

Extending to a model structure is not a first-order property

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ABSTRACT. Let \mathcal{C} be a finitely bicomplete category and \mathcal{W} a subcategory. We prove that the existence of a model structure on \mathcal{C} with \mathcal{W} as the subcategory of weak equivalence is not first order expressible. Along the way we characterize all model structures where \mathcal{C} is a partial order and show that these are determined by the homotopy categories.

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Introduction

What is a “homotopy theory”? Colloquially, it is a context in which one classifies objects up to “weak equivalence” instead of up to isomorphism: chain complexes up to quasiisomorphism [Qui67, Section 4 Remarks], topological spaces up to homotopy equivalence [Str72] or weak equivalence [Qui67, Theorem II.3.1], categories up to functors which are homotopy equivalences on geometric realization [Tho80], etc. These can be modeled and formalized in many different ways. However, if we wish to show that homotopy theories are equivalent (possibly up to “homotopy”, inside a “homotopy theory of homotopy theories”) then we often use Quillen’s model

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categories [Qui67] to prove this. A model category has three distinguished classes of morphisms—weak equivalences, cofibrations and fibrations—of which only the weak equivalences characterize the homotopy theory. The cofibrations and fibrations are there purely to assist in calculations and constructions. Due to the presence of this extra structure, model categories are rigid and have many computational and formal methods for working with them. However—and also due to the presence of extra structure—they generally do not appear fully formed: they often arise in situations where the weak equivalences are known, but the choice of cofibrations and fibrations is not.

More concretely, we know that model categories naturally produce models of homotopy theories, such as quasi-categories [Joy02, Lur09], simplicially enriched categories [Qui67, DK80, DK87] and complete Segal spaces [Ber07]. However, there is no known way of identifying which quasi-categories (for instance) arise from model structures.¹ Thus model categories live in a strange gray area of homotopy theory: we know that all models of the homotopy theory of homotopy theories form model categories, and we know that they are all equivalent as model categories. However, we do not know which parts of the homotopy theory of homotopy theories can be explored purely using the theory of model categories.

One easy place to start the comparison would be with Barwick–Kan’s model of relative categories [BK12, DHKS04]. A relative category is simply a pair $(\mathcal{C}, \mathcal{W})$ of a category and a subcategory of weak equivalences. It is known that the category of relative categories is a model category, which is Quillen equivalent to the other models of the homotopy theory of homotopy theories. To try to identify which homotopy theories arise from model categories is to answer the question of when a pair $(\mathcal{C}, \mathcal{W})$ of a category and a subcategory of weak equivalences extends to a model structure. In a few cases, specialized techniques can be used to construct model structures, for example cofibrant generation [Hov99, Section 2.1], Bousfield localization [Hir03, Chapter 4], Cisinski’s minimal model structures [Cis06] or one-dimensional model structures studied in [RT07, BG19]. However, there is no practically useful necessary and sufficient criterion for determining whether it is possible to complete a pair $(\mathcal{C}, \mathcal{W})$ to a model structure.

In this paper, we show that in a well defined sense such a criterion does not exist, and therefore that there is no simple way to characterize which homotopy theories can be accessed by model categories. More concretely, we show that there is no set of first order formulae (i.e. formulae quantifying only over elements but not over sets or functions) which can identify those relative categories that can extend to model categories. Note that if we allow quantifying over sets (or proper classes), we can identify model structures by simply stating that there exist sets of cofibrations and fibrations that

¹For *combinatorial* model categories there is: if the ∞ -category represented by the pair is locally presentable. [Lur09, Proposition A.3.7.6]

satisfy the model category axioms. One could hope, however, that there exist simpler formulas, which only quantified over objects or morphisms in categories, that could identify which relative categories arise from model categories. This is unfortunately not the case.

Theorem A. *In the language of categories with a designated subcategory of weak equivalence there is no first order characterization of those that extend to model structures.*

To prove this theorem we construct two different pairs $(\mathcal{C}, \mathcal{W})$ and $(\mathcal{C}', \mathcal{W}')$ that satisfy all of the same first-order statements but such that $(\mathcal{C}, \mathcal{W})$ does not extend to a model structure while $(\mathcal{C}', \mathcal{W}')$ does. To accomplish this we produce a complete characterization of all model categories whose underlying categories are posets: those skeletal categories for which $|\text{Hom}(A, B)| \leq 1$ for all A and B . We then show that a simpler characterization exists when the underlying category is countable, and use this to produce the desired pairs.

The characterization of model categories on posets is the following.

Theorem B. *Let \mathcal{C} be a preorder closed under finite limits and colimits, and let \mathcal{W} be a subcategory of \mathcal{C} . A model structure exists on \mathcal{C} with weak equivalences \mathcal{W} if and only if the following conditions hold.*

- (a) *For any two composable morphisms f and g in \mathcal{C} , if gf is in \mathcal{W} then f and g are in \mathcal{W} .*
- (b) *There exists a functor $\chi: \mathcal{C} \rightarrow \mathcal{C}$ such that $\chi(\mathcal{W}) \subseteq \text{iso}\mathcal{C}$, and for every object $A \in \mathcal{C}$, the diagram*

$$\begin{array}{ccc}
 A \times \chi(A) & \longrightarrow & \chi(A) \\
 \downarrow & & \downarrow \\
 A & \longrightarrow & A \cup \chi(A)
 \end{array}$$

lies in \mathcal{W} .

When \mathcal{C} is countable we prove a stronger statement:

Theorem C. *If \mathcal{W} has only a countable number of connected components then there is a first-order characterization of when $(\mathcal{C}, \mathcal{W})$ extends to a model structure.*

Thus when \mathcal{C} is countable the existence of a model structure extending $(\mathcal{C}, \mathcal{W})$ is a first-order condition. Therefore in order to construct the desired counterexample it suffices to construct a pair $(\mathcal{C}, \mathcal{W})$ that satisfies the first order characterization from Theorem C (but not the condition of countability!) and does not satisfy the conditions of Theorem B. By the Löwenheim–Skolem [Mal36] Theorem there exists a countable model $(\mathcal{C}', \mathcal{W}')$ which satisfies all of the same first-order statements that $(\mathcal{C}, \mathcal{W})$ does. By Theorem C the pair $(\mathcal{C}', \mathcal{W}')$ extends to a model structure while $(\mathcal{C}, \mathcal{W})$ does not.

As an interesting aside, we also prove in Theorem 5.4 that any two model structures on a poset that have the same weak equivalences are Quillen equivalent. The proof of this theorem also allows us to construct examples of model categories which are not cofibrantly generated: see Corollary 5.7. Moreover, we show that when the poset and weak equivalences are especially nice the zigzag of equivalences can be taken to consist only of the identity functor on the underlying category; see Theorem 5.9. It would be highly desirable to be able to prove that such zigzags exist in general, and the existence of notable special cases (see for example [Dug01, Theorem 5.7] or [Ber07, Theorem 7.5]) shows that it ought to be possible.

This paper is organized as follows. Section 1 contains technical preliminaries on lifting systems, model structures, and the particular ways they behave in posets. Section 2 introduces the notion of a center and explores the interactions of centers and model structures. Section 3 constructs a model structure given a choice of centers and proves Theorem B. Section 4 provides an alternate characterization of the existence of model structures on countable posets and proves Theorem A. Lastly, Section 5 compares different model structures extending a given pair and shows that in many cases all such model structures are equivalent.

Notation. All categories are assumed to be skeletal, in the sense that if $A \rightarrow B$ is an isomorphism in \mathcal{C} then $A = B$. As equivalence of categories preserves model structures and all categories are equivalent to a skeletal category, this does not lose any generality for our results. A *poset* is a skeletal category \mathcal{C} such that for all objects A and B , $\#\text{Hom}_{\mathcal{C}}(A, B) \leq 1$. When \mathcal{C} is small then it uniquely defines a poset in the classical sense, with underlying set $\text{ob}\mathcal{C}$ and relation $A \geq B$ if $\#\text{Hom}_{\mathcal{C}}(A, B) = 1$. Conversely, given a classical poset P we can define a category \mathcal{C} with $\text{ob}\mathcal{C} = P$ and $\text{Hom}_{\mathcal{C}}(A, B) = \{*\}$ if $A \geq B$ and \emptyset otherwise. Thus our notion of a poset corresponds exactly to the classical notion of a poset except that we allow the class of objects to be a proper class, not simply a set.²

In a poset, for any diagram

$$B \longleftarrow A \longrightarrow C$$

the pushout $B \cup_A C$ is equal to $B \cup C$. For concision we write both of these as $B \cup C$. Dually, we write $Y \times Z$ for $Y \times_X Z$.

A category \mathcal{C} is *finitely bicomplete* if it contains all finite limits and colimits; it is *bicomplete* if it contains all small limits and colimits.

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²To be completely consistent we may want to use the word “poclass” instead of “poset” to emphasize this fact, but as “poclass” is a much more nonstandard term we avoid its usage.

of “semi-(co)fibrant” and \mathcal{W}_f^X) greatly improved the paper. Zakharevich was supported in part by NSF grant DMS-1654522.

1. Lifting systems, model structures, and posets

We begin by recalling the definition of maximal lifting system and weak factorization system. For more background on these, especially in relation to model categories, see for example [MP12, Chapter 14] or [Rie14, Section 11].

Definition 1.1. For any two morphisms $f: A \rightarrow B$ and $g: X \rightarrow Y$ in \mathcal{C} we say that f *lifts on the left of g* or g *lifts on the right of f* if for all commutative squares

$$\begin{array}{ccc} A & \longrightarrow & X \\ f \downarrow & & \downarrow g \\ B & \longrightarrow & Y \end{array}$$

there exists a morphism $h: B \rightarrow X$ which makes the diagram commute. If f lifts on the left of g we write $f \square g$.

For any class S of morphisms of \mathcal{C} , we write

$$S^\square = \{g \in \mathcal{C} \mid f \square g \text{ for all } f \in S\}, \text{ and}$$

$$\square S = \{f \in \mathcal{C} \mid f \square g \text{ for all } g \in S\}.$$

Note that both S^\square and $\square S$ can be proper classes.

Definition 1.2. A *maximal lifting system* (henceforth written MLS) in \mathcal{C} is a pair of classes of morphisms $(\mathcal{L}, \mathcal{R})$ satisfying the following three conditions:

- (1) $\mathcal{L} \square \mathcal{R}$.
- (2) $\square \mathcal{R} \subseteq \mathcal{L}$.
- (3) $\mathcal{L}^\square \subseteq \mathcal{R}$.

A *weak factorization system* (henceforth written WFS) is a MLS such that every morphism f in \mathcal{C} can be factored as $f_R f_L$ with $f_R \in \mathcal{R}$ and $f_L \in \mathcal{L}$.

Lemma 1.3. *Let J be any class of morphisms in \mathcal{C} . Then J^\square is closed under composition, pullbacks in \mathcal{C} and arbitrary products. Dually, $\square J$ is closed under composition, pushouts in \mathcal{C} and arbitrary coproducts.*

For a proof, see for example [MP12, 14.1.8].

We now recall the definition of a model category, using the WFS definition (as presented in, for example, [MP12] and [Rie14]).

Definition 1.4. A *model structure* \mathbb{C} on a finitely bicomplete category \mathcal{C} is the specification of three subcategories of \mathcal{C} called the *weak equivalences* (\mathbb{C}_{we}), the *cofibrations* (\mathbb{C}_{cof}) and the *fibrations* (\mathbb{C}_{fib}). Those three subcategories should respect the following axioms.

WFS: The pairs

$$(\mathbb{C}_{\text{cof}}, \mathbb{C}_{\text{fib}} \cap \mathbb{C}_{\text{we}}) \quad (\mathbb{C}_{\text{cof}} \cap \mathbb{C}_{\text{we}}, \mathbb{C}_{\text{fib}})$$

are WFSs.

2OF3: For morphisms f and g , if two of the morphisms f , g and gf are weak equivalences, then so is the third.

We call a morphism which is both a cofibration (resp. fibration) and a weak equivalence an *acyclic cofibration* (resp. *acyclic fibration*). An object A such that the morphism $\emptyset \rightarrow A$ is a cofibration (resp. fibration) is called *cofibrant* (resp. *fibrant*). An object which is both cofibrant and fibrant is called *bifibrant*. We call any connected component of \mathbb{C}_{we} a *weak equivalence class*.

Remark. The definition above is an equivalent restatement of Quillen's original definition of a closed model category. In more modern treatments it is customary to assume that \mathcal{C} is bicomplete, not finitely bicomplete, as the construction of factorizations generally requires small limits and colimits, not just finite ones. In Section 5 we will need this assumption to compare model structures. However, for the main theorem in this paper this assumption is counterproductive, since the existence of small limits and colimits is not a first-order assumption. However, the existence of *finite* limits and colimits is, since it only requires the existence of an initial object, a terminal object, binary (co)products and (co)equalizers.

From this point onwards, \mathcal{C} is a finitely bicomplete poset. We begin with a lemma which is used repeatedly to prove lifting properties.

Lemma 1.5. *Let J be a class of morphisms in \mathcal{C} , closed under pushouts along morphisms in \mathcal{C} . Then $J \boxtimes f$ if and only if for all factorizations of $f: A \rightarrow B$ as $A \xrightarrow{f'} C \rightarrow B$, if $f' \in J$ then $f' = 1_A$.*

Proof. First, suppose that $J \boxtimes f$ and consider any factorization of f as $A \xrightarrow{f'} C \rightarrow B$ where $f' \in J$. We then have a diagram

$$\begin{array}{ccc} A & \xrightarrow{=} & A \\ f' \downarrow & & \downarrow \\ C & \longrightarrow & B \end{array}$$

which must have a lift; thus $A = C$.

Conversely, suppose that the condition in the lemma holds, and consider any diagram

$$\begin{array}{ccc} X & \longrightarrow & A \\ g \downarrow & & \downarrow f \\ Y & \longrightarrow & B \end{array}$$

with $g \in J$. As J is closed under pushouts, the morphism $g': A \rightarrow A \cup Y$ is also in J . Since f factors through g' and $g' \in J$ we must have $A \cup Y = A$. Thus the morphism $Y \rightarrow A \cup Y = A$ is a lift in the diagram, and $J \boxtimes f$. \square

We now turn to a uniqueness lemma.

Lemma 1.6. *In \mathcal{C} , factorizations into an acyclic cofibration and a fibration or a cofibration and an acyclic fibration are unique.*

Each weak equivalence class has a unique fibrant and cofibrant object. In addition, in each weak equivalence class all elements in the class are at zigzag distance at most two from this object. The zigzags can be chosen to consist of an inverse acyclic fibration and an acyclic cofibration.

Proof. Let $f: A \rightarrow B$ be any morphism, and consider two factorization of f :

$$\begin{array}{ccc} A & \xrightarrow{\subset} & B' \\ \downarrow & & \downarrow \sim \\ B'' & \xrightarrow{\sim} & B \end{array} .$$

Since cofibrations lift against acyclic fibrations, there exist morphisms $B' \rightarrow B$ and $B \rightarrow B'$. Since \mathcal{C} is a poset these must both be identities, and the factorization is unique. The statement for factorizations into an acyclic cofibration followed by a fibration follows analogously.

Suppose that X and Y are two bifibrant objects which are in the same weak equivalence class. Since they are isomorphic in $\text{Ho } \mathcal{C}$, there exist morphisms $X \rightarrow Y$ and $Y \rightarrow X$ in \mathcal{C} ; since \mathcal{C} is a poset these must be identities, and $X = Y$. Thus each weak equivalence class contains a unique bifibrant object.

Now suppose that A is any object. Then there is a diagram

$$\begin{array}{ccc} A & \xleftarrow{\sim} & A^c \\ \sim \downarrow & & \sim \downarrow \\ A^f & \xleftarrow{\sim} & A^{cf} \end{array}$$

where A^c is a cofibrant replacement for A , A^f is a fibrant replacement for A , and A^{cf} is both a cofibrant replacement for A^f and a fibrant replacement for A^c (which will end up being equal because there is a unique bifibrant object in the weak equivalence class of A). This constructs the length-two zigzags. \square

The following condition on a subclass of morphisms is a strengthening of the usual 2-of-3 property for weak equivalences.

Definition 1.7. We say that a class \mathcal{E} of morphisms in \mathcal{C} is *decomposable* if for any morphism $f \in \mathcal{E}$, if $f = gh$ for some morphisms g and h , then both g and h are in \mathcal{E} .

Proposition 1.8. \mathbb{C}_{we} is decomposable.

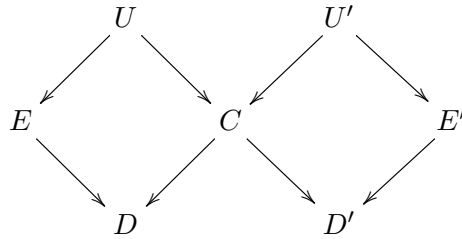
Proof. Fix $f: A \xrightarrow{\sim} B$ in \mathbb{C}_{we} . Write $f = hg$. Factor g as a cofibration followed by an acyclic fibration, and factor f as an acyclic cofibration followed by an acyclic fibration, as illustrated in the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{f_{ac}} & A' \\ g_c \downarrow & \sim & \downarrow f_{af} \\ C' & \xrightarrow{g_{af}} & C \xrightarrow{h} B \end{array}$$

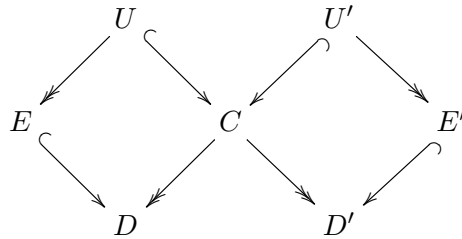
Then this diagram has a lift $\alpha: C' \rightarrow A'$. As \mathcal{C} is a poset, α is the pushout of f_{ac} along g_c , so it must also be an acyclic cofibration. By (2OF3) g_c is also a weak equivalence. Thus g is also a weak equivalence, and by (2OF3) h is as well. \square

We mention an important example of a particular type of weak equivalence class.

Example 1.9. Suppose that \mathbb{C} contains a seven-object weak equivalence class with the following diagram (and no other morphisms between these seven objects):



Then the model structure must assign the morphisms as follows:



C must be the cofibrant fibrant object, as it is the only object with zigzag distance 2 from all other objects in the weak equivalence class. U and U' must be cofibrant, as they receive no weak equivalences; dually, D and D' must be fibrant. The morphisms $U \rightarrow C$ and $C \rightarrow D$ are cofibrations and fibrations, respectively, as U, U' cannot be fibrant and D, D' cannot be cofibrant. By Proposition 1.8, the morphism $U \rightarrow E$ is the pullback of the morphism $C \rightarrow D$ along $E \rightarrow D$, so it is also a fibration; dually, the morphism $E \rightarrow D$ is the pushout of $U \rightarrow C$ and must be a cofibration.

Bifibrant objects are vitally important to model structures, as they are “good choices” for both mapping into and mapping out of. In a poset the choice of bifibrant objects is uniquely functorial, and thus these give a “good” retract of the category.

Proposition 1.10. *The map $C \mapsto C^{cf}$ extends to a functor $\mathcal{C} \rightarrow \mathcal{C}$.*

This observation is the key to the definition of a *center*, given in the following section. We finish up this section with two technical observations which motivate the definition of the model structure in Section 3.

Proposition 1.11. *If B is any cofibrant object in \mathcal{C} and $f: A \rightarrow B$ is any morphism in \mathcal{C} , then f is a cofibration in \mathcal{C} . Dually, if A is fibrant then f is a fibration.*

Proof. Factor f into a cofibration followed by an acyclic fibration and consider the following diagram:

$$\begin{array}{ccccc}
 \emptyset & \longrightarrow & A & \hookrightarrow & A' \\
 \downarrow & & & \searrow & \downarrow \sim \\
 B & \xrightarrow{=} & & & B
 \end{array}$$

By (WFS) this has a lift $B \rightarrow A'$. As \mathcal{C} is a poset we conclude that $B = A'$, so f is equal to the cofibration $A \hookrightarrow A'$. The second part follows by duality. \square

Corollary 1.12. *If $C = C^{cf}$ and $f: U \rightarrow C$ is in \mathcal{C}_{we} then f is an acyclic cofibration. Dually, if $g: C \rightarrow D$ is in \mathcal{C}_{we} then g is an acyclic fibration.*

2. Centers

We now turn to encoding properties of bifibrant objects in a more direct manner. Inspired by Proposition 1.10 we define a “center” of a weak equivalence class to be given by a choice of retraction which is compatible with weak equivalences. Such a retraction will encode all of the relevant properties of bifibrant objects and will allow us to construct a model structure. For the rest of this section, fix a finitely bicomplete poset \mathcal{C} and a subcategory \mathcal{W} that is decomposable. We denote morphisms in \mathcal{W} by $\xrightarrow{\sim}$.

Definition 2.1. A *choice of centers* is a functor

$$\chi: \mathcal{C} \rightarrow \mathcal{C}$$

such that the following properties hold:

C1: The image of $\chi|_{\mathcal{W}}$ only contains identity morphisms.

C2: For all $A \in \mathcal{C}$ the diagram

$$\begin{array}{ccc}
 A \times \chi(A) & \longrightarrow & \chi(A) \\
 \downarrow & & \downarrow \\
 A & \longrightarrow & A \cup \chi(A)
 \end{array}$$

lies in \mathcal{W} .

Condition (C1) implies that if $f: A \rightarrow B$ is in \mathcal{W} then $\chi(f) = 1_{\chi(A)}$. In particular, if there exists a zigzag of morphisms in \mathcal{W} connecting A and B then $\chi(A) = \chi(B)$. In particular, χ must be idempotent: $\chi(\chi(A)) = \chi(A)$.

We can now make our claim that centers are akin to bifibrant objects precise by showing that any model structure produces a choice of centers by taking any objects to its bifibrant approximation.

Lemma 2.2. *Every model structure on \mathcal{C} gives a choice of centers.*

Proof. Let \mathbb{C} be any model structure on \mathcal{C} . We define $\chi(A) = A^{cf}$, the bifibrant object in the same weak equivalence class as A ; this is unique by Lemma 1.6 so χ is well-defined and satisfies the first condition for a choice of centers. To check the second one, let A^c be a cofibrant replacement of A and A^f be a fibrant replacement of A ; then we have a diagram

$$\begin{array}{ccc} A^c & \xrightarrow{\sim} & \chi(A) \\ \sim \downarrow & & \downarrow \sim \\ A & \xrightarrow{\sim} & A^f \end{array}$$

in \mathbb{C}_{we} . By Proposition 1.8, \mathbb{C}_{we} is decomposable, so the square

$$\begin{array}{ccc} A \times \chi(A) & \xrightarrow{\sim} & \chi(A) \\ \sim \downarrow & & \downarrow \sim \\ A & \xrightarrow{\sim} & A \cup \chi(A) \end{array}$$

must also be in \mathbb{C}_{we} . □

Even though χ is uniquely determined by \mathbb{C} , the model structure \mathbb{C} is not uniquely determined by χ .

Example 2.3. The following two model structures have the same choice of centers. All cofibrant objects (other than \emptyset) are marked with \cdot^c and all fibrant objects (other than $*$) are marked with \cdot^f .

$$\begin{array}{ccc} & B^c & \\ \sim \nearrow & & \searrow \sim \\ \emptyset \hookrightarrow A^c & \xrightarrow{\sim} & C^{cf} \twoheadrightarrow * \\ \sim \searrow & & \nearrow \sim \\ & B'^c & \end{array} \qquad \begin{array}{ccc} & B & \\ \sim \nearrow & & \searrow \sim \\ \emptyset \hookrightarrow A^c & \xrightarrow{\sim} & C^{cf} \twoheadrightarrow * \\ \sim \searrow & & \nearrow \sim \\ & B' & \end{array}$$

Just as bifibrant objects record the homotopical information in a model structure, the choice of centers records homotopical information in a poset. In particular, choices of centers identify the weak equivalences.

Lemma 2.4. *Any morphism $f: A \rightarrow B$ in \mathcal{C} such that $\chi(A) = \chi(B)$ is in \mathcal{W} .*

In particular, this implies that in a model structure, any morphism between two objects in the same weak equivalence class is itself a weak equivalence.

Proof. Let $C = \chi(A) = \chi(B)$. By (C2) the morphisms $A \times C \rightarrow C$ and $C \rightarrow B \cup C$ are both in \mathcal{W} . Thus $A \times C \rightarrow C \rightarrow B \cup C$ is in \mathcal{W} . But we can also factor this morphism as

$$A \times C \longrightarrow A \xrightarrow{f} B \longrightarrow B \cup C,$$

so, since \mathcal{W} is decomposable, f is a weak equivalence. □

It is also the case that choices of centers are all closely related.

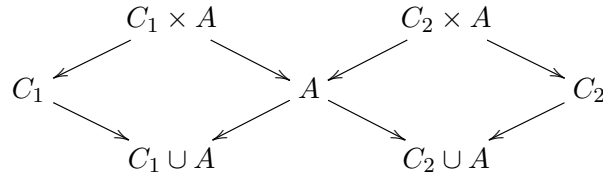
Lemma 2.5. *If χ_1 and χ_2 are choices of centers then $\chi_1 \times \chi_2$ is also a choice of centers. Dually, $\chi_1 \cup \chi_2$ is also a choice of centers.*

Proof. We prove the first part; the second follows by duality.

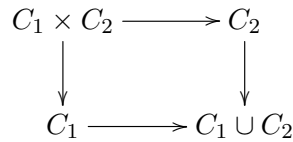
Since \mathcal{C} is closed under products, $\chi_1 \times \chi_2$ is clearly a well-defined functor $\mathcal{C} \rightarrow \mathcal{C}$. We just need to check the other conditions.

(C1) We need to show that $\chi_1 \times \chi_2|_{\mathcal{W}}$ hits only identity morphisms. If $A \xrightarrow{\sim} B$ then $\chi_1(A) = \chi_1(B)$ and $\chi_2(A) = \chi_2(B)$, so $\chi_1 \times \chi_2(A) = \chi_1(B) \times \chi_2(B)$, as desired.

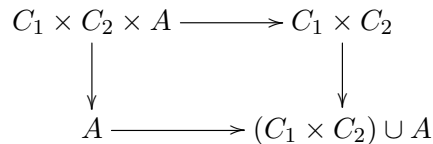
(C2) We write $C_i = \chi_i(A)$ for $i = 1, 2$ in the interests of space. We know that there exists a diagram



in \mathcal{W} ; thus C_1 and C_2 are connected by a zigzag of morphisms in \mathcal{W} , and in particular we know that $\chi_1(C_2) = C_1$. Thus we also have a diagram



in \mathcal{W} . We want to show that the diagram



is in \mathcal{W} . Note that $\chi_2(C_1 \times A) = C_2$, so the morphism $(C_1 \times A) \times C_2 \rightarrow C_1 \times A$ is in \mathcal{W} . Thus we have the following diagram,

$$\begin{array}{ccccc}
 & & (C_1 \times C_2) \times A & \longrightarrow & C_1 \times C_2 & \xrightarrow{\sim} & C_1 \\
 & \nearrow \sim & \downarrow & & \downarrow & & \downarrow \sim \\
 A \times C_1 & \xrightarrow{\sim} & A & \longrightarrow & (C_1 \times C_2) \cup A & \longrightarrow & C_1 \cup A \\
 & & & \searrow \sim & & &
 \end{array}$$

where the morphisms that we know are in \mathcal{W} are marked with \sim . The fact that the middle square is in \mathcal{W} follows because \mathcal{W} is decomposable. \square

To finish the discussion of centers we prove a technical lemma which will help in the future for constructing WFS. Classically, factorizations are constructed using a small object argument in some fashion. In our case we do not do this, as we want to choose “bifibrant generators” rather than cofibrant generators. It turns out that when we are working with a poset, rather than a more complicated category, this is fairly straightforward. To assist with clarity, we introduce an extra definition.

Definition 2.6. An object A is defined to be *semi-fibrant* (resp. *semi-cofibrant*) if there exists a morphism $\chi(A) \rightarrow A$ (resp. $A \rightarrow \chi(A)$).

Directly from the definition it follows that any object of the form $\chi(A) \times A$ (resp. $\chi(A) \cup A$) is semi-cofibrant (resp. semi-fibrant). In particular, $\chi(A)$ is both semi-fibrant and semi-cofibrant.

Lemma 2.7. Let χ be a choice of centers for $(\mathcal{C}, \mathcal{W})$. Suppose that $(\mathcal{L}, \mathcal{R})$ is a pair of classes of morphisms such that

- (1) Both \mathcal{L} and \mathcal{R} are closed under composition and $\mathcal{L} \boxtimes \mathcal{R}$,
- (2) \mathcal{L} is closed under pushouts along morphisms in \mathcal{C} and \mathcal{R} is closed under pullbacks along morphisms in \mathcal{C} ,
- (3) All morphisms with semi-fibrant domain are in \mathcal{R} or all morphisms with semi-cofibrant codomain are in \mathcal{L} , and
- (4) All morphisms in \mathcal{W} factor as a morphism in \mathcal{L} followed by a morphism in \mathcal{R} .

Then $(\mathcal{L}, \mathcal{R})$ is a WFS.

Proof. We prove this assuming that the first part of condition (3) holds. Since the other conditions are self-dual, the proof for the other part follows by duality.

As $\mathcal{L} \boxtimes \mathcal{R}$, if all morphisms in \mathcal{C} factor as a morphism in \mathcal{L} followed by a morphism in \mathcal{R} then by [MP12, 14.1.13] $(\mathcal{L}, \mathcal{R})$ is a WFS. Consider any morphism $f: A \rightarrow B$ in \mathcal{C} . We can factor f as

$$A \xrightarrow{f'} (A \cup \chi(A)) \times B \xrightarrow{f''} B;$$

we claim that f' is in \mathcal{W} and f'' is in \mathcal{R} . Then using condition (4) on f' we can write $f' = f'_R f'_L$ and the desired factorization is then

$$f = \underbrace{f'' f'_R}_{\in \mathcal{R}} \underbrace{f'_L}_{\in \mathcal{L}}.$$

The morphism $A \rightarrow A \cup \chi(A)$ —which is in \mathcal{W} —factors as $A \rightarrow (A \cup \chi(A)) \times B \rightarrow A \cup \chi(A)$, so since \mathcal{W} is decomposable f' is in \mathcal{W} . It remains only to check that $(A \cup \chi(A)) \times B \rightarrow B$ is in \mathcal{R} .

Because χ is a functor, there is a morphism $A \cup \chi(A) \rightarrow B \cup \chi(B)$. By hypothesis (3), this morphism is in \mathcal{R} ; thus by hypothesis (2) its pullback along the morphism $B \rightarrow B \cup \chi(B)$ must also be in \mathcal{R} . Thus $(A \cup \chi(A)) \times B \rightarrow B$ is in \mathcal{R} . \square

3. Construction of model structures

The goal of this section is to prove Theorem B. We therefore fix a relative category $(\mathcal{C}, \mathcal{W})$ and a choice of centers χ and use these to construct a model structure. As before, we assume that \mathcal{C} is finitely bicomplete and \mathcal{W} is decomposable.

Lemma 3.1. *Let $\{A_i\}_{i \in I}$ be a family of semi-cofibrant objects such that $\coprod_{i \in I} A_i$ and $\coprod_{i \in I} \chi(A_i)$ exist. Then $\coprod_{i \in I} A_i$ is also semi-cofibrant. Dually, if $\{A_i\}_{i \in I}$ is a family of semi-fibrant objects such that $\prod_{i \in I} A_i$ and $\prod_{i \in I} \chi(A_i)$ exist, then $\prod_{i \in I} A_i$ is semi-fibrant.*

Proof. We prove the first part of the lemma; the second follows by duality. For all $i \in I$ there is a morphism $A_i \rightarrow \coprod_i A_i$, and thus a morphism $A_i \rightarrow \chi(A_i) \rightarrow \chi(\coprod_i A_i)$. Thus there exists a morphism $\coprod_i A_i \rightarrow \chi(\coprod_i A_i)$, as desired. \square

Recall that, in a poset, in any composition $A \xrightarrow{f} B \xrightarrow{g} C$, g is a pushout of gf and f is a pullback of gf . Thus the semi-fibrant and semi-cofibrant objects contain a lot of information about which morphisms “ought” to be acyclic cofibrations/fibrations.

Definition 3.2. Write Q_χ for the full subcategory of \mathcal{W} with semi-fibrant domain and codomain, and J_χ for the full subcategory of \mathcal{W} with semi-cofibrant domain and codomain.

Note that if \mathcal{W} is decomposable then the class Q_χ is decomposable and the class J_χ is decomposable. In addition, a morphism in \mathcal{W} with semi-fibrant domain (resp. semi-cofibrant codomain) automatically has semi-fibrant codomain (resp. semi-cofibrant domain).

Lemma 3.3. *Suppose $f: A \rightarrow B$ is a morphism with B semi-cofibrant. Then $f \square Q_\chi$. Dually, if A is semi-fibrant then $J_\chi \square f$. In particular, $J_\chi \square Q_\chi$.*

In particular, for any object A in \mathcal{C} ,

$$(\emptyset \rightarrow \chi(A)) \square Q_\chi \quad \text{and} \quad J_\chi \square (\chi(A) \rightarrow *).$$

Proof. We prove the first statement; the second follows by duality. Let $p: X \rightarrow Y \in Q_\chi$, and consider a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ f \downarrow & & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

Applying χ to the square takes p to the identity morphism on $\chi(X)$, and by the defining properties of Q_χ and f we get a diagram

$$\begin{array}{ccccc} A & & \chi(X) & \longrightarrow & X \\ & \searrow & \downarrow = & & \downarrow p \\ f \downarrow & & \chi(X) & \longrightarrow & X \\ B & \longrightarrow & \chi(B) & \longrightarrow & \chi(Y) & \longrightarrow & Y \end{array}$$

This gives the desired lift. \square

We would like to identify those morphisms which “behave like” acyclic cofibrations. Acyclic cofibrations lift on the left of all fibrations; Proposition 1.11 shows that, in a model structure on a poset, all morphisms with fibrant domain are fibrations. We thus take our definition of acyclic cofibrations to be exactly those that lift on the left of the morphisms with semi-fibrant domain.³

Definition 3.4. Let χ be a choice of centers. We define

$$\mathcal{W}_c^\chi = \square \{f: A \rightarrow B \in \mathcal{C} \mid A \text{ semi-fibrant}\}$$

and

$$\mathcal{W}_f^\chi = \{f: A \rightarrow B \in \mathcal{C} \mid B \text{ semi-cofibrant}\}^\square.$$

In particular \mathcal{W}_c^χ is closed under pushouts and \mathcal{W}_f^χ is closed under pullbacks.

By Lemma 3.3 $J_\chi \subseteq \mathcal{W}_c^\chi$ and $Q_\chi \subseteq \mathcal{W}_f^\chi$. As implied by the notation, all morphisms in \mathcal{W}_c^χ and \mathcal{W}_f^χ are weak equivalences:

Lemma 3.5.

$$\mathcal{W}_c^\chi \cup \mathcal{W}_f^\chi \subseteq \mathcal{W}.$$

Proof. We prove that $\mathcal{W}_c^\chi \subseteq \mathcal{W}$; the result for \mathcal{W}_f^χ follows analogously. Let $f: X \rightarrow Y$ be in \mathcal{W}_c^χ . Then it must lift on the left of $X \cup \chi(X) \rightarrow Y \cup \chi(Y)$. In particular, $X \rightarrow X \cup \chi(X)$ factors through f ; thus by decomposition f is a weak equivalence, as desired. \square

We now have the following factorization result:

Lemma 3.6. *Every morphism in \mathcal{W} factors as a morphism in \mathcal{W}_c^χ followed by a morphism which is a pullback of a morphism in Q_χ .*

³We would like to extend our sincerest thanks to the anonymous referee, who pointed out this characterization and greatly simplified this portion of the exposition.

Proof. Suppose that $X \rightarrow Y$ is in \mathcal{W} and let $C = \chi(X) = \chi(Y)$. We claim that

$$X \rightarrow Y \times (X \cup C) \rightarrow Y$$

is the desired factorization. The morphism $Y \times (X \cup C) \rightarrow Y$ is a pullback of $X \cup C \rightarrow Y \cup C$, which is in Q_χ . It remains to show that $X \rightarrow Y \times (X \cup C)$ is in \mathcal{W}_c^χ . Let $A \rightarrow B$ be a morphism with A semi-fibrant, and consider a square

$$\begin{array}{ccc} X & \longrightarrow & A \\ \downarrow & & \downarrow \\ Y \times (X \cup C) & \longrightarrow & B \end{array}$$

To check that a lift exists it suffices to check that a morphism $Y \times (X \cup C) \rightarrow A$ exists. This is given by the composition $Y \times (X \cup C) \rightarrow X \cup C \rightarrow A$, where the morphism $X \cup C \rightarrow A$ exists because the square gives a morphism $X \rightarrow A$, and the fact that $C = \chi(X)$ gives a morphism $C = \chi(X) \rightarrow \chi(A) \rightarrow A$ (since A is semi-fibrant). \square

We are now ready to construct a model structure that depends only on a choice of centers. The fibrations in this model structure are defined to be the naïve set of fibrations making the semi-fibrant objects fibrant.

Definition 3.7. Given a choice of centers χ , the model structure \mathbb{C}^χ is defined by

$$\mathbb{C}_{we}^\chi = \mathcal{W} \quad \mathbb{C}_{fib}^\chi = (\mathcal{W}_c^\chi)^\square \quad \mathbb{C}_{cof}^\chi = \square(\mathbb{C}_{fib}^\chi \cap \mathbb{C}_{we}^\chi).$$

Proposition 3.8. \mathbb{C}^χ is a model structure.

Proof. We need to show that $(\mathbb{C}_{cof}^\chi \cap \mathbb{C}_{we}^\chi, \mathbb{C}_{fib}^\chi)$ and $(\mathbb{C}_{cof}^\chi, \mathbb{C}_{fib}^\chi \cap \mathbb{C}_{we}^\chi)$ are WFSs. We will use Lemma 2.7 for both, and check the conditions simultaneously.

(1) \mathbb{C}_{we}^χ is closed under composition by definition; \mathbb{C}_{cof}^χ and \mathbb{C}_{fib}^χ are defined by lifting properties, and thus are closed under composition by Lemma 1.3. The lifting condition holds by definition for $(\mathbb{C}_{cof}^\chi, \mathbb{C}_{fib}^\chi \cap \mathbb{C}_{we}^\chi)$. To prove the lifting condition for $(\mathbb{C}_{cof}^\chi \cap \mathbb{C}_{we}^\chi, \mathbb{C}_{fib}^\chi)$ it suffices to show that $\mathbb{C}_{cof}^\chi \cap \mathbb{C}_{we}^\chi \subseteq \mathcal{W}_c^\chi$. Let f be in $\mathbb{C}_{cof}^\chi \cap \mathbb{C}_{we}^\chi$. By Lemma 3.6 we can write $f = f_r f_c$ with $f_c \in \mathcal{W}_c^\chi$ and f_r a pullback of a morphism in Q_χ . Every morphism in Q_χ is in $\mathcal{W}_f^\chi \cap (\mathcal{W}_c^\chi)^\square \subseteq \mathbb{C}_{we}^\chi \cap \mathbb{C}_{fib}^\chi$. Thus we have a diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{f_c} & \bullet \\ f \downarrow & \sim & \downarrow f_r \\ \bullet & \xrightarrow{=} & \bullet \end{array}$$

which has a lift because f is in \mathbb{C}_{cof}^χ . Thus $f = f_c \in \mathcal{W}_c^\chi$, as desired.

(2) First consider $(\mathbb{C}_{cof}^{\mathcal{X}}, \mathbb{C}_{fib}^{\mathcal{X}} \cap \mathbb{C}_{we}^{\mathcal{X}})$. We have $\mathbb{C}_{cof}^{\mathcal{X}} = \square(\mathbb{C}_{fib}^{\mathcal{X}} \cap \mathbb{C}_{we}^{\mathcal{X}})$, so it is automatically closed under pushouts. Now let f be in $\mathbb{C}_{fib}^{\mathcal{X}} \cap \mathbb{C}_{we}^{\mathcal{X}}$. Since by definition $\mathbb{C}_{fib}^{\mathcal{X}}$ is closed under pullbacks, it suffices to show that $f \in \mathcal{W}_f^{\mathcal{X}}$, which is closed under pullbacks by definition. By Lemma 3.6 we can factor f as $f_2 f_1$, with $f_2 \in \mathcal{W}_f^{\mathcal{X}}$ and $f_1 \in \mathcal{W}_c^{\mathcal{X}}$. Then we have the following diagram:

$$\begin{array}{ccc} \bullet & \xrightarrow{=} & \bullet \\ f_1 \downarrow & & \downarrow f \\ \bullet & \xrightarrow{f_2} & \bullet \end{array}$$

Since f is in $\mathbb{C}_{fib}^{\mathcal{X}} = (\mathcal{W}_c^{\mathcal{X}})^{\square}$, a lift exists in this diagram, and we see that $f = f_2$ which is in $\mathcal{W}_f^{\mathcal{X}}$, as desired.

Second consider $(\mathbb{C}_{cof}^{\mathcal{X}} \cap \mathbb{C}_{we}^{\mathcal{X}}, \mathbb{C}_{fib}^{\mathcal{X}})$. By definition we know that $\mathbb{C}_{cof}^{\mathcal{X}}$ is closed under pushouts. Since $\mathbb{C}_{cof}^{\mathcal{X}} \cap \mathbb{C}_{we}^{\mathcal{X}} \subseteq \mathcal{W}_c^{\mathcal{X}}$ we know that the pushout of any morphism in $\mathbb{C}_{cof}^{\mathcal{X}} \cap \mathbb{C}_{we}^{\mathcal{X}}$ is a weak equivalence, and thus $\mathbb{C}_{cof}^{\mathcal{X}} \cap \mathbb{C}_{we}^{\mathcal{X}}$ is closed under pushouts. $\mathbb{C}_{fib}^{\mathcal{X}}$ is closed under pullbacks by construction.

(3) The condition is satisfied for $(\mathbb{C}_{cof}^{\mathcal{X}} \cap \mathbb{C}_{we}^{\mathcal{X}}, \mathbb{C}_{fib}^{\mathcal{X}})$ by the definition of $\mathbb{C}_{fib}^{\mathcal{X}}$. Now consider $(\mathbb{C}_{cof}^{\mathcal{X}}, \mathbb{C}_{fib}^{\mathcal{X}} \cap \mathbb{C}_{we}^{\mathcal{X}})$. We show that all morphisms $f: A \rightarrow B$ with B semi-cofibrant lift on the left of $\mathbb{C}_{fib}^{\mathcal{X}} \cap \mathbb{C}_{we}^{\mathcal{X}}$, and thus are in $\mathbb{C}_{cof}^{\mathcal{X}}$. Consider a factorization of f as $A \rightarrow C \rightarrow B$ with $C \rightarrow B$ in $\mathbb{C}_{fib}^{\mathcal{X}} \cap \mathbb{C}_{we}^{\mathcal{X}}$. Since $C \rightarrow B \in \mathbb{C}_{we}^{\mathcal{X}}$, $\chi(C) = \chi(B)$ and thus $C \rightarrow B$ is in $J_{\chi} \subseteq \mathcal{W}_c^{\mathcal{X}}$. On the other hand, $C \rightarrow B \in \mathbb{C}_{fib}^{\mathcal{X}} = (\mathcal{W}_c^{\mathcal{X}})^{\square}$, so $C = B$. Thus by Lemma 1.5, $f \in \square(\mathbb{C}_{fib}^{\mathcal{X}} \cap \mathbb{C}_{we}^{\mathcal{X}})$.

(4) By Lemma 3.6 all morphisms in \mathcal{W} factor as a morphism in $\mathcal{W}_c^{\mathcal{X}}$ followed by a morphism which is a pullback of a morphism in Q_{χ} . As $\mathcal{W}_c^{\mathcal{X}} \subseteq \mathbb{C}_{cof}^{\mathcal{X}} \cap \mathbb{C}_{we}^{\mathcal{X}}$, which was shown in (1) of this proof, and pullbacks of morphisms in Q_{χ} are in $\mathbb{C}_{we}^{\mathcal{X}} \cap \mathbb{C}_{fib}^{\mathcal{X}}$ the condition is satisfied for both WFSs. \square

By duality we have the following.

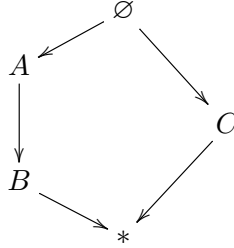
Corollary 3.9. *Suppose that \mathcal{C} is a finitely bicomplete poset, \mathcal{W} is decomposable, and χ is a choice of centers. Then the structure ${}^{\mathcal{X}}\mathcal{C}$ defined by*

$${}^{\mathcal{X}}\mathcal{C}_{we} = \mathcal{W} \quad {}^{\mathcal{X}}\mathcal{C}_{cof} = \square(\mathcal{W}_f^{\mathcal{X}}) \quad {}^{\mathcal{X}}\mathcal{C}_{fib} = ({}^{\mathcal{X}}\mathcal{C}_{cof} \cap {}^{\mathcal{X}}\mathcal{C}_{we})^{\square}$$

is a model structure on \mathcal{C} .

Remark. Before this section, all definitions and results that we have discussed have been self-dual. The model structures constructed in this section are not and this asymmetry is unavoidable. It arises even when both \mathcal{C} , \mathcal{W} and the choice of centers are self-dual.

The following preorder on 5 objects with every morphism considered a weak equivalence provides an example.



The object C is chosen as center. Then \emptyset is the only semi-cofibrant object and $*$ is the only semi-fibrant object, and the morphism $A \rightarrow B$ is in both \mathcal{W}_c^X and \mathcal{W}_f^X . In any model structure on the category, this morphism must be either an acyclic cofibration or an acyclic fibration—but not both!—breaking the symmetry.

We are ready to prove Theorem B.

Proof of Theorem B. By Lemma 2.2, any model structure gives a choice of centers. By Proposition 3.8 a choice of centers gives rise to at least one model structure. \square

4. Model structures on countable posets

In this section we restrict our attention to pairs $(\mathcal{C}, \mathcal{W})$ where \mathcal{C} is countable and \mathcal{W} is decomposable, and show that in this case we can give a first-order characterization of those pairs that extend to a model structure.

Definition 4.1. let W be a weak equivalence class in \mathcal{C} . A *proto-center* P for W is an object in W such that for all $X \in W$, $X \times P \rightarrow X$ and $X \rightarrow X \cup P$ are weak equivalences. For an object A , a *proto-center for A* is a proto-center in the weak equivalence class of A .

A proto-center P is *locally compatible* if

- (1) for any morphism $A' \rightarrow A$ such that A is in the same weak equivalence class as P there exists a morphism $P' \rightarrow P$ where P' is a proto-center in the weak equivalence class of A' , and
- (2) for any morphism $A \rightarrow A'$ such that A is in the same weak equivalence class as P there exists a morphism $P \rightarrow P'$ where P' is a proto-center in the weak equivalence class of A' .

The following lemma shows that proto-centers locally behave the way choices of centers do: the product of two proto-centers is a proto-center and so is the coproduct. (For comparison, see Lemma 2.5.)

Lemma 4.2. *The set of proto-centers of a weak equivalence class is closed under binary products and coproducts.*

Proof. We prove that the product of two proto-centers is a proto-center; the closure by coproduct follows by duality.

Let P_1 and P_2 be two proto-centers in a weak equivalence class W , and consider $Q = P_1 \times P_2$. We need to show that for any $X \in W$, $X \times Q \rightarrow X$ and $X \rightarrow X \cup Q$ are in \mathcal{W} . The morphism $X \times Q \rightarrow X$ factors as

$$(X \times P_1) \times P_2 \rightarrow X \times P_1 \rightarrow X.$$

The first of these is in \mathcal{W} because P_2 is a proto-center, and the second is in \mathcal{W} because P_1 is a proto-center.

Now consider $X \cup Q$. The morphism $X \rightarrow X \cup P_1$ is in \mathcal{W} , since P_1 is a proto-center; but this morphism factors as $X \rightarrow X \cup Q \rightarrow X \cup P_1$. Since \mathcal{W} is decomposable each of these must be a weak equivalence and we have $X \rightarrow X \cup Q \in \mathcal{W}$. \square

Lemma 4.3. *Let W and W' be weak equivalence classes, and suppose that there exists $f: A \rightarrow A'$ with $A \in W$ and $A' \in W'$. For any two locally compatible proto-centers $Q \in W$ and $Q' \in W'$, $Q \times Q'$ is a locally compatible proto-center in W and $Q \cup Q'$ is a locally compatible proto-center in W' .*

Proof. We begin by showing that $Q \times Q'$ and $Q \cup Q'$ are proto-centers in the appropriate weak equivalence classes. We prove only the statement for $Q \times Q'$; the second statement follows by duality.

First, consider the morphism $Q \times Q' \rightarrow Q$; we wish to show that this is in \mathcal{W} . Since Q' is locally compatible there exists a proto-center $P \in W$ and a morphism $P \rightarrow Q'$. By Lemma 4.2 $P \times Q$ is also a proto-center, and thus $P \times Q \rightarrow Q$ is in \mathcal{W} . The morphism $P \times Q \rightarrow Q$ factors as $P \times Q \rightarrow Q \times Q' \rightarrow Q$; thus since \mathcal{W} is decomposable, $Q \times Q' \rightarrow Q$ is in \mathcal{W} .

We now check that $Q \times Q'$ is a proto-center. Let $A \in W$, and consider $A \times Q \times Q' \rightarrow A$. We have a composition

$$A \times Q \times P \rightarrow A \times Q \times Q' \rightarrow A,$$

which is in \mathcal{W} because $Q \times P$ is a proto-center. Since \mathcal{W} is decomposable, $A \times Q \times Q' \rightarrow A$ is in \mathcal{W} , as desired. Now consider $A \rightarrow A \cup (Q \times Q')$. The morphism $A \rightarrow A \cup Q$ (which is in \mathcal{W}) factors through $A \cup (Q \times Q')$; thus it is in \mathcal{W} , as desired.

We now need to check local compatibility of $Q \times Q'$; the statement for $Q \cup Q'$ follows by duality. Let $B'' \rightarrow B$ be any morphism with $B \in W$; let W'' be the weak equivalence class of B'' . Since Q is a locally compatible proto-center there exists a proto-center P'' in W'' with a morphism $P'' \rightarrow Q$. Thus there exists a morphism $P'' \rightarrow Q \cup Q'$. Since Q' is locally compatible there exists a proto-center $R'' \in W''$ with a morphism $R'' \rightarrow Q'$. Then $P'' \times R''$ is a proto-center in W'' ; since there exist morphisms $P'' \rightarrow Q$ and $R'' \rightarrow Q'$ there exists a morphism $P'' \times R'' \rightarrow Q \times Q'$, as desired.

Now suppose that $B \rightarrow B''$ is any morphism with $B \in W$; let W'' be the weak equivalence class of B'' . Since Q is a locally compatible proto-center there exists a proto-center $R'' \in W''$ with a morphism $Q \rightarrow R''$. Then $Q \times Q' \rightarrow Q \rightarrow R''$ gives the desired morphism. \square

The existence of locally compatible proto-centers implies that there is a well-defined ordering on weak equivalence classes.

Lemma 4.4. *Suppose that W and W' are distinct weak equivalence classes containing locally compatible proto-centers $Q \in W$ and $Q' \in W'$. If there exists a morphism $A \rightarrow A'$ with $A \in W$ and $A' \in W'$ then there does not exist a morphism $B' \rightarrow B$ with $B' \in W'$ and $B \in W$.*

Proof. Suppose that both $A \rightarrow A'$ and $B' \rightarrow B$ exist. Then by Lemma 4.3 applied to $A \rightarrow A'$, $Q \times Q' \in W$. On the other hand, by Lemma 4.3 applied to $B' \rightarrow B$, $Q \times Q' \in W'$. Thus $W \cap W' \neq \emptyset$, a contradiction. Thus both $A \rightarrow A'$ and $B' \rightarrow B$ cannot exist. \square

The point of locally compatible proto-centers is that they can be used to construct approximations to choices of centers.

Definition 4.5. A *partial choice of centers* is a functor $\tilde{\chi}: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ such that the following properties hold:

- PC1:** $\tilde{\mathcal{C}}$ is a full subcategory of \mathcal{C} , and if A and A' are in the same weak equivalence class and $A \in \tilde{\mathcal{C}}$ then $A' \in \tilde{\mathcal{C}}$.
- PC2:** The image of $\tilde{\chi}|_{W \cap \tilde{\mathcal{C}}}$ only contains identity morphisms.
- PC3:** $\tilde{\chi}(A)$ is a proto-center for A for all $A \in \tilde{\mathcal{C}}$.

In particular, a partial choice of centers with $\tilde{\mathcal{C}} = \mathcal{C}$ is a choice of centers.

When we are given a partial choice of centers and a locally compatible proto-center we can use the proto-center to extend the partial choice of centers. We encode the conditions for doing so in the following lemma.

Lemma 4.6. *Let $\tilde{\chi}: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ be a partial choice of centers and let Q be a locally compatible proto-center for a weak equivalence class $W \subseteq \mathcal{W}$ which is not in $\tilde{\mathcal{C}}$. Suppose that the following two conditions hold:*

- (1) *For all $A \in \tilde{\mathcal{C}}$ and $A' \in W$, if there exists a morphism $A \rightarrow A'$ then there exists a morphism $\tilde{\chi}(A) \rightarrow Q$.*
- (2) *for all $A \in \tilde{\mathcal{C}}$ and $A' \in W$, if there exists a morphism $A' \rightarrow A$ then there exists a morphism $Q \rightarrow \tilde{\chi}(A)$.*

Then the functor

$$\tilde{\chi}'(A) = \begin{cases} \tilde{\chi}(A) & \text{if } A \in \tilde{\mathcal{C}} \\ Q & \text{if } A \in W \end{cases}$$

defined on the full subcategory of \mathcal{C} generated by $\tilde{\mathcal{C}}$ and W is a partial choice of centers.

Proof. We first check that it is a functor. We have defined it on objects. To check that it is well-defined we must check that it takes morphisms to morphisms. For a morphism $A \rightarrow A'$ in $\tilde{\mathcal{C}}$ it is well-defined because $\tilde{\chi}$ is well-defined. Given any morphism $A \rightarrow A'$ with $A \in \tilde{\mathcal{C}}$ and $A' \in W$, $\tilde{\chi}'(A \rightarrow A') = \tilde{\chi}(A) \rightarrow Q$ exists by condition (1), thus $\tilde{\chi}'$ is well-defined on such morphisms. Analogously it is well-defined on morphisms $A' \rightarrow A$ with $A \in \tilde{\mathcal{C}}$ and $A' \in W$ by condition (2). It is compatible with composition because all maps between posets which are well-defined on objects and morphisms are functors. It satisfies the conditions to be a partial choice of centers by definition. \square

We now use the machinery we have built to construct a choice of centers out of locally compatible proto-centers.

Theorem 4.7. *If each weak equivalence class of \mathcal{C} has a locally compatible proto-center and there is only a countable number of weak equivalence classes then there exists a choice of centers.*

Proof. Let $\{W_i\}_{i=1}^\infty$ be an enumeration of the weak equivalence classes in \mathcal{C} ; let \mathcal{C}_n be the full subcategory of \mathcal{C} containing $\bigcup_{i=1}^n W_i$. In the interest of conciseness, we also define $\mathcal{C}_{n,m}$ for $m > n$ to be the full subcategory of \mathcal{C} containing both \mathcal{C}_n and W_m .

We prove the following statement: for each $n \geq 0$ we can construct a pair

$$(\tilde{\chi}_n: \mathcal{C}_n \rightarrow \mathcal{C}, \{Q_m\}_{m=n+1}^\infty)$$

where $\tilde{\chi}_n$ is a partial choice of centers and for each m , Q_m and $\tilde{\chi}_n$ satisfy the conditions of Lemma 4.6. We construct these pairs in such a way so that for all $n' > n$, $\tilde{\chi}_{n'}(A) = \tilde{\chi}_n(A)$ for all $A \in \mathcal{C}_n$. Using this sequence we then define a choice of centers $\chi: \mathcal{C} \rightarrow \mathcal{C}$ by

$$\chi(A) = \chi_n(A) \quad \text{if } A \in W_n.$$

This will prove the theorem.

For our base case $n = 0$, we let $\tilde{\chi}_0: \emptyset \rightarrow \mathcal{C}$ be the trivial map, and we let $\{Q_m\}_{m=1}^\infty$ be a choice of locally compatible proto-centers for each weak equivalence class. These exist by assumption.

Now consider a general n , and suppose that we are given $\tilde{\chi}_{n-1}: \mathcal{C}_{n-1} \rightarrow \mathcal{C}$ and a sequence $\{Q_m\}_{m=n}^\infty$ such that each Q_m satisfies the conditions of Lemma 4.6. We let $\tilde{\chi}_n$ be the functor constructed in Lemma 4.6 for $\tilde{\chi}_{n-1}$ and Q_n . We then define the sequence $\{Q'_m\}_{m=n+1}^\infty$ by

$$Q'_m = \begin{cases} Q_m \times Q_n & \exists A_m \rightarrow A_n \text{ with } A_m \in W_m \text{ and } A_n \in W_n, \\ Q_m \cup Q_n & \exists A_n \rightarrow A_m \text{ with } A_m \in W_m \text{ and } A_n \in W_n, \\ Q_m & \text{otherwise.} \end{cases}$$

These conditions are mutually exclusive by Lemma 4.4. We need to check that this pair satisfies the conditions required by the inductive hypothesis.

In particular, all we need to check is that for all $m > n$, $\tilde{\chi}_n$ and Q'_m satisfy the conditions of Lemma 4.6.

Q'_m is a locally compatible proto-center in W_m by Lemma 4.3. Now suppose that $A \in \mathcal{C}_n$ and A' is in W_m , and suppose that there exists a morphism $A \rightarrow A'$. If $A \in W_n$ we need to show that there exists a morphism $Q_n \rightarrow Q'_m$; but by definition $Q'_m = Q_m \cup Q_n$, so this exists. Now suppose that $A \in W_i$ for $i < n$. By the inductive hypothesis there exists a morphism $\tilde{\chi}_n(A) \rightarrow Q_m$. When there does not exist a morphism $A_m \rightarrow A_n$ (with $A_n \in W_n$ and $A_m \in W_m$) there exists a morphism $Q_m \rightarrow Q'_m$, so there exists a morphism $\tilde{\chi}_n(A) \rightarrow Q'_m$, as desired. If such a morphism $A_m \rightarrow A_n$ exists then $Q'_m = Q_n \times Q_m$, so it suffices to check that there exists a morphism $\tilde{\chi}_n(A) \rightarrow Q_n$. Since Q_m is a locally compatible proto-center, there exists a proto-center $P_i \in W_i$ and a morphism $P_i \rightarrow Q_m$. There must also exist a proto-center $P_n \in W_n$ and a morphism $Q_m \rightarrow P_n$. Thus there is a morphism $P_i \rightarrow P_n$ which $\tilde{\chi}_n$ takes to $\tilde{\chi}_n(A) \rightarrow Q_n$. Thus condition (1) of Lemma 4.6 holds. Condition (2) holds by symmetry. \square

Since the property of being a proto-center and the property of being a locally compatible proto-center are first-order properties, we get the following:

Corollary 4.8. *The existence of a model structure extending $(\mathcal{C}, \mathcal{W})$ when \mathcal{W} only has countably many weak equivalence classes is first order definable.*

We are ready to tackle Theorem A.

Proof of Theorem A. We construct a pair $(\mathcal{P}, \mathcal{W})$ where \mathcal{P} is an uncountable poset and $(\mathcal{P}, \mathcal{W})$ satisfies all of the conditions of Theorem 4.7 other than the countability of \mathcal{P} , but which does not extend to a model structure. By the downward Löwenheim–Skolem theorem, the pair $(\mathcal{P}, \mathcal{W})$ has an elementarily equivalent countable model $(\mathcal{P}', \mathcal{W}')$. By Theorem 4.7 there is a Quillen model structure extending $(\mathcal{P}', \mathcal{W}')$. This gives two pairs with the same first order theory where only one extends to a Quillen model structure; the statement of the theorem follows.

Let \mathcal{P} be the poset of subsets of $\mathbb{N} \times \mathbb{N}$ ordered by inclusion regarded as a category, so that there is a morphism $A \rightarrow B$ if $A \subseteq B$. Let \mathcal{W} be the subcategory taking the morphisms in \mathcal{P} for which the domain and the codomain differ by a finite number of elements.

We claim that the pair $(\mathcal{P}, \mathcal{W})$ satisfies all conditions of Theorem 4.7 except countability. First, since the weak classes are closed under products and coproducts, we deduce that in every weak class all elements are proto-centers. Second, if one weak class A contains an element above an element of a weak class B , then every element of A is above some element of B and every element of B is below some element of A . It follows that every weak class contains a locally compatible proto-center.

We claim that there is no model structure on \mathcal{P} with \mathcal{W} as category of weak equivalences. By Theorem B, it suffices to prove the nonexistence of a choice of center.

Suppose for the sake of contradiction that a choice of centers χ for the pair $(\mathcal{P}, \mathcal{W})$ exists. Let $R_i = \{i\} \times \mathbb{N}$, and let (i, k_i) be any element in $\chi(R_i) \cap R_i$. Let

$$X = \bigcup_{i \in \mathbb{N}} (\chi(R_i) \cap R_i - \{(i, k_i)\}).$$

There is a diagram

$$R_i \xleftarrow{\sim} \chi(R_i) \cap R_i - \{(i, k_i)\} \longrightarrow X$$

for all i ; applying χ to this produces a morphism $f_i: \chi(R_i) \rightarrow \chi(X)$. Since the symmetric difference between X and $\chi(X)$ is finite, there exists an N such that for all $n \geq N$,

$$\chi(X) \cap R_n = \chi(R_n) \cap R_n - \{(n, k_n)\}.$$

Thus $\chi(R_n) \not\subseteq \chi(X)$ and f_n cannot exist; contradiction. \square

5. Classification of model category structures on posets up to Quillen equivalence

We end this paper with an aside on uniqueness of model structures. We begin by recalling the definition of Quillen equivalence:

Definition 5.1. Given two categories \mathcal{C} and \mathcal{D} together with model structures \mathbb{C} and \mathbb{D} , an adjoint pair of functors $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ is a *Quillen adjunction* if F preserves cofibrations and acyclic cofibrations and G preserves fibrations and acyclic fibrations. It is a *Quillen equivalence* if moreover whenever $A \in \mathcal{C}$ is cofibrant and $B \in \mathcal{D}$ is fibrant then the morphism $A \rightarrow G(B)$ is a weak equivalence if and only if its adjoint $F(A) \rightarrow B$ is a weak equivalence. \mathbb{C} and \mathbb{D} are called *Quillen equivalent* if there exists a chain of Quillen equivalences between them. For a model structure \mathbb{C} , write $\text{Ho } \mathbb{C} := \mathcal{C}[\mathbb{C}_{we}^{-1}]$. If \mathbb{C} and \mathbb{D} are Quillen equivalent then $\text{Ho } \mathbb{C}$ and $\text{Ho } \mathbb{D}$ are equivalent.

We recall without proof some basic properties of Quillen equivalences. For more details, see [MP12, Section 16.2].

Lemma 5.2. *Let $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ be an adjoint pair of functors between model categories \mathbb{C} and \mathbb{D} .*

- (1) *F preserves cofibrations and acyclic cofibrations if and only if G preserves fibrations and acyclic fibrations.*
- (2) *If the adjunction is a Quillen adjunction, F reflects weak equivalences and the counit of the adjunction is a weak equivalence for all fibrant objects then it is a Quillen equivalence.*

Even if we know that a pair $(\mathcal{C}, \mathcal{W})$ extends to a model structure there is still the possibility for non-uniqueness: there might be two model structures \mathbb{C} and \mathbb{C}' extending $(\mathcal{C}, \mathcal{W})$ that are not Quillen equivalent. Thus we have the following question:

Question 5.3.

- (1) If \mathbb{C} and \mathbb{C}' are two model structures extending the pair $(\mathcal{C}, \mathcal{W})$, are they Quillen equivalent?
- (2) Moreover, if \mathbb{C} and \mathbb{C}' are Quillen equivalent, is it possible to construct a chain of Quillen equivalences in which every underlying functor is the identity functor?

We expect that the answer to (1) is “yes”, even when \mathcal{C} is not a poset, and that the answer to (2) is “yes” when \mathcal{C} is nice. Intuitively, if we think of a model structure as a “choice of coordinates” on a relative pair $(\mathcal{C}, \mathcal{W})$, this says that all choices of coordinates are equivalent.

Although we cannot answer the question in general, in this section we prove that when \mathcal{C} is a poset the answer to (1) is “yes,” (Theorem 5.4) and when \mathcal{C} is bicomplete and all weak equivalence classes in \mathcal{W} are small the answer to (2) is “yes” (Theorem 5.9).

Theorem 5.4. *Let \mathbb{C} be a model structure on a preorder \mathcal{C} , and let \mathcal{D} be the full subcategory of the cofibrant fibrant objects in \mathcal{C} . Then \mathbb{C} is Quillen equivalent to the model structure \mathbb{D} on \mathcal{D} given by*

$$\mathbb{D}_{we} = \text{iso } \mathcal{D} \quad \mathbb{D}_{cof} = \mathbb{D}_{fib} = \mathcal{D}.$$

In particular, this theorem shows that any two model structures on posets with isomorphic homotopy categories are Quillen equivalent. Embedded in the statement of this theorem is the observation that $\text{Ho } \mathbb{C}$ must be finitely bicomplete. In fact, $\text{Ho } \mathbb{C}$ will have all limits and colimits that \mathcal{C} does.

Most of the proof of this theorem is contained in the following proposition:

Proposition 5.5. *Let \mathcal{C}^c be the full subcategory of cofibrant objects in \mathcal{C} . We define*

$$\mathbb{C}_{we}^c = \mathbb{C}_{we} \cap \mathcal{C}^c \quad \mathbb{C}_{cof}^c = \mathbb{C}_{cof} \cap \mathcal{C}^c \quad \mathbb{C}_{fib}^c = \mathbb{C}_{fib} \cap \mathcal{C}^c.$$

Then \mathbb{C}^c is model structure on \mathcal{C}^c and the inclusion $\iota: \mathcal{C}^c \rightarrow \mathcal{C}$ is the left adjoint in a Quillen equivalence $\mathbb{C}^c \rightleftarrows \mathbb{C}$.

This proposition is a special case of [BG19, Proposition 17]; we present the proof here as the poset case is easier to visualize than the one-dimensional model structures in [BG19].

Proof. First, note that \mathcal{C}^c is bicomplete⁴. It suffices to check that it has all products and coproducts, since equalizers and coequalizers are trivial in a poset. An arbitrary coproduct of cofibrant objects is still cofibrant, so it suffices to check that \mathcal{C}^c has all products. Let $\{A_i\}_{i \in I}$ be a tuple of objects of \mathcal{C}^c , and let $B = \prod_{i \in I} A_i \in \mathcal{C}$. We claim that the cofibrant replacement (unique by Lemma 1.6) B^c of B is the product of A_i in \mathcal{C}^c . Indeed, suppose

⁴We do not distinguish between finite and small in this case; \mathcal{C}^c will have the same ones that \mathcal{C} does.

that a cofibrant object D has morphisms $D \rightarrow A_i$ for all i . Then we have a diagram

$$\begin{array}{ccc} \emptyset & \hookrightarrow & B^c \\ \downarrow & & \downarrow \sim \\ D & \longrightarrow & B \end{array}$$

which has a lift $h: D \rightarrow B^c$. Thus all cofibrant objects with morphisms to B have morphisms to B^c . As morphisms are uniquely determined by their source and target this makes B^c into the product of the A_i inside \mathcal{C}^c , as desired. Factorizations in \mathcal{C} yield factorizations in \mathcal{C}^c , so by [MP12, 14.1.13] \mathcal{C}^c is a model structure. We define a right adjoint γ to ι by sending each object A to its cofibrant replacement; by Lemma 1.6, this is well-defined. By Lemma 1.6 again, $\gamma(\iota(A)) = A$, so the unit of the adjunction is the identity transformation. The counit of the transformation is the acyclic fibration $\gamma(A) \xrightarrow{\sim} A$. As ι preserves cofibrations and weak equivalences by definition, it is the left adjoint in a Quillen adjunction. Since ι reflects weak equivalences and the counit of the adjunction is a natural weak equivalence, the adjunction is a Quillen equivalence, as desired. \square

We can now prove Theorem 5.4.

Proof of Theorem 5.4. Let \mathcal{C}^c be the full subcategory of \mathcal{C} containing all cofibrant objects and \mathbb{C}^c be the model structures defined on it by Proposition 5.5. By Proposition 5.5 \mathcal{C} is Quillen equivalent to \mathbb{C}^c . Note that $\mathbb{D}_{cof} = \mathbb{C}_{cof}^c \cap \mathcal{D}$ and $\mathbb{D}_{fib} = \mathbb{C}_{fib}^c \cap \mathcal{D}$. By the dual of Proposition 5.5, \mathbb{C}^c is Quillen equivalent to \mathbb{D} . \square

As an amusing aside, this allows us to classify which model structures on posets are cofibrantly generated:

Theorem 5.6. *Let \mathcal{C} be any bicomplete poset, and let \mathbb{C} a model structure on \mathcal{C} . If \mathcal{C} is small then \mathbb{C} is cofibrantly generated; conversely, if \mathbb{C} is cofibrantly generated then \mathcal{C} is right Quillen equivalent to a small model category.*

Proof. If \mathcal{C} is small then \mathbb{C} is trivially cofibrantly generated: we define the set of generating cofibrations to be the set of all cofibrations, and the set of generating acyclic cofibrations to be the set of all acyclic cofibrations.

Now suppose that \mathbb{C} is cofibrantly generated. By Proposition 5.5 it suffices to show that \mathcal{C}^c is small. Let $S = \{f_i: A_i \rightarrow B_i\}$ be the set of generating cofibrations. By the small object argument, for any object $X \in \mathcal{C}$ we construct its cofibrant replacement $\gamma(X)$ by defining $\gamma_0(X) = \emptyset$ and setting $\gamma_{n+1}(X)$ to be the pushout of

$$\coprod_{f_i \in S_n} B_i \longleftarrow \coprod_{f_i \in S_n} A_i \longrightarrow \gamma_n(X)$$

where

$$S_n = \{f_i \in S \mid \text{Hom}(A_i, \gamma_n(X)) \times \text{Hom}(B_i, X) \neq \emptyset\}.$$

Then there is a cofibration $\gamma_n(X) \rightarrow \gamma_{n+1}(X)$ and the cofibrant replacement of X is $\text{colim}_n \gamma_n(X)$. Note that by definition, $S_n \subseteq S_{n+1}$ for all n . Observe that for any nonempty set T and any object $A \in \mathcal{C}$ we have $\coprod_T A = A$; therefore

$$\gamma_{n+1}(X) = \gamma_n(X) \amalg \coprod_{f_i \in S_n} B_i \cong \coprod_{f_i \in S_n} B_i.$$

Thus if we set $S_\infty = \bigcup_{n \geq 0} S_n$ it follows that $\gamma(X) = \coprod_{f_i \in S_\infty} B_i$. In particular, all cofibrant replacements correspond to subsets of S ; as S is a set, the class of cofibrant objects must also be a set. Therefore \mathcal{C}^c is small. \square

This theorem allows us to construct model categories which are neither cofibrantly nor fibrantly generated.

Corollary 5.7. *Let \mathbb{C} be a model structure on a bicomplete poset \mathcal{C} such that \mathcal{C}^c has size 2^κ for some cardinal κ . Then if \mathbb{C} is cofibrantly generated it must have at least κ generators. In particular, if \mathcal{C}^c is not small then \mathbb{C} is not cofibrantly generated. Dually, if \mathcal{C}^f is not small then \mathbb{C} is not fibrantly generated.*

Example 5.8. Corollary 5.7 and Theorem B give an interesting method for producing non-cofibrantly or fibrantly generated model structures on posets. Let \mathcal{C} be any large bicomplete poset. Take any collection $\{A_i\}_{i \in \text{ob } \mathcal{C}}$, where each A_i is a bicomplete poset with terminal object $*_i$. Write

$$i: \coprod_{i \in \text{ob } \mathcal{C}} \text{ob } A_i \rightarrow \text{ob } \mathcal{C}$$

for the map taking $X \in A_i$ to i . Let $\tilde{\mathcal{C}}$ be the poset whose objects are $\coprod_{i \in \text{ob } \mathcal{C}} \text{ob } A_i$ and where

$$\text{Hom}(X, Y) = \begin{cases} \text{Hom}_{A_i}(X, Y) & \text{if } i(X) = i(Y) \\ \text{Hom}_{\mathcal{C}}(i(X), i(Y)) & \text{if } i(X) \neq i(Y). \end{cases}$$

Then i extends to a functor $i: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$; we define \mathcal{W} to be the preimage of the identity morphisms.

To check that \mathcal{C} is bicomplete it suffices to check that all products and coproducts exist. We check products, as coproducts follow analogously. Let $\{B_\alpha\}_{\alpha \in A}$ be a collection of objects in $\tilde{\mathcal{C}}$. Let $i: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ be the functor taking an object to its indexing element in \mathcal{C} . Let $i' = \prod_{\alpha \in A} i(A_\alpha)$, and let $*_{i'}$ be the terminal object in $A_{i'}$. Then

$$\prod_{\alpha \in A} B_\alpha = \prod_{\substack{\alpha \in A \\ i(\alpha) = i'}} B_\alpha \times *_i.$$

The subcategory \mathcal{W} is decomposable. To see this, note that if $X \rightarrow Y$ is a morphism in \mathcal{W} then we must have $i(X) = i(Y)$, and thus this is a morphism in A_i . If a composition $X \rightarrow Y \rightarrow Z$ is in \mathcal{W} then the composition

$i(X) \rightarrow i(Y) \rightarrow i(Z)$ is an identity; in particular, since \mathcal{C} is a poset we must have $i(Y) = i(X)$, and thus $X \rightarrow Y$ and $Y \rightarrow Z$ are also in \mathcal{W} .

The map $X \mapsto *_i(X)$ is a choice of centers, and by Theorem B this pair extends to a model structure. However, since \mathcal{C} is large this model is neither cofibrantly nor fibrantly generated.

We turn to the second half of Question 5.3. In the case of posets where every weak equivalence class of \mathcal{W} is small and \mathcal{C} is bicomplete we answer it.

Theorem 5.9. *If all weak equivalence classes of \mathcal{W} are small and \mathcal{C} is bicomplete then any two model structures extending $(\mathcal{C}, \mathcal{W})$ are Quillen equivalent via a zigzag of equivalences each of whose underlying functors is the identity.*

We prove this theorem by constructing a “minimal” model structure in which a fixed class J of weak equivalences are actually acyclic cofibrations. The construction of this structure is where the assumptions on \mathcal{W} and \mathcal{C} are necessary. We begin with a couple of extra technical results about lifting systems.

Definition 5.10. A poset \mathcal{C} is *left-small* with respect to class \mathcal{L} if for all objects $A \in \mathcal{C}$, the class $\{f \in \mathcal{L} \mid \text{dom } f = A\}$ is a set. Dually, \mathcal{C} is *right-small* with respect to \mathcal{R} if for all objects $A \in \mathcal{C}$ the class $\{f \in \mathcal{R} \mid \text{codom } f = A\}$ is a set.

Lemma 5.11. *Suppose that $(\mathcal{L}, \mathcal{R})$ is a pair of classes of morphisms such that $\mathcal{L}^\square = \mathcal{R}$ and \mathcal{L} contains all isomorphisms. If \mathcal{C} is left-small with respect to \mathcal{L} and \mathcal{L} is closed under compositions, pushouts in \mathcal{C} and arbitrary coproducts then $(\mathcal{L}, \mathcal{R})$ is a WFS.*

Proof. By [MP12, 14.1.13], in order to show that $(\mathcal{L}, \mathcal{R})$ is a WFS it suffices to show that every morphism $f: A \rightarrow B$ in \mathcal{C} can be factored as $f_R f_L$, where $f_L \in \mathcal{L}$ and $f_R \in \mathcal{R}$.

Let $f: A \rightarrow B$ be any morphism in \mathcal{C} . Let

$$S = \{A' \in \mathcal{C} \mid A \rightarrow A' \rightarrow B, A \rightarrow A' \in \mathcal{L}\}$$

and let $\tilde{A} = \text{colim } S$; note that this is well-defined since \mathcal{C} is left-small with respect to \mathcal{L} , S is a set and \mathcal{C} contains all colimits. The morphism $A \rightarrow \tilde{A}$ can be written as $\coprod_{A' \in S} (A \rightarrow A')$, so $A \rightarrow \tilde{A} \in \mathcal{L}$. We claim that $A \rightarrow \tilde{A} \rightarrow B$ gives the desired factorization. By the definition of \tilde{A} there are no factorizations of $\tilde{A} \rightarrow B$ through non-invertible morphisms $\tilde{A} \rightarrow Z \in \mathcal{L}$; thus by Lemma 1.5 $\tilde{A} \rightarrow B \in \mathcal{L}^\square$, as desired. \square

Corollary 5.12. *If $(\mathcal{L}, \mathcal{R})$ is a MLS (see Definition 1.2) and \mathcal{C} is left-small with respect to \mathcal{L} or right-small with respect to \mathcal{R} then $(\mathcal{L}, \mathcal{R})$ is a WFS.*

We are now ready to construct our minimal model structure.

Proposition 5.13. *Let \mathcal{C} be a bicomplete poset, \mathcal{W} a decomposable subcategory and J a class of morphisms in \mathcal{W} . We define*

$$\mathbb{C}_{we}^J = \mathcal{W} \quad \mathbb{C}_{fib}^J = J^\square \quad \mathbb{C}_{cof}^J = \square(\mathbb{C}_{we}^J \cap \mathbb{C}_{fib}^J).$$

\mathbb{C}^J is a model structure if the following extra assumptions hold:

- (1) All connected components of \mathcal{W} are small.
- (2) $(\mathbb{C}_{cof}^J)^\square \subseteq \mathcal{W}$.
- (3) $\square\mathbb{C}_{fib}^J \subseteq \mathcal{W}$.

This proposition also has a dual version, which defines a model structure ${}^J\mathbb{C}$ with ${}^J\mathbb{C}_{cof} = \square J$.

Proof. Since \mathcal{W} is decomposable it must also satisfy (2OF3). By (1) \mathcal{C} is left-small with respect to $\mathbb{C}_{cof}^J \cap \mathbb{C}_{we}^J$ and right-small with respect to $\mathbb{C}_{fib}^J \cap \mathbb{C}_{we}^J$. Thus by Corollary 5.12 in order to show that (WFS) holds it suffices to check that both $(\mathbb{C}_{cof}^J, \mathbb{C}_{fib}^J \cap \mathbb{C}_{we}^J)$ and $(\mathbb{C}_{cof}^J \cap \mathbb{C}_{we}^J, \mathbb{C}_{fib}^J)$ are MLSs.

To check that a pair $(\mathcal{L}, \mathcal{R})$ is an MLS we must check that $\mathcal{L} = \square\mathcal{R}$ and that $\mathcal{L}^\square \subseteq \mathcal{R}$. We begin by showing that the first of these holds for both pairs. For $(\mathbb{C}_{cof}^J, \mathbb{C}_{fib}^J \cap \mathbb{C}_{we}^J)$, this is true by definition. Now consider $(\mathbb{C}_{cof}^J \cap \mathbb{C}_{we}^J, \mathbb{C}_{fib}^J)$. Suppose that $i: A \rightarrow B$ is in $\mathbb{C}_{cof}^J \cap \mathbb{C}_{we}^J$. By the dual of Lemma 1.5, $i \in \square\mathbb{C}_{fib}^J$ if and only if all factorizations $A \rightarrow Z \rightarrow B$ with $Z \rightarrow B$ in \mathbb{C}_{fib}^J have $Z = B$. As \mathcal{W} is decomposable such a factorization has $Z \rightarrow B$ in \mathbb{C}_{we}^J ; since $i \in \mathbb{C}_{cof}^J$ it lifts on the left of $Z \rightarrow B$ and we must have $Z = B$, as desired. Thus $\mathbb{C}_{cof}^J \cap \mathbb{C}_{we}^J \subseteq \square\mathbb{C}_{fib}^J$. On the other hand, by assumption (3), $\square\mathbb{C}_{fib}^J \subseteq \mathcal{W}$ and

$$\square\mathbb{C}_{fib}^J \subseteq \square(\mathbb{C}_{fib}^J \cap \mathbb{C}_{we}^J) = \mathbb{C}_{cof}^J.$$

Thus $\mathbb{C}_{cof}^J \cap \mathbb{C}_{we}^J \supseteq \square\mathbb{C}_{fib}^J$ and equality holds.

We now turn to proving that $\mathcal{L}^\square \subseteq \mathcal{R}$ for both pairs.

First consider $(\mathbb{C}_{cof}^J, \mathbb{C}_{fib}^J \cap \mathbb{C}_{we}^J)$. Note that

$$J \subseteq \square(J^\square) = \square\mathbb{C}_{fib}^J \subseteq \square(\mathbb{C}_{fib}^J \cap \mathbb{C}_{we}^J) = \mathbb{C}_{cof}^J.$$

Thus $(\mathbb{C}_{cof}^J)^\square \subseteq J^\square = \mathbb{C}_{fib}^J$. By assumption (2) we know that it is also a subset of \mathbb{C}_{we}^J , as desired. Now consider $(\mathbb{C}_{cof}^J \cap \mathbb{C}_{we}^J, \mathbb{C}_{fib}^J)$. From before, $J \subseteq \mathbb{C}_{cof}^J \cap \mathbb{C}_{we}^J$, from which it follows that

$$(\mathbb{C}_{cof}^J \cap \mathbb{C}_{we}^J)^\square \subseteq J^\square = \mathbb{C}_{fib}^J.$$

□

We can now start comparing different model structures on a poset.

Proposition 5.14. *Suppose that \mathbb{C} is any model structure on \mathcal{C} with $\mathbb{C}_{we} = \mathcal{W}$ and χ is any choice of centers for \mathcal{W} . Let $J = (\mathbb{C}_{cof} \cap \mathbb{C}_{we}) \cup J_\chi$. If all connected components in \mathcal{W} are small, then \mathbb{C}^J is another model structure on \mathcal{C} , and the identity functor gives a Quillen equivalence $\mathbb{C} \rightleftarrows \mathbb{C}^J$.*

Proof. We use Proposition 5.13. Condition (1) is assumed, so we just need to check (2) and (3).

(2) Note that $\mathbb{C}_{fib}^J \subseteq \mathbb{C}_{fib}$. Thus $\mathbb{C}_{cof}^J \supseteq \mathbb{C}_{cof}$ and $(\mathbb{C}_{cof}^J)^\square \subseteq \mathbb{C}_{cof}^\square \subseteq \mathcal{W}$.

(3) Suppose $f: A \rightarrow B$ is such that $1_A \in Q_\chi$ and $f \square \mathbb{C}'_{fib}$. Factor $f = f_f f_{ac}$ with $f_f: B' \rightarrow B \in \mathbb{C}_{fib}$ and $f_{ac} \in \mathbb{C}_{cof} \cap \mathbb{C}_{we}$. We claim that $f_f \in \mathbb{C}_{fib}^J$. Because $\mathbb{C}_{cof} \cap \mathbb{C}_{we} \square f_f$, it suffices to show that $\mathcal{W}_c^\chi \square f_f$. By Lemma 1.5, it suffices to check that any factorization of f_f as $B' \rightarrow Z \rightarrow B$ with $B' \rightarrow Z$ in \mathcal{W}_c^χ must have $B' = Z$. But $B' \rightarrow Z$ is in Q_χ , so by definition it is an identity and $B' = Z$. Since $f_f \in \mathbb{C}_{fib}^J$, $f \square f_f$ and we must have $f = f_{ac} \in \mathbb{C}'_{we}$.

Now suppose $f \in \square \mathbb{C}_{fib}^J$ is arbitrary. Let f' be the pushout of f along $A \rightarrow A \cup \chi(A)$. Since f' is the pushout of f it is also in $\square \mathbb{C}'_{fib}$, and as $A \rightarrow A \cup \chi(A)$ is in \mathbb{C}_{we}^J , f is a weak equivalence if and only if f' is. By the above, $f' \in \mathbb{C}'_{we}$, so we conclude that so is f .

The identity functor gives a Quillen equivalence $\mathbb{C} \rightleftarrows \mathbb{C}^J$ because the weak equivalences of the two structures are the same, and $\mathbb{C}_{cof} \subseteq \mathbb{C}_{cof}^J$. \square

We are now ready to prove Theorem 5.9.

Proof of Theorem 5.9. Let $\mathbb{C}_1, \mathbb{C}_2$ be any model structures on \mathcal{C} with weak equivalences \mathcal{W} , and let χ_i be the choice of centers given by \mathbb{C}_i . Let $J_i = ((\mathbb{C}_i)_{cof} \cap \mathcal{W}) \cup J_{\chi_i}$ and let \mathbb{C}^{χ_i} be the model structure constructed in Definition 3.7. Let $\chi = \chi_1 \times \chi_2$; by Lemma 2.5 this is another choice of centers. Note that $\mathcal{W}_c^{\chi_i} \subseteq \mathcal{W}_c^\chi$ since $Q_\chi \subseteq Q_{\chi_i}$. Thus $(\mathcal{W}_c^{\chi_i})^\square \supseteq (\mathcal{W}_c^\chi)^\square$, and the identity functor gives a Quillen equivalence $\mathbb{C}^{\chi_i} \rightleftarrows \mathbb{C}^\chi$. Thus for $i = 1, 2$ the identity functor gives a zigzag of Quillen equivalences

$$\mathbb{C}_i \rightleftarrows \mathbb{C}^{J_i} \rightleftarrows \mathbb{C}^{\chi_i} \rightleftarrows \mathbb{C}^\chi,$$

and the two model structures are equivalent. \square

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