n(C)$ over a point $D \in \text{Sym}^n(C)$ is isomorphic to the fiber of the analogous morphism $\mathbb{Q}_C^n(\mathcal{O}_C^{\oplus r}) \rightarrow \text{Sym}^n(C)$ over the same point.*

Proof. From Remark 2.5 we have that $\sigma^{-1}(\text{Sym}^n(U))$ depends only on $\mathcal{E}|_U$, for any $U \subseteq C$ open. Then the proposition follows from the fact that for any $D \in \text{Sym}^n(C)$, there exists an open subset $U \subseteq C$ such that $D \in \text{Sym}^n(U)$ and $\mathcal{E}|_U$ is trivial.

In order to see this, let V be an open affine subset of C , such that $D \in \text{Sym}^n(V)$. Then, by [Ser58, Thm. 1], $\mathcal{E}|_V \cong \mathcal{O}_V^{\oplus(r-1)} \oplus \mathcal{L}$, where \mathcal{L} is an invertible \mathcal{O}_V -module. Let us consider the short exact sequence $0 \rightarrow \mathcal{L}(-D) \rightarrow \mathcal{L} \rightarrow \mathcal{L} \otimes \mathcal{O}_D \rightarrow 0$, and the associated exact cohomology sequence, $0 \rightarrow H^0(V, \mathcal{L}(-D)) \rightarrow H^0(V, \mathcal{L}) \rightarrow H^0(V, \mathcal{L} \otimes \mathcal{O}_D) \rightarrow 0$. We deduce that there exists $s \in H^0(V, \mathcal{L})$ such that $s(x) \neq 0$, for all $x \in \text{Supp}(D)$. Hence \mathcal{L} (and consequently \mathcal{E}) is trivial on the open set $U = V \setminus \{x : s(x) = 0\}$. □

Definition 3.2. For any $p \in C$, let us define $F_{n,r}(p) := \sigma^{-1}(np)$. More generally, for any $D \in \text{Sym}^n(C)$, we define $F_r(D) := \sigma^{-1}(D)$.

Proposition 3.3. *Let $D = a_1p_1 + \dots + a_m p_m \in \text{Sym}^n(C)$, with p_1, \dots, p_m pairwise distinct. Then*

$$F_r(D) \cong F_{a_1,r}(p_1) \times \dots \times F_{a_m,r}(p_m).$$

Proof. There is a natural morphism $F_r(D) \rightarrow F_{a_1,r}(p_1) \times \dots \times F_{a_m,r}(p_m)$, which is defined in the following way on k -rational points. For any quotient $[q : \mathcal{E} \rightarrow \mathcal{H}]$ in $F_r(D)$, we have a splitting $\mathcal{H} = \bigoplus_{i=1}^m \mathcal{H}_{p_i}$, where \mathcal{H}_{p_i} is a skyscraper sheaf on C , which is supported in $\{p_i\}$. Therefore $q = \bigoplus_{i=1}^m q_i$, with $q_i : \mathcal{E} \rightarrow \mathcal{H}_{p_i}$. Then the point $[q : \mathcal{E} \rightarrow \mathcal{H}] \in F_r(D)$ is mapped to

$[q_1 : \mathcal{E} \rightarrow \mathcal{H}_{p_1}] \times \cdots \times [q_m : \mathcal{E} \rightarrow \mathcal{H}_{p_m}] \in F_{a_1, r}(p_1) \times \cdots \times F_{a_m, r}(p_m)$. Clearly this is an isomorphism. \square

4. The class of $\mathbf{Q}_C^n(\mathcal{E})$ in $\mathbf{K}_0(\mathbf{Var})$

In this section we prove our main theorem.

Theorem 4.1. *Let C be a smooth projective curve over k . Let \mathcal{E} be a coherent locally free \mathcal{O}_C -module of rank r . Then, for any non-negative integer n , the equality*

$$[\mathbf{Q}_C^n(\mathcal{E})] = [\mathbf{Q}_C^n(\mathcal{O}_C^{\oplus r})]$$

holds true in the Grothendieck group $\mathbf{K}_0(\mathbf{Var})$ of k -varieties.

The proof will be divided into several steps.

Step 1. In order to make the proof clearer, we first fix our notation (see also Section 2). Let $U \subseteq C$ be a fixed open subset such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$, and let $C \setminus U = \{p_1, \dots, p_N\}$. Then $\mathrm{Sym}^n(C)$ is the set-theoretic disjoint union of the locally closed subsets

$$Z_{\mathbf{a}} := \{E \in \mathrm{Sym}^n(C) \mid \mathrm{Supp}(E - a_1 p_1 - \dots - a_N p_N) \subset U\},$$

for $\mathbf{a} \in A := \{(a_1, \dots, a_N) \in \mathbb{N}^N \mid a_1 + \dots + a_N \leq n\}$. Notice that

$$Z_{\mathbf{a}} \cong \mathrm{Sym}^{n-|\mathbf{a}|}(U),$$

where $|\mathbf{a}| := a_1 + \dots + a_N$.

For any $\mathbf{a} \in A$, we denote by $\mathbf{Q}_{\mathbf{a}}(\mathcal{E})$ the preimage of $Z_{\mathbf{a}}$ under the morphism $\sigma : \mathbf{Q}_C^n(\mathcal{E}) \rightarrow \mathrm{Sym}^n(C)$ of Section 2.1, with the reduced subscheme structure.

Remark 4.2. Using the relations in the Grothendieck group of varieties, the decomposition of $\mathbf{Q}_C^n(\mathcal{E})$ into its locally closed subsets $\mathbf{Q}_{\mathbf{a}}(\mathcal{E})$ yields the equality

$$[\mathbf{Q}_C^n(\mathcal{E})] = \sum_{\mathbf{a} \in A} [\mathbf{Q}_{\mathbf{a}}(\mathcal{E})]$$

in $\mathbf{K}_0(\mathbf{Var})$.

Finally, we denote by D the divisor $a_1 p_1 + \dots + a_N p_N \in \mathrm{Sym}^{|\mathbf{a}|}(C)$ corresponding to $\mathbf{a} \in A$. Associated to this effective divisor we have the fiber $F_r(D) \subset \mathbf{Q}_C^{|\mathbf{a}|}(\mathcal{E})$, as in Definition 3.2.

Step 2. The core of our proof is the following proposition.

Proposition 4.3. *For any $\mathbf{a} \in A$, there is a natural isomorphism*

$$\mathbf{Q}_{\mathbf{a}}(\mathcal{E}) \xrightarrow{\sim} \mathbf{Q}_U^{n-|\mathbf{a}|}(\mathcal{E}) \times F_r(D).$$

The idea behind this proposition is that any quotient of \mathcal{E} in $\mathbf{Q}_{\mathbf{a}}(\mathcal{E})$ can be obtained by glueing a quotient supported in U and a quotient supported on $\{p_1, \dots, p_N\}$.

Before proving Proposition 4.3 in Step 3, we need the following result in order to define the morphism $\mathbf{Q}_{\mathbf{a}}(\mathcal{E}) \rightarrow \mathbf{Q}_U^{n-|\mathbf{a}|}(\mathcal{E})$.

Lemma 4.4. *Let \mathcal{Q} be the universal quotient associated to $\mathbf{Q}_C^n(\mathcal{E})$, and let $i : U \times \mathbf{Q}_{\mathbf{a}}(\mathcal{E}) \hookrightarrow C \times \mathbf{Q}_C^n(\mathcal{E})$ be the inclusion. For any $\mathbf{a} \in A$, the support $\text{Supp}(i^*\mathcal{Q})$ is proper over $\mathbf{Q}_{\mathbf{a}}(\mathcal{E})$.*

Proof. We shall apply the valuative criterion of properness to the restriction $\varphi : \text{Supp}(i^*\mathcal{Q}) \rightarrow \mathbf{Q}_{\mathbf{a}}(\mathcal{E})$ of the projection $U \times \mathbf{Q}_{\mathbf{a}}(\mathcal{E}) \rightarrow \mathbf{Q}_{\mathbf{a}}(\mathcal{E})$. So, let R be a valuation ring, and let K be its quotient field. Assume we are given a commutative diagram

$$\begin{array}{ccc} \text{Spec}(K) & \longrightarrow & \text{Supp}(i^*\mathcal{Q}) \\ \downarrow & & \downarrow \varphi \\ \text{Spec}(R) & \longrightarrow & \mathbf{Q}_{\mathbf{a}}(\mathcal{E}) \end{array}$$

where $\text{Spec}(K) \rightarrow \text{Spec}(R)$ is induced by the inclusion $R \hookrightarrow K$.

Since $\text{Supp}(i^*\mathcal{Q}) \subset C \times \mathbf{Q}_{\mathbf{a}}(\mathcal{E})$ and $C \times \mathbf{Q}_{\mathbf{a}}(\mathcal{E})$ is proper over $\mathbf{Q}_{\mathbf{a}}(\mathcal{E})$, there is a unique morphism $\psi : \text{Spec}(R) \rightarrow C \times \mathbf{Q}_{\mathbf{a}}(\mathcal{E})$ such that the diagram

$$\begin{array}{ccccc} \text{Spec}(K) & \longrightarrow & \text{Supp}(i^*\mathcal{Q}) & \hookrightarrow & C \times \mathbf{Q}_{\mathbf{a}}(\mathcal{E}) \\ \downarrow & & \downarrow \varphi & \nearrow & \\ \text{Spec}(R) & \xrightarrow{\psi} & \mathbf{Q}_{\mathbf{a}}(\mathcal{E}) & & \end{array}$$

commutes. The claim follows, if we prove that the image of ψ is contained in $U \times \mathbf{Q}_{\mathbf{a}}(\mathcal{E})$.

Let us consider the composition

$$\psi' : \text{Spec}(R) \xrightarrow{\psi} C \times \mathbf{Q}_{\mathbf{a}}(\mathcal{E}) \xrightarrow{\text{id}_C \times (\tau|_{\mathbf{Q}_{\mathbf{a}}(\mathcal{E})})} C \times \rho^{-1}(Z_{\mathbf{a}}),$$

where $\tau : \mathbf{Q}_C^n(\mathcal{E}) \rightarrow \text{Hilb}_{C/k}^n$ is the morphism defined in Section 2.1. The universal ideal sheaf of $\text{Hilb}_{C/k}^n$ restricted to $C \times \rho^{-1}(Z_{\mathbf{a}})$ is of the form $\mathcal{O}_{C \times \rho^{-1}(Z_{\mathbf{a}})}(-\mathcal{D} - \mathcal{D}')$, for $\mathcal{D} = a_1\{p_1\} \times \rho^{-1}(Z_{\mathbf{a}}) + \dots + a_N\{p_N\} \times \rho^{-1}(Z_{\mathbf{a}})$ and \mathcal{D}' an effective Weil divisor (notice that $C \times \rho^{-1}(Z_{\mathbf{a}})$ is smooth), such that $\mathcal{D} \cap \mathcal{D}' = \emptyset$. Since the image of the composition

$$\text{Spec}(K) \rightarrow \text{Supp}(i^*\mathcal{Q}) \hookrightarrow C \times \mathbf{Q}_{\mathbf{a}}(\mathcal{E}) \xrightarrow{\text{id}_C \times (\tau|_{\mathbf{Q}_{\mathbf{a}}(\mathcal{E})})} C \times \rho^{-1}(Z_{\mathbf{a}})$$

is contained in $\text{Supp}(\mathcal{D}') \subset U \times \rho^{-1}(Z_{\mathbf{a}})$, the same holds for the image of ψ' . Therefore, the image of ψ lies in $U \times \mathbf{Q}_{\mathbf{a}}(\mathcal{E})$, as claimed. \square

Step 3. Using Lemma 4.4, we can now prove Proposition 4.3.

Proof of Proposition 4.3. As above, let $u : \mathcal{E}_{\mathbb{Q}_C^n(\mathcal{E})} \rightarrow \mathcal{Q}$ be the universal quotient associated to $\mathbb{Q}_C^n(\mathcal{E})$, and let $i : U \times \mathbb{Q}_a(\mathcal{E}) \hookrightarrow C \times \mathbb{Q}_C^n(\mathcal{E})$ be the inclusion. By Lemma 4.4, the quotient $i^*u : i^*(\mathcal{E}_{\mathbb{Q}_C^n(\mathcal{E})}) \rightarrow i^*\mathcal{Q}$ yields a natural morphism $f_0 : \mathbb{Q}_a(\mathcal{E}) \rightarrow \mathbb{Q}_U^{n-|\mathbf{a}|}(\mathcal{E})$.

In order to define a morphism $f_\infty : \mathbb{Q}_a(\mathcal{E}) \rightarrow F_r(D)$, let us consider the open neighbourhood $(C \times \mathbb{Q}_a(\mathcal{E})) \setminus \text{Supp}(i^*\mathcal{Q})$ of $\cup_{\lambda=1}^N \{p_\lambda\} \times \mathbb{Q}_a(\mathcal{E})$, together with its inclusion j into $C \times \mathbb{Q}_a(\mathcal{E})$. By composing the pullback of u to $\mathbb{Q}_a(\mathcal{E})$ with the unit of the adjunction $j^* \dashv j_*$, we get a surjective morphism

$$q : \mathcal{E}_{\mathbb{Q}_a(\mathcal{E})} \rightarrow \mathcal{Q}_{\mathbb{Q}_a(\mathcal{E})} \rightarrow j_*j^*(\mathcal{Q}_{\mathbb{Q}_a(\mathcal{E})})$$

of coherent $\mathcal{O}_{C \times \mathbb{Q}_a(\mathcal{E})}$ -modules. Notice that $j_*j^*(\mathcal{Q}_{\mathbb{Q}_a(\mathcal{E})})$ has constant Hilbert polynomial $|\mathbf{a}|$, hence it is flat over $\mathbb{Q}_a(\mathcal{E})$. Therefore, q is associated to a natural morphism $\mathbb{Q}_a(\mathcal{E}) \rightarrow \mathbb{Q}_C^{|\mathbf{a}|}(\mathcal{E})$, whose image is contained in $F_r(D)$. Thus we obtain a morphism $f_\infty : \mathbb{Q}_a(\mathcal{E}) \rightarrow F_r(D)$, and the morphism in Proposition 4.3 is (f_0, f_∞) .

To prove that (f_0, f_∞) is an isomorphism, we exhibit its inverse, as follows. Let $u_0 : \mathcal{E}_{\mathbb{Q}_U^{n-|\mathbf{a}|}(\mathcal{E})} \rightarrow \mathcal{Q}_0$ be the universal quotient of $\mathbb{Q}_U^{n-|\mathbf{a}|}(\mathcal{E})$, and let $u_\infty : \mathcal{E}_{F_r(D)} \rightarrow \mathcal{Q}_\infty$ be the pullback of the universal quotient of $\mathbb{Q}_C^{|\mathbf{a}|}(\mathcal{E})$ to $C \times F_r(D) \subset C \times \mathbb{Q}_C^{|\mathbf{a}|}(\mathcal{E})$. In the following, we view u_0 as a family of quotients of \mathcal{E} supported in U (see Lemma 2.1 above). Let us denote the projections by $\text{pr}_{12} : C \times \mathbb{Q}_U^{n-|\mathbf{a}|}(\mathcal{E}) \times F_r(D) \rightarrow C \times \mathbb{Q}_U^{n-|\mathbf{a}|}(\mathcal{E})$ and $\text{pr}_{13} : C \times \mathbb{Q}_U^{n-|\mathbf{a}|}(\mathcal{E}) \times F_r(D) \rightarrow C \times F_r(D)$. Then

$$(\text{pr}_{12})^*u_0 \oplus (\text{pr}_{13})^*u_\infty : \mathcal{E}_{\mathbb{Q}_U^{n-|\mathbf{a}|}(\mathcal{E}) \times F_r(D)} \rightarrow (\text{pr}_{12})^*\mathcal{Q}_0 \oplus (\text{pr}_{13})^*\mathcal{Q}_\infty$$

is precisely a family of quotients of \mathcal{E} , parametrized by $\mathbb{Q}_U^{n-|\mathbf{a}|}(\mathcal{E}) \times F_r(D)$, with constant Hilbert polynomial equal to n . The associated morphism $\mathbb{Q}_U^{n-|\mathbf{a}|}(\mathcal{E}) \times F_r(D) \rightarrow \mathbb{Q}_C^n(\mathcal{E})$ is the inverse morphism of (f_0, f_∞) . \square

Step 4. We can finally conclude the proof of Theorem 4.1.

By Remark 4.2 and Proposition 4.3, we have the following equalities in $\mathbf{K}_0(\mathbf{Var})$:

$$[\mathbb{Q}_C^n(\mathcal{E})] = \sum_{\mathbf{a} \in A} [\mathbb{Q}_a(\mathcal{E})] = \sum_{\mathbf{a} \in A} [\mathbb{Q}_U^{n-\mathbf{a}}(\mathcal{E})][F_r(D)].$$

In particular, this is true also for $\mathcal{E} = \mathcal{O}_C^{\oplus r}$.

Now, $\mathcal{E}|_U$ is trivial, therefore $[\mathbb{Q}_U^{n-\mathbf{a}}(\mathcal{E})] = [\mathbb{Q}_U^{n-\mathbf{a}}(\mathcal{O}_C^{\oplus r})]$. Theorem 4.1 thus follows from Proposition 3.1.

4.1. Explicit computation. We provide an explicit formula for the class $[\mathbb{Q}_C^n(\mathcal{E})] \in \mathbf{K}_0(\mathbf{Var})$ in terms of the classes $[\text{Sym}^m(C)]$.

Proposition 4.5. *For any non-negative integer n , the equality*

$$[\mathbb{Q}_C^n(\mathcal{E})] = \sum_{\mathbf{n} \in \mathbb{N}^r, |\mathbf{n}|=n} [\mathrm{Sym}^{n_1}(C)] \cdots [\mathrm{Sym}^{n_r}(C)] \cdot [\mathbb{A}_k^1]^{d_{\mathbf{n}}}$$

holds true in $K_0(\mathbf{Var})$, where $d_{\mathbf{n}} := \sum_{i=1}^r (i-1)n_i$.

Proof. From Theorem 4.1 we have that $[\mathbb{Q}_C^n(\mathcal{E})] = [\mathbb{Q}_C^n(\mathcal{O}_C^{\oplus r})]$. Then the result follows directly from [Bif89]. \square

Remark 4.6. From the previous formula we can determine the Poincaré polynomial of $\mathbb{Q}_C^n(\mathcal{E})$ (for ℓ -adic cohomology, where $\ell \neq \mathrm{char}(k)$ is a prime) as follows (cf. also [BGL94]). By [Mac62], the Poincaré polynomial $P(\mathrm{Sym}^m(C); t)$ of $\mathrm{Sym}^m(C)$ is the coefficient of u^m in the expansion of

$$\frac{(1+tu)^{2g}}{(1-u)(1-t^2u)},$$

where g is the genus of C . Then, for $E(t, u) := \sum_{n=0}^{\infty} P(\mathbb{Q}_C^n(\mathcal{E}); t) u^n$, we have:

$$\begin{aligned} E(t, u) &= \sum_{n=0}^{\infty} \sum_{\mathbf{n} \in \mathbb{N}^r, |\mathbf{n}|=n} P(\mathrm{Sym}^{n_1}(C); t) \cdots P(\mathrm{Sym}^{n_r}(C); t) t^{2d_{\mathbf{n}}} u^n \\ &= \sum_{\mathbf{n} \in \mathbb{N}^r} \prod_{i=1}^r P(\mathrm{Sym}^{n_i}(C); t) u^{n_i} t^{2(i-1)n_i} \\ &= \prod_{i=0}^{r-1} \frac{(1+t^{2i+1}u)^{2g}}{(1-t^{2i}u)(1-t^{2i+2}u)}. \end{aligned}$$

Corollary 4.7. *The Quot scheme $\mathbb{Q}_C^n(\mathcal{E})$ is irreducible.*

Proof. Since $\mathbb{Q}_C^n(\mathcal{E})$ is smooth, it suffices to show that the coefficient of t^0 in the Poincaré polynomial of $\mathbb{Q}_C^n(\mathcal{E})$ is 1. To this aim, notice that, for any $\mathbf{n} \in \mathbb{N}^r$ with $|\mathbf{n}| = n$, we have that $d_{\mathbf{n}} = \sum_{i=1}^r (i-1)n_i = 0$ only for $\mathbf{n} = (n, 0, \dots, 0)$. The claim now follows from Proposition 4.5. \square

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(Massimo Bagnarol) SISSA – INTERNATIONAL SCHOOL FOR ADVANCED STUDIES, VIA BONOMEA 265, 34136 TRIESTE, ITALY
 mbagnarol@sissa.it

(Barbara Fantechi) SISSA – INTERNATIONAL SCHOOL FOR ADVANCED STUDIES, VIA BONOMEA 265, 34136 TRIESTE, ITALY
 fantechi@sissa.it

(Fabio Perroni) DIPARTIMENTO DI MATEMATICA E GEOSCIENZE, UNIVERSITÀ DEGLI STUDI DI TRIESTE, VIA VALERIO 12/1, 34127 TRIESTE, ITALY
 fperroni@units.it

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