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On some cohomological invariants for large families of infinite groups

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ABSTRACT. Over the ring of integers, groups of type Φ were first introduced by Olympia Talelli as a possible algebraic characterisation of groups that admit finite dimensional models for classifying spaces for proper actions. In this short article, we make the same definition over arbitrary commutative rings of finite global dimension and prove a number of properties pertaining to cohomological invariants of these groups with the extra condition that the groups belong to a large hierarchy of groups introduced by Peter Kropholler in the nineties. We prove most of Talelli's conjecture of equivalent statements for type Φ groups for these groups, and expand the scope of a few existing results in the literature.

CONTENTS

Acknowledgement 819
1. Background on cohomological invariants and a new result 819
2. Background on the classes of groups 824
3. Main results 827
4. Results on Conjecture 2.5 and other applications 832
References 838

Cohomological invariants are a useful tool in studying various cohomological and homological properties of infinite groups. It is often helpful to cluster these groups into various families and classes and study properties of certain cohomological invariants for all groups belonging to those classes. In this short article, we will be dealing with two classes of groups - one called groups of type Φ over various rings which were introduced over the ring of integers by Talelli in [23], and our other class is derived from a hierarchy of groups first introduced by Kropholler in the nineties in [18] - we will also be forming a class of groups mixing ideas behind the formation of both these classes. One of our aims is to prove an array of equalities of a bunch of cohomological invariants extending some results by Cornick and Kropholler.

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As mentioned in the abstract, we also prove a large part of a conjecture for type Φ groups proposed by Talelli with the extra assumption that the groups in question are in our large mixed class mentioned earlier.

For clarity, we provide two separate background sections - Section 1 on the cohomological invariants that we shall be using, and Section 2 on the classes of groups, because cohomological invariants often need to be accompanied with a lot of context and significance. Our original results are mostly collected in Section 2, Section 3 and Section 4. This work can be studied in conjunction with another paper of the author[4] where related questions on some of the cohomological invariants and some of the classes of modules studied in this article are studied, and some other important papers by Emmanouil and Talelli [12][13][23].

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1. Background on cohomological invariants and a new result

We begin by defining the following two invariants that were introduced by Gedrich and Gruenberg in [14].

Definition 1.1. *Let R be a ring. Define* $\mathrm{spli}(R)$ *and* $\mathrm{silp}(R)$ *to be respectively the supremum over the projective lengths (dimensions) of injective R-modules and the supremum over the injective lengths (dimensions) of projective R-modules.*

For any ring R, the finiteness of either $\mathrm{spli}(R)$ or $\mathrm{silp}(R)$ is connected to the question of whether R-modules admit complete projective resolutions (usually called just "complete resolutions") or complete injective resolutions. We shall not be dealing with complete injective resolutions in this article. So, we shall be using the term "complete resolutions" to mean "complete projective resolutions". Before going forward, we need to define complete resolutions.

Definition 1.2. *Let R be a ring. For any R-module M, a complete resolution of M, alternatively called a complete resolution admitted by M, is defined to be an infinite exact complex of projective R-modules, $(F_i, d_i)_{i\in\mathbb{Z}}$, that satisfies the following properties.*

a) There exists $n \geqslant 0$ such that for some projective resolution $(P_*, \delta_*) \twoheadrightarrow M$, $(P_i, \delta_i)_{i\geqslant n} = (F_i, d_i)_{i\geqslant n}$. The smallest such n is called the coincidence *index of the complete resolution* (F_*, d_*) *with respect to the projective resolution* (P_*, δ_*).

b) $Hom_R(F_*, Q)$ *is acyclic for any R-projective module Q.*

If (b) is not satisfied, we call (F_i, d_i) *a weak complete resolution of M.*

An R-module is said to be (weak) Gorenstein projective if it occurs as a kernel in a (weak) complete resolution.

If R is a group ring $A\Gamma$, *with* Γ *some group, it is said that* Γ *admits (weak) complete resolutions (over A) if the trivial module A admits (weak) complete resolutions as an* $A\Gamma$*-module.*

A quite handy example of a class of Gorenstein projectives is given in the following result which we state over the ring of integers.

Lemma 1.3. *(Lemma 2.21 of [1]) For any group* Γ, *all permutation* $\mathbb{Z}\Gamma$*-modules with finite stabilisers, i.e. modules that are direct sums of modules of the form* $Ind_G^{\Gamma}\mathbb{Z}$ *for any finite* $G \leqslant \Gamma$, *are Gorenstein projective.*

The following result was proved in [14].

Theorem 1.4. *(Result 4.1 of [14]) Let R be a ring. If* $\operatorname{spli}(R) < \infty$, *then every R-module admits a weak complete resolution.*

Remark 1.5. Whether a group or a module admitting weak complete resolutions over a ring is equivalent to the same admitting complete resolutions over the same ring is an interesting question (see Theorem 3.4).

Definition 1.2 contained the definition of Gorenstein projectives, which is a very useful class of modules in this theory. Using it, we make the following definitions.

Definition 1.6. *Let R be a ring. For any R-module M, the Gorenstein projective dimension of M with respect to R, denoted* $Gpd_R(M)$, *is defined to be the smallest integer n such that there is an exact sequence* $0 \to G_n \to G_{n-1} \to ... \to G_0 \to M \to 0$, *where each* G_i *is a Gorenstein projective R-module. If* $R = A\Gamma$, *where A is a commutative ring and* Γ *a group, then the Gorenstein cohomological dimension of* Γ *with respect to A, denoted* $\operatorname{Gcd}_A(\Gamma)$, *is defined to be* $Gpd_{A\Gamma}(A)$.

Remark 1.7. It is easy to see that, for any ring R, an R-module M admits a complete resolution iff it has finite Gorenstein projective dimension: if M admits a complete resolution F_* which has coincidence index say n with respect to a projective resolution $P_* \twoheadrightarrow M$, then the n-th kernel in P_*, which we can denote by $\Omega^n(M)$, is a kernel in the complete resolution F_*, which means $\Omega^n(M)$ is Gorenstein projective. We now have an exact sequence $0 \to \Omega^n(M) \to P_{n-1} \to .. \to P_0 \to M \to 0$, where each term other than M is Gorenstein projective (projectives are Gorenstein projective), and so $Gpd_R M \leqslant n$.

Now, let M satisfy $Gpd_R M \leqslant n$. Take $P_* \twoheadrightarrow M$ be a projective resolution of M. Then by Theorem 2.20 of [15], $\Omega^n(M)$, the n-the kernel in P_*, is Gorenstein projective. So, $\Omega^n(M)$ admits a complete resolution of coincidence index 0, and M admits a complete resolution of coincidence index $\leqslant n$.

Like the spli invariant defined earlier, the Gorenstein cohomological dimension of a group is a good indicator of whether the group admits complete resolutions. Here, it helps if the base ring is of finite global dimension:

Theorem 1.8. *(Theorem 1.7 of [13]) For any commutative ring A of finite global dimension and any group Γ, the following are equivalent.*

a) $\mathrm{Gcd}_A(\Gamma) < \infty$, i.e. the trivial module A admits complete resolutions as an $A\Gamma$-module.

b) $\mathrm{silp}(A\Gamma) = \mathrm{spli}(A\Gamma) < \infty$.

c) $Gpd_{A\Gamma}(M) < \infty$, for all $A\Gamma$-modules M, i.e. all $A\Gamma$-modules admit complete resolutions.

Also of use is the fact that one can put an upper bound on the spli and silp invariants using the Gorenstein cohomological dimension if the base ring is of finite global dimension:

Lemma 1.9. *(Corollary 1.6 of [13]) For any commutative ring A of global dimension t and any group Γ, $\mathrm{silp}(A\Gamma), \mathrm{spli}(A\Gamma) \leqslant \mathrm{Gcd}_A(\Gamma) + t$.*

Remark 1.10. There are no known examples of group rings where the silp and spli invariants differ. It was shown in [14] (Result 1.6) that if they are both finite over a ring then they are equal. Result 2.4 of [14] showed that if A is a Noetherian commutative ring of global dimension t and Γ is any group, then $\mathrm{silp}(A\Gamma) \leqslant \mathrm{spli}(A\Gamma) + t$. It is possible that one might be able to prove this result without the Noetherian condition. In [12], Emmanouil showed that, under the same conditions, $\mathrm{silp}(A\Gamma) = \mathrm{spli}(A\Gamma)$. It follows from Lemma 2.2 of [21], although they only work over the integers, that if a group Γ admits weak complete resolutions over A and $\mathrm{silp}(A\Gamma) < \infty$, then Γ also admits complete resolutions. More generally, any weak complete resolution is a complete resolution, provided that all projective modules have finite injective dimension.

Very similar in use and purpose to the Gorenstein cohomological dimension, is the invariant "generalized cohomological dimension" which was introduced by Ikenaga in [16] over the integers.

Definition 1.11. *For any commutative ring A and any group Γ, define the generalized cohomological dimension of Γ with respect to A, denoted $\underline{\mathrm{cd}}_A(\Gamma)$, to be $\sup\{n \in \mathbb{Z}_{\geqslant 0} : \mathrm{Ext}^n_{A\Gamma}(M, F) \neq 0$, for some A-free M and some $A\Gamma$-free $F\}$.*

For any group, the Gorenstein cohomological dimension, when finite, coincides with its generalized cohomological dimension over rings of finite global dimension - this result was proved over the integers in [2] without the finiteness condition, and the same proof works for rings of finite global dimension albeit with the extra finiteness condition. We record this result below.

Theorem 1.12. *(follows from Theorem* 2.5 *of* [2]*) Let A be a commutative ring of finite global dimension. Then, for any group Γ, $\mathrm{Gcd}_A(\Gamma) = \underline{cd}_A(\Gamma)$ if $\mathrm{Gcd}_A(\Gamma)$ is finite.*

Proof. Let M be an $A\Gamma$-module such that $Gpd_{A\Gamma}(M) < \infty$. It therefore follows from Theorem 2.20 of [15] that $Gpd_{A\Gamma}(M) = \sup\{i \in \mathbb{Z} : \mathrm{Ext}^i_{A\Gamma}(M, P) \neq 0$, for some $A\Gamma$-projective $P\}$. This gives us the following:

a) Noting that $\mathrm{Gcd}_A(\Gamma) := Gpd_{A\Gamma}(A)$, it follows from Definition 1.11 that $\underline{cd}_A(\Gamma) \geqslant \mathrm{Gcd}_A(\Gamma)$.

b) Note that from Theorem 1.8 and Lemma 3.9.a. (which we prove later), it follows that $\mathrm{Gcd}_A(\Gamma) < \infty$ implies $\underline{cd}_A(\Gamma) < \infty$. As noted in the second paragraph of the proof of Theorem 2.5 of [2], it follows from Definition 1.11 and the above characterisation of finite Gorenstein projective dimension that $\underline{cd}_A(\Gamma) = \sup\{Gpd_{A\Gamma}(M) : M\ A\text{-free}\}$. Proposition 2.4.$c$ of [2] shows that if a $\mathbb{Z}\Gamma$-module N is $\mathbb{Z}$-free, then $Gpd_{\mathbb{Z}\Gamma}(N) \leqslant \mathrm{Gcd}_{\mathbb{Z}}(\Gamma)$. The same proof works when $\mathbb{Z}$ is replaced by A, and so we have $\underline{cd}_A(\Gamma) \leqslant \mathrm{Gcd}_A(\Gamma)$. □

Remark 1.13. We can use the proof of Theorem 2.5 of [2] to say that if A is a Noetherian commutative ring of finite global dimension, then $\mathrm{Gcd}_A(\Gamma) = \underline{cd}_A(\Gamma)$, for any group Γ. The Noetherian assumption becomes useful in handling the case when $\mathrm{Gcd}_A(\Gamma)$ might not be finite. That is because it follows from Theorem 4.4 of [12] that for any commutative Noetherian A of finite global dimension and any group Γ, $\mathrm{silp}(A\Gamma) = \mathrm{spli}(A\Gamma)$ (the Noetherian assumption is required because, here, one needs to invoke Result 2.4 of [14] that we mentioned in Remark 1.10), and this result is crucial to show that $\mathrm{Gcd}_A(\Gamma) < \infty$ iff $\underline{cd}_A(\Gamma) < \infty$, as noted in the first paragraph of the proof of Theorem 2.5 of [2]. We are able to not have to use the Noetherian assumption in Theorem 1.12 because we are focusing only on the case where the Gorenstein cohomological dimension is known to be finite.

In Section 3, we shall see how in some cases to achieve bounds, it is more helpful to work with the generalized cohomological dimension instead of the Gorenstein cohomological dimension.

We now introduce two more interesting cohomological invariants, one of which, the finitistic dimension, is quite well-studied in representation theory. As a matter of common notation, throughout this article, for any ring R, $Mod(R)$ will denote the category of all R-modules whose morphisms are all module homomorphisms between R-modules.

Definition 1.14. *Let A be a commutative ring and let Γ be a group.*

$k(A\Gamma) := \sup\{\mathrm{proj.dim}_{A\Gamma} M : M \in Mod(A\Gamma), \mathrm{proj.dim}_{AG} M < \infty$ *for all finite* $G \leqslant \Gamma\}$.

$\mathrm{fin.dim}(A\Gamma) := \sup\{\mathrm{proj.dim}_{A\Gamma} M : M \in Mod(A\Gamma), \mathrm{proj.dim}_{A\Gamma} M < \infty\}$.

The two invariants introduced in Definition 1.14 will be playing a major role in our dealings with type Φ groups in Section 3 and Section 4.

The last invariant we want to introduce in this section is defined as the projective dimension of a specific module.

Definition 1.15. *For any commutative ring A and any group Γ, denote by $B(\Gamma, A)$ the module of those functions $\Gamma \to A$ that are only allowed to take finitely many values in A. The $A\Gamma$-module structure on $B(\Gamma, A)$ is given the following way: for any $f \in B(\Gamma, A)$, $(\gamma_1 \cdot f)(\gamma) := f(\gamma_1^{-1}\gamma)$, for all $\gamma, \gamma_1 \in \Gamma$.*

Following [3], we define an $A\Gamma$-module M to be a Benson's cofibrant if $M \otimes_A B(\Gamma, A)$ is a projective $A\Gamma$-module.

The following is an important set of properties of the module defined above.

Lemma 1.16. *(Lemma 3.4 of [3]) For any group Γ and any commutative ring A, $B(\Gamma, A)$ is A-free and is AG-free, for any finite $G \leqslant \Gamma$.*

We make the following conjecture and prove it (see Theorem 1.18) under a finiteness condition.

Conjecture 1.17. *For any commutative ring A of finite global dimension and any group Γ, $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) = \operatorname{Gcd}_A(\Gamma)$.*

Theorem 1.18. *Let A be a commutative ring of finite global dimension and let Γ be a group. Then, Conjecture 1.17 is satisfied for A and Γ if $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) < \infty$.*

To prove Theorem 1.18, we need the following two lemmas.

Lemma 1.19. *If, for some commutative ring A and for some group Γ, $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A)$ is finite, then $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) \leqslant \underline{cd}_A(\Gamma)$.*

Proof. We can assume that $\underline{cd}_A(\Gamma)$ is finite.

Now, let us assume that $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) = k > \underline{cd}_A(\Gamma)$. There exists an $A\Gamma$-module M such that $\operatorname{Ext}^k_{A\Gamma}(B(\Gamma, A), M) \neq 0$ because otherwise $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) \leqslant k - 1$. Let F be the $A\Gamma$-free module on M. We have a short exact sequence $0 \to \Omega(M) \to F \to M \to 0$. We now look at the following long exact Ext-sequence associated to this short exact sequence and get $.. \to \operatorname{Ext}^k_{A\Gamma}(B(\Gamma, A), \Omega(M)) \to \operatorname{Ext}^k_{A\Gamma}(B(\Gamma, A), F) \to \operatorname{Ext}^k_{A\Gamma}(B(\Gamma, A), M) \to \operatorname{Ext}^{k+1}_{A\Gamma}(B(\Gamma, A), \Omega(M)) \to ...$ Here, $\operatorname{Ext}^k_{A\Gamma}(B(\Gamma, A), F) = 0$ because $k > \underline{cd}_A(\Gamma)$ (see Definition 1.11) and $B(\Gamma, A)$ is A-free by Lemma 1.16 and F is $A\Gamma$-free. Also, we have that $\operatorname{Ext}^{k+1}_{A\Gamma}(B(\Gamma, A), \Omega(M)) = 0$ since $\operatorname{proj.dim}_{AG} B(\Gamma, A) = k$. So, $\operatorname{Ext}^k_{A\Gamma}(B(\Gamma, A), M) = 0$ which gives us a contradiction. □

Before we state our next result regarding comparison of the invariants that we have introduced, we state the following result which gives a sufficient condition on a module for it to admit complete resolutions.

Theorem 1.20. *(Theorem 3.5 of [7]) Let A be a commutative ring and Γ a group. If $M \otimes_A B(\Gamma, A)$ is projective, then M is Gorenstein projective, i.e. it admits a complete resolution of coincidence index 0.*

Remark 1.21. Note that in [7] when Theorem 1.20 was proved, it was stated in different language. What we state in Theorem 1.20 is exactly what was proved in proving Theorem 3.5 in [7].

Lemma 1.22. *For any commutative ring A and any group Γ, $\mathrm{Gcd}_A(\Gamma) \leqslant \operatorname{proj.dim}_{A\Gamma} B(\Gamma, A)$.*

Proof. We can assume that $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A)$ is finite because otherwise we have nothing to prove.

Let M be an $A\Gamma$-module satisfying $\operatorname{proj.dim}_{A\Gamma} M \otimes_A B(\Gamma, A) = n$. Since $B(\Gamma, A)$ is A-free by Lemma 1.16.a, if we take a projective resolution $(P_*, d_*) \twoheadrightarrow M$ of $A\Gamma$-projective modules P_i with the kernels given by $\Omega^*(M)$, we get a projective resolution $(P_* \otimes_A B(\Gamma, A), d_* \otimes id) \twoheadrightarrow M \otimes_A B(\Gamma, A)$ where the kernels are given by $\Omega^*(M) \otimes_A B(\Gamma, A)$. So, $\Omega^n(M) \otimes_A B(\Gamma, A)$ is projective as an $A\Gamma$-module. And, now we can use Theorem 1.20 to deduce that $\Omega^n(M)$ is Gorenstein projective; it therefore follows that $Gpd_{A\Gamma}(M) \leqslant n$. If we replace M by the trivial module A, the hypothesis $\operatorname{proj.dim}_{A\Gamma} M \otimes_A B(\Gamma, A) = n$ becomes $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) = n$, and we get that $n \geqslant Gpd_{A\Gamma}(A) = \mathrm{Gcd}_A(\Gamma)$. ☐

We can finish the proof of Theorem 1.18 now.

Proof of Theorem 1.18. Theorem 1.18 now follows from Lemma 1.19, Lemma 1.22 and Theorem 1.12. ☐

2. Background on the classes of groups

We first define groups of type Φ as those groups will play a crucial role in our treatment.

Definition 2.1. *(made over $\mathbb{Z}$ in [23]) For any commutative ring A, a group Γ is said to be of type Φ over A if, for any $A\Gamma$-module M, the following two statements are equivalent.*

a) $\operatorname{proj.dim}_{A\Gamma} M < \infty$.

b) $\operatorname{proj.dim}_{AG} M < \infty$*, for all finite $G \leqslant \Gamma$.*

We denote the class of all groups of type Φ over A by $\mathcal{F}_{\phi,A}$.

Examples of groups of type Φ over all commutative rings of finite global dimension are groups of finite virtual cohomological dimension, groups acting on trees with finite stabilisers, etc. (see [20] or [22]).

Another important class of groups comes from Kropholler's hierarchy:

Definition 2.2. *([18]) Let $\mathcal{X}$ be a class of groups. Define a hierarchy of groups in the following way: $H_0\mathcal{X} := \mathcal{X}$, and for any successor ordinal (like an integer) α, a group $\Gamma \in H_\alpha\mathcal{X}$ iff there exists a finite dimensional contractible CW-complex on which Γ acts by permuting the cells with all the* *cell stabilisers in* $H_{\alpha-1}\mathfrak{X}$*. If* α *is a limit ordinal,* $H_\alpha\mathfrak{X} := \bigcup_{\beta<\alpha} H_\beta\mathfrak{X}$*. A group is said to be in* $H\mathfrak{X}$ *iff it is in* $H_\alpha\mathfrak{X}$ *for some ordinal* α*. Also, for any ordinal* α*,* $H_{<\alpha}\mathfrak{X} := \bigcup_{\beta<\alpha} H_\beta\mathfrak{X}$*.*

The class $L\mathfrak{X}$ *is defined to be the class of all groups* Γ *such that every finitely generated subgroup of* Γ *is in* $\mathfrak{X}$*.*

Throughout this article, $\mathfrak{F}$ *denotes the class of all finite groups.*

Regarding groups acting on finite dimensional contractible CW-complexes, the following is a standard trick which will be of use to us later.

Lemma 2.3. *Let* Γ *be a group acting cellularly on a finite dimensional contractible* CW*-complex with stabilisers in the class of groups* $\mathfrak{X}$*, and let* R *be a commutative ring. Then, any* $R\Gamma$*-module* M *admits a finite length resolution with modules from the class* $\{Ind_{\Gamma'}^{\Gamma}Res_{\Gamma'}^{\Gamma}M : \Gamma' \in \mathfrak{X}\}^{\oplus}$*; here the superscript "*$\oplus$*" means that we are taking the smallest direct-sum closed class of modules containing the given class.*

Proof. Let the dimension of X be n. From the action of Γ on X, we get the augmented cellular complex $0 \to C_n \to ... \to C_0 \to \mathbb{Z} \to 0$, where each C_i is a permutation module that we get from the action of Γ as a group of permutations of the i-dimensional cells of X. So, C_i can be written as a direct sum of the trivial module induced up to Γ from subgroups of Γ that are of the form Γ_σ, where Γ_σ denotes the stabiliser of the cell σ, with σ running over the set of Γ-representatives of the i-dimensional cells; note that each $\Gamma_\sigma \in \mathfrak{X}$.

Tensoring the augmented cellular complex by M, for any $R\Gamma$-module M, we get an exact sequence $0 \to C_n \otimes_{\mathbb{Z}} M \to ... \to C_0 \otimes_{\mathbb{Z}} M \to M \to 0$, where each $C_i \otimes_{\mathbb{Z}} M$ is a direct sum of modules of the form $Ind_{\Gamma'}^{\Gamma}Res_{\Gamma'}^{\Gamma}M$ with $\Gamma' \in \mathfrak{X}$, and we are done. □

A very useful property admitted by $H_1\mathfrak{F}$-groups is the admission of complete resolutions over any commutative ring. This result was proved by Cornick and Kropholler in [7], but we are now in a position to give a much shorter direct proof of this result. It is noteworthy that the fact that $H_1\mathfrak{F}$-groups admit complete resolutions is useful in constructing stable module categories of modules over those groups and proving important generation properties of those stable module categories (see Section 6 of [5]).

Proposition 2.4. *(different proof in* [7]*) Let* A *be a commutative ring. Then, all* $H_1\mathfrak{F}$*-groups admit complete resolutions over* A*. If additionally,* A *has finite global dimension, then we can prove that for any group* $\Gamma \in H_1\mathfrak{F}$*, all* $A\Gamma$*-modules admit complete resolutions.*

Proof. Let $\Gamma \in H_1\mathfrak{F}$. Then, there is an n-dimensional contractible CW-complex, for some integer n, on which Γ acts with finite stabilisers. The augmented cellular complex looks like an exact sequence $0 \to C_n \to .. \to C_0 \to \mathbb{Z} \to 0$ where each C_i is a direct sum of permutation modules with finite stabilisers, which are all Gorenstein projective by Lemma 1.3. Each

C_i is Gorenstein projective, and therefore $\mathrm{Gcd}_{\mathbb{Z}}(\Gamma) = Gpd_{\mathbb{Z}\Gamma}(\mathbb{Z}) < \infty$. By Proposition 2.1 of [13], we have $\mathrm{Gcd}_R(\Gamma) < \infty$, for all commutative rings R. So, for our given commutative ring A, it follows by Remark 1.7 that the trivial $A\Gamma$-module A admits complete resolutions and therefore Γ admits complete resolutions.

Now, if A has finite global dimension, using Theorem 1.8, we can say that all $A\Gamma$-modules admit complete resolutions. □

We can make the following conjecture mixing a part of Conjecture A of [23] (where the base ring was the ring of integers) and Conjecture 43.1 of [6] and adding a few extra conditions.

Conjecture 2.5. *For any group Γ and any commutative ring A of finite global dimension, the following are equivalent.*

a) Γ *is of type* Φ *over* A.
b) $\mathrm{silp}(A\Gamma) < \infty$.
c) $\mathrm{spli}(A\Gamma) < \infty$.
d) $\mathrm{proj.dim}_{A\Gamma} B(\Gamma, A) < \infty$.
e) $\mathrm{Gcd}_A(\Gamma) < \infty$.
f) $\mathrm{fin.dim}(A\Gamma) < \infty$.
g) $k(A\Gamma) < \infty$.
When $A = \mathbb{Z}$, *we can add the condition*
h) $\Gamma \in H_1\mathfrak{F}$, *where* $\mathfrak{F}$ *is the class of all finite groups.*

In Section 3, we prove that statements (a) to (g) are equivalent if $\Gamma \in LH\mathfrak{F}_{\phi,A}$.

Since the statement of Conjecture 1.17 deals with two of the invariants mentioned in Conjecture 2.5, the following connection between them is worth noting.

Proposition 2.6. *Let $\mathfrak{X}$ be a class of groups such that, for a fixed commutative A of finite global dimension, $(a) \Leftrightarrow (e)$ in Conjecture 2.5 for all groups $\Gamma \in \mathfrak{X}$. Then, Conjecture 1.17 holds true over A for all groups $\Gamma \in \mathfrak{X}$.*

Proof. Let $\Gamma \in \mathfrak{X}$. We can assume that $\mathrm{proj.dim}_{A\Gamma} B(\Gamma, A)$ is not finite because if it is finite we are done due to Theorem 1.18. Now, if $\mathrm{Gcd}_A(\Gamma)$ is finite, then by our hypothesis, Γ is of type Φ over A, and therefore it follows from Definition 2.1 that $\mathrm{proj.dim}_{A\Gamma} B(\Gamma, A) < \infty$ due to Lemma 1.16, and we have a contradiction. □

We end this section with the following remark on the size of Kropholler's hierarchy.

Remark 2.7. It follows from the definition of Kropholler's hierarchy that $H_\alpha\mathfrak{X} \subseteq H_\beta\mathfrak{X}$, for any $\mathfrak{X}$ and for any two ordinals α and β satisfying $\alpha \leqslant \beta$. It is shown in [17] that $H_\alpha\mathfrak{F} \neq H_{\alpha+1}\mathfrak{F}$ for every ordinal α smaller than the first infinite ordinal, i.e. starting with the class of finite groups, with every iteration of the operator H, one gets a strictly bigger class than the class they started with. All known examples of groups in $H_n\mathfrak{F}\backslash H_{n-1}\mathfrak{F}$, for any integer $n > 1$, do not satisfy the conditions (a) to (g) of Conjecture 2.5 for any commutative ring A of finite global dimension (see Remark 4.8).

We have $LH\mathfrak{F} \subseteq LH\mathfrak{F}_{\phi,A}$ as all finite groups are of type Φ over A. However, if $(a) \Longrightarrow (h)$ in Conjecture 2.5 with $A = \mathbb{Z}$ is true, then $\mathfrak{F}_{\phi,\mathbb{Z}} = H_1\mathfrak{F}$ (it is known that $(h) \Longrightarrow (a)$, see Proposition 2.4 of [22]), and $LH\mathfrak{F}_{\phi,\mathbb{Z}} = LH\mathfrak{F}$.

3. Main results

Our main result in this section is the following.

Theorem 3.1. *Let $\Gamma \in LH\mathfrak{F}_{\phi,A}$ with A being a commutative ring of global dimension t. Then,*

$$\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) = \operatorname{Gcd}_A(\Gamma)$$

and, denoting the above common value by Θ, we have

$$\Theta \leqslant \operatorname{fin.dim}(A\Gamma) = \operatorname{silp}(A\Gamma) = \operatorname{spli}(A\Gamma) = k(A\Gamma) \leqslant \Theta + t.$$

To prove the first equality in Theorem 3.1, we need to first state a conjecture involving the class of Benson's cofibrants and Gorenstein projectives.

Conjecture 3.2. *(see [4] or [10]) For any commutative ring A of finite global dimension and any group Γ, the class of Benson's cofibrant $A\Gamma$-modules (see Definition 1.15) and the class of Gorenstein projective $A\Gamma$-modules coincide.*

The following connection can be proved between Conjecture 3.2 and Conjecture 1.17.

Proposition 3.3. *Let Γ be a group and let A be a commutative ring of finite global dimension. If the class of Benson's cofibrant $A\Gamma$-modules coincides with the class of Gorenstein projective $A\Gamma$-modules, then* $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) = \operatorname{Gcd}_A(\Gamma)$.

Proof. In light of Theorem 1.18, we can assume that $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A)$ is not finite. Now, let us assume that $\operatorname{Gcd}_A(\Gamma) = n < \infty$. Then, $\Omega^n(A)$ is Gorenstein projective, and therefore from our hypothesis, $\Omega^n(A) \otimes_A B(\Gamma, A)$ is projective as an $A\Gamma$-module, and since $B(\Gamma, A)$ is A-free by Lemma 1.16, we get that $\Omega^n(A \otimes_A B(\Gamma, A)) = \Omega^n(B(\Gamma, A))$ is projective as an $A\Gamma$-module. Therefore, $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A)$ is finite, and we have a contradiction. □

The following result is important because its first part will be useful to deduce that, over any commutative ring A of finite global dimension, Conjecture 1.17 is satisfied for all $\Gamma \in LH\mathfrak{F}_{\phi,A}$. All the material between Theorem 3.4 and the end of its proof is from [4] which, in turn, is derived from the treatment in [10].

Theorem 3.4. *Let $\Gamma \in LH\mathfrak{F}_{\phi,A}$ where A is a commutative ring of finite global dimension. Then,*

a) The class of Benson's cofibrant $A\Gamma$-modules and Gorenstein projective $A\Gamma$-modules coincide.

b) M admits a weak complete resolution iff it admits a complete resolution, for any $M \in Mod(A\Gamma)$.

To prove Theorem 3.4, we need a few technical results:

Lemma 3.5. *(standard knowledge, see Lemma 2.1.c. of [10]) Let R be a ring and let $(F_i, d_i)_{i\in\mathbb{Z}}$ be an infinite exact complex of R-projective modules with a finite bound on the projective dimensions of the kernels as R-modules. Then, each kernel is R-projective.*

Proof. Let m be the bound on the projective dimensions of the kernels, and let us denote the kernels as $K_i = Ker(d_i)$, for all $i \in \mathbb{Z}$. Let $K_t := Ker(d_t)$ be of projective dimension $n > 0$. Then, from the short exact sequence $0 \to K_t \hookrightarrow F_t \twoheadrightarrow K_{t-1} \to 0$, it follows that $\operatorname{proj.dim}_R K_{t-1} = n + 1$. Going on like this, we get that $\operatorname{proj.dim}_R K_{t-m} = n + m > m$, which is not possible. □

Lemma 3.6. *Let A be a commutative ring of finite global dimension t and let Γ be a group, and let $WGProj(A\Gamma)$ denote the class of all weak Gorenstein projective $A\Gamma$-modules. If $\operatorname{proj.dim}_{A\Gamma} M \otimes_A B(\Gamma, A) < \infty$ for all $M \in WGProj(A\Gamma)$, then $M \otimes_A B(\Gamma, A)$ is projective for all $M \in WGProj(A\Gamma)$.*

Proof. Now, let $M \in WGProj(A\Gamma)$ such that $\operatorname{proj.dim}_{A\Gamma} M \otimes_A B(\Gamma, A) = n > 0$. There exists a weak complete resolution with $A\Gamma$-projectives, which we shall denote by $(F_i, d_i)_{i\in\mathbb{Z}}$, where M is a kernel. Since $B(\Gamma, A)$ is A-free, $M \otimes_A B(\Gamma, A)$ too occurs as a kernel in a weak complete resolution by $A\Gamma$-projectives, $(F_i \otimes_A B(\Gamma, A), d_i \otimes_A id)_{i\in\mathbb{Z}}$. Let $M = Ker(d_p)$. It follows from the proof of Lemma 3.5 that $\operatorname{proj.dim}_{A\Gamma} Ker(d_{p-k}) \otimes_A B(\Gamma, A) = n + k$, for all $k > 0$. Now, $K := \bigoplus_{m\leqslant p} Ker(d_m) \in WGProj(A\Gamma)$ as $WGProj(A\Gamma)$ is closed under arbitrary direct sums (this is obvious from Definition 1.2). But, for any $k > 0$, we have $\operatorname{proj.dim}_{A\Gamma} K \otimes_A B(\Gamma, A) \geqslant \operatorname{proj.dim}_{A\Gamma} Ker(d_{p-k}) \otimes B(\Gamma, A) = n + k$, and we have a contradiction. □

Proof of Theorem 3.4. a) We start with the observation that if M is a (weak) Gorenstein projective $A\Gamma$-module, then it is A-projective. This is easy to see for the following reason. We know that M occurs as a kernel in a doubly infinite acyclic complex of projectives, say $(F_i, d_i)_{i\in\mathbb{Z}}$. If $M = Ker(d_n)$, then M can be written as a t-th syzygy of $Ker(d_{n-t})$, where t is the global dimension of A, and therefore M has to be A-projective.

Now fix a weak Gorenstein projective $A\Gamma$-module M. Note that we have $\operatorname{proj.dim}_{A\Gamma'} B(\Gamma, A) < \infty$ for all $\mathfrak{F}_{\phi,A}$-subgroups Γ' of Γ by Lemma 1.16 and Definition 2.1. Therefore, $\operatorname{proj.dim}_{A\Gamma'} N \otimes_A B(\Gamma, A) < \infty$, for any weak Gorenstein projective N. So, $M \otimes_A B(\Gamma, A)$ is projective over all $\mathfrak{F}_{\phi,A}$-subgroups of Γ by Lemma 3.6.

Now, we make the induction hypothesis that for all ordinals $\beta < \alpha$, $M \otimes_A B(\Gamma, A)$ is projective over $H_\beta\mathcal{F}_{\phi,A}$-subgroups of Γ. The base for $\beta = 0$ has already been checked above. Let Γ' be an $H_\alpha\mathcal{F}_{\phi,A}$-subgroup of Γ. Then, by Lemma 2.3, $M \otimes_A B(\Gamma, A)$, as an $A\Gamma'$-module, admits a finite-length resolution with modules that are direct sums of modules of the form $Ind^{\Gamma'}_{\Gamma''}Res^{\Gamma'}_{\Gamma''}M \otimes_A B(\Gamma, A)$ with $\Gamma'' \in H_{<\alpha}\mathcal{F}_{\phi,A}$, and since by our induction hypothesis, $Res^{\Gamma'}_{\Gamma''}M \otimes_A B(\Gamma, A)$ is $A\Gamma''$-projective for any $\Gamma'' \in H_{<\alpha}\mathcal{F}_{\phi,A}$ (note that an $H_{<\alpha}\mathcal{F}_{\phi,A}$-subgroup of Γ' is also an $H_{<\alpha}\mathcal{F}_{\phi,A}$-subgroup of Γ), we have $\operatorname{proj.dim}_{A\Gamma'} M \otimes_A B(\Gamma, A) < \infty$. Since the above conclusion is true for any weak Gorenstein projective M, by Lemma 3.5, $M \otimes_A B(\Gamma, A)$ is $A\Gamma'$-projective. Thus, we have proved that $M \otimes_A B(\Gamma, A)$ is projective over all $H\mathcal{F}_{\phi,A}$-subgroups of Γ.

Since $\Gamma \in LH\mathcal{F}_{\phi,A}$, we can assume that it is uncountable because if it is countable, then $\Gamma \in H\mathcal{F}_{\phi,A}$ (this follows from Lemma 2.5 of [17]), and we are done by the previous paragraph. We now make the induction hypothesis that over all subgroups Γ' of Γ that have cardinality strictly smaller than that of Γ, $M \otimes_A B(\Gamma, A)$ is projective. As Γ is uncountable, it can be expressed as an ascending union of subgroups $\bigcup_{\lambda<\delta} \Gamma_\lambda$, for some ordinal δ, where each Γ_λ has cardinality strictly smaller than that of Γ. By our induction hypothesis, $M \otimes_A B(\Gamma, A)$ is projective over each Γ_λ, and so by Lemma 5.6 of [3], $\operatorname{proj.dim}_{A\Gamma} M \otimes_A B(\Gamma, A) \leqslant 1$, and since this is true for all $M \in WGProj(A\Gamma)$ (here again, $WGProj(A\Gamma)$ denotes the class of all weak Gorenstein projective $A\Gamma$-modules), we have by Lemma 3.6 that $M \otimes_A B(\Gamma, A)$ is $A\Gamma$-projective.

We have thus showed that weak Gorenstein projective $A\Gamma$-modules are Benson's cofibrants. Theorem 1.20 tells us that Benson's cofibrants are Gorenstein projectives. So, we have a coincidence between weak Gorenstein projectives, Gorenstein projectives and Benson's cofibrants.

b) Part (b) follows directly from the coincidence between weak Gorenstein projectives and Benson's cofibrants, as noted in the proof of Corollary D in [10] over the integers and exactly the same proof works over A in our case. □

Remark 3.7. A relevant observation to make for the proof of Theorem 3.4, that we have provided above, is that we have a coincidence between weak Gorenstein projectives and Gorenstein projectives with Benson's cofibrants playing an auxiliary role.

However, we do mention Benson's cofibrants in the statement of Conjecture 3.2 because (a) in that exact form, the conjecture has been studied in the literature in the past [10], and (b) having "Benson's cofibrants" in the statement of Conjecture 3.2 helps us show, in Proposition 3.3, how Conjecture 1.17 and Conjecture 3.2 can be related.

Now, we are in a position to prove the following result. Note that a proof of the same was claimed for $H\mathcal{F}$-groups in the proof of Theorem C of [8] but the authors of that paper overlooked a condition on the base ring that was present in the hypothesis of a key theorem that they were citing. We expand on this more towards the end of this section in Remark 3.12.

Lemma 3.8. *Let $\Gamma \in LH\mathfrak{F}_{\phi,A}$ where A is a commutative ring of finite global dimension. Then,* $\mathrm{silp}(A\Gamma) \leqslant \mathrm{spli}(A\Gamma)$.

Proof. Assume $\mathrm{spli}(A\Gamma) < \infty$ because otherwise we have nothing to prove. $\mathrm{spli}(A\Gamma) < \infty$ implies that all $A\Gamma$-modules admit weak complete resolutions by Theorem 1.4. By Theorem 3.4.b., since $\Gamma \in LH\mathfrak{F}_{\phi,A}$, it now follows that every $A\Gamma$-module admits complete resolutions. It now follows directly from the (b)-(c) equivalence in Theorem 1.8 that $\mathrm{silp}(A\Gamma) = \mathrm{spli}(A\Gamma) < \infty$. □

We now prove the following three inequalities involving five different invariants that are known in the literature.

Lemma 3.9. *([21], [8]) Let A be a commutative ring and let Γ be a group. Then,*

a) $\underline{cd}_A(\Gamma) \leqslant \mathrm{silp}(A\Gamma)$.
b) $\mathrm{fin.\,dim}(A\Gamma) \leqslant \mathrm{silp}(A\Gamma)$.
If, in addition, A is of finite global dimension, then
c) $\mathrm{spli}(A\Gamma) \leqslant k(A\Gamma)$.

Proof. a) This has been noted in [21]. It is obvious from the definitions - it follows from the definition of $\mathrm{silp}(A\Gamma)$ that it is $\sup\{n \in \mathbb{N} : \mathrm{Ext}^n_{A\Gamma}(X, P) \neq 0$ for some $A\Gamma$-module X and some $A\Gamma$-projective $P\}$, and $\underline{cd}_A(\Gamma) := \sup\{n \in \mathbb{Z}_{\geqslant 0} : \mathrm{Ext}^n_{A\Gamma}(M, F) \neq 0$ for some A-free M and some $A\Gamma$-free $F\}$. Since, free modules are projective, the inequality follows.

b) This again follows from definitions and has been noted in the proof of Theorem C of [8]. We can assume that $\mathrm{silp}(A\Gamma) = r < \infty$ because otherwise we have nothing to prove. From the definition of injective dimension, it follows that $r = \sup\{n \in \mathbb{Z} : \mathrm{Ext}^n_{A\Gamma}(M, P) \neq 0$ for some $A\Gamma$-module M and some $A\Gamma$-projective $P\}$.

Take an $A\Gamma$-module T of finite projective dimension, say k. There exists an $A\Gamma$-module X such that $\mathrm{Ext}^k_{A\Gamma}(T, X) \neq 0$ because otherwise we will have that $\mathrm{proj.\,dim}_{A\Gamma} T \leqslant k - 1$. Take F to be the $A\Gamma$-free module on X and we get a short exact sequence $0 \to \Omega(X) \to F \to X \to 0$, that gives us a long exact Ext-sequence $.. \to \mathrm{Ext}^k_{A\Gamma}(T, \Omega(X)) \to \mathrm{Ext}^k_{A\Gamma}(T, F) \to \mathrm{Ext}^k_{A\Gamma}(T, X) \to \mathrm{Ext}^{k+1}_{A\Gamma}(T, \Omega(X)) \to ...$ Here, $\mathrm{Ext}^{k+1}_{A\Gamma}(T, \Omega(X)) = 0$ as $\mathrm{proj.\,dim}_{A\Gamma} T = k$, so if $\mathrm{Ext}^k_{A\Gamma}(T, F) = 0$, we get an embedding $\mathrm{Ext}^k_{A\Gamma}(T, X) \hookrightarrow \mathrm{Ext}^{k+1}_{A\Gamma}(T, \Omega(X)) = 0$ implying $\mathrm{Ext}^k_{A\Gamma}(T, X) = 0$, which is not possible. So, $\mathrm{Ext}^k_{A\Gamma}(T, F) \neq 0$, and since F is $A\Gamma$-projective as it is free, we get from the definition of $\mathrm{silp}(A\Gamma)$ that $k \leqslant r = \mathrm{silp}(A\Gamma)$. Thus, $\mathrm{fin.\,dim}(A\Gamma) \leqslant \mathrm{silp}(A\Gamma)$.

c) This, again, has been covered in the proof of Theorem C of [8]. We can assume that $k(A\Gamma) = n < \infty$. If I is an injective $A\Gamma$-module, then for any finite $G \leqslant \Gamma$, I is an injective AG-module with finite projective dimension as an AG-module since A has finite global dimension. So, by definition of $k(A\Gamma)$, $\operatorname{proj.dim}_{A\Gamma} I \leqslant n$, and therefore, $\operatorname{spli}(A\Gamma) \leqslant n$. □

Before proving our last major inequality involving the invariants, we record the following result which is now considered standard knowledge.

Lemma 3.10. *(done over $\mathbb{Z}$ in Lemma 3.3.ii of [23], same proof works here) For any group Γ and any commutative ring A of finite global dimension, $\operatorname{fin.dim}(A\Gamma') \leqslant \operatorname{fin.dim}(A\Gamma)$, for all subgroups $\Gamma' \leqslant \Gamma$.*

Note that in the proof of Theorem C of [8], the following result has been proved by Cornick and Kropholler for $\Gamma \in H\mathfrak{F}$. The first part of our proof of Lemma 3.11 is very similar to their treatment which again revolves around the standard trick highlighted in Lemma 2.3.

Lemma 3.11. *Let $\Gamma \in LH\mathfrak{F}_{\phi,A}$ where A is a commutative ring of finite global dimension. Then, $k(A\Gamma) \leqslant \operatorname{fin.dim}(A\Gamma)$.*

Proof. Assume that $\operatorname{fin.dim}(A\Gamma) = r < \infty$.

Fix an $A\Gamma$-module M that has finite projective dimension over all finite subgroups of Γ.

We first want to prove that M has finite projective dimension over all $H\mathfrak{F}_{\phi,A}$-subgroups of Γ. Let Γ' be an $H\mathfrak{F}_{\phi,A}$-subgroup of Γ, and say, α is the smallest ordinal such that $\Gamma' \in H_\alpha\mathfrak{F}_{\phi,A}$.

We make the following induction hypothesis - for all ordinals $\beta < \alpha$, M has finite projective dimension over all $H_\beta\mathfrak{F}_{\phi,A}$-subgroups of Γ. For the base case $\beta = 0$, note that as M has finite projective dimension over all finite subgroups of Γ, it also has finite projective dimension over all finite subgroups of any $\mathfrak{F}_{\phi,A}$-subgroup of Γ, and thus it follows directly from Definition 2.1 that M has finite projective dimension over all $\mathfrak{F}_{\phi,A}$-subgroups. Now, since there is a finite dimensional contractible CW-complex on which Γ' acts cellularly with stabilisers in $H_{<\alpha}\mathfrak{F}_{\phi,A}$, using Lemma 2.3, we get that, M, as an $A\Gamma'$-module, admits a finite length resolution with modules that are direct sums of modules of the form $Ind_{\Gamma''}^{\Gamma'} Res_{\Gamma''}^{\Gamma'} M$ with $\Gamma'' \in H_{<\alpha}\mathfrak{F}_{\phi,A}$. For any $\Gamma'' \in H_{<\alpha}\mathfrak{F}_{\phi,A}$, $Res_{\Gamma''}^{\Gamma'} M$ has finite projective dimension by our induction hypothesis (note that an $H_{<\alpha}\mathfrak{F}_{\phi,A}$-subgroup of Γ' is also an $H_{<\alpha}\mathfrak{F}_{\phi,A}$-subgroup of Γ) and this projective dimension is at most r by Lemma 3.10. Thus, it follows that M has finite projective dimension over Γ', and again this projective dimension can be at most r by Lemma 3.10.

Like in the proof of Theorem 3.4, we can assume now that Γ is uncountable, because if it is countable, it will be in $H\mathfrak{F}_{\phi,A}$ (as noted in the proof of Theorem 3.4.a., this follows from Lemma 2.5 of [17]), and we are done. Again, as in the proof of Theorem 3.4.a., we make the induction hypothesis that over all subgroups $\Gamma' < \Gamma$ of cardinality strictly smaller than that of Γ, M has finite projective dimension. As Γ is uncountable, it can be expressed as an ascending union of subgroups $\bigcup_{\lambda<\delta} \Gamma_\lambda$, for some ordinal δ, where each Γ_λ has cardinality strictly smaller than that of Γ. Take an r-th syzygy of M over $A\Gamma$, $\Omega^r(M)$. By our induction hypothesis and Lemma 3.10, M has projective dimension at most r over each Γ_λ, and therefore $\Omega^r(M)$ is projective over each Γ_λ. Now, again by Lemma 5.6 of [3], it follows that $\operatorname{proj.dim}_{A\Gamma} \Omega^r(M) \leqslant 1$. It now follows from the definition of $\operatorname{fin.dim}(A\Gamma)$ that $\operatorname{proj.dim}_{A\Gamma} M \leqslant r$. □

We can finish the proof of Theorem 3.1 now.

Proof of Theorem 3.1. The first equality in the statement of Theorem 3.1 follows from Theorem 3.4.a and Proposition 3.3. Putting together the results of Lemma 3.8, Lemma 3.9.b., Lemma 3.9.c. and Lemma 3.11, we get that $\operatorname{fin.dim}(A\Gamma) = \operatorname{silp}(A\Gamma) = \operatorname{spli}(A\Gamma) = k(A\Gamma)$.

To prove the inequality in the statement of Theorem 3.1, we look at two possibilities that can arise based on the finiteness of $\operatorname{silp}(A\Gamma)$. If $\operatorname{silp}(A\Gamma)$ is not finite, then $\operatorname{Gcd}_A(\Gamma)$ is not finite by Lemma 1.9, and therefore we can say that $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A)$ is not finite by Lemma 1.22, and we are done. If $\operatorname{silp}(A\Gamma)$ is finite, then since we already have $\operatorname{spli}(A\Gamma) = \operatorname{silp}(A\Gamma) < \infty$, Theorem 1.8 gives us $\operatorname{Gcd}_A(\Gamma) < \infty$, and the first equality of Theorem 3.1 gives us $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) = \operatorname{Gcd}_A(\Gamma) < \infty$. Now, by Theorem 1.12, we get $\underline{cd}_A(\Gamma) = \operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) = \operatorname{Gcd}_A(\Gamma) < \infty$, and the inequality follows using Lemma 1.9 and Lemma 3.9.a. □

Remark 3.12. In [8], Theorem C states that for $\Gamma \in H\mathfrak{F}$ and for any commutative ring A of finite global dimension, $\operatorname{fin.dim}(A\Gamma) = \operatorname{silp}(A\Gamma) = \operatorname{spli}(\Gamma) = k(A\Gamma)$. The authors proved, without using the assumption $G \in H\mathfrak{F}$, that $\operatorname{fin.dim}(A\Gamma) \leqslant \operatorname{silp}(A\Gamma)$, $\operatorname{silp}(A\Gamma) \leqslant \operatorname{spli}(A\Gamma)$ and $\operatorname{spli}(A\Gamma) \leqslant k(A\Gamma)$. The proofs of these results except $\operatorname{silp}(A\Gamma) \leqslant \operatorname{spli}(A\Gamma)$ that we provided while proving Lemma 3.9 were achieved using their tactics, as we have noted before. However, their proof of $\operatorname{silp}(A\Gamma) \leqslant \operatorname{spli}(A\Gamma)$ had a logical fallacy - they used Result 2.4 of [14] to say that $\operatorname{silp}(A\Gamma)$ must be finite if $\operatorname{spli}(A\Gamma)$ is finite, but that result of [14] requires A to be Noetherian, as noted in Remark 1.10. We resolved this problem with Lemma 3.8 and we broadened the class of groups for which those invariants would concur, going from groups in the hierarchy to groups locally in the hierarchy and changing the base class of groups from the class of finite groups to groups of type Φ over A.

4. Results on Conjecture 2.5 and other applications

We first note the following complete characterisation of groups of type Φ in terms of the finiteness of one cohomological invariant.

Lemma 4.1. *Let A be a commutative ring of finite global dimension. Then, Γ is of type Φ over A iff $k(A\Gamma) < \infty$.*

Proof. It is obvious from the definition of $k(A\Gamma)$ and type Φ groups that if $k(A\Gamma) = n < \infty$, then for any $A\Gamma$-module M that has finite projective dimension over finite subgroups, $\operatorname{proj.dim}_{A\Gamma} M \leqslant n$, so Γ is of type Φ over A.

Now, assume that Γ is of type Φ over A. Then, by definition of type Φ groups, $k(A\Gamma) = \operatorname{fin.dim}(A\Gamma)$ as the class of $A\Gamma$-modules with finite projective dimension is precisely the class of $A\Gamma$-modules with finite projective dimension over finite subgroups. If we assume that $\operatorname{fin.dim}(A\Gamma)$ is not finite, then for any integer n, we have an $A\Gamma$-module M_n such that $n \leqslant \operatorname{proj.dim}_{A\Gamma} M_n < \infty$. Over finite subgroups, M_n has finite projective dimension bounded by the global dimension of A. Therefore, $\bigoplus_{n\in\mathbb{N}} M_n$ does not have finite projective dimension as an $A\Gamma$-module but has finite projective dimension over finite subgroups which cannot be possible as Γ is of type Φ over A. □

Proposition 4.2. *Let $\Gamma \in LH\mathfrak{F}_{\phi,A}$ where A is a commutative ring of finite global dimension. Then, statements (a) to (g) are equivalent in Conjecture 2.5.*

Proof. Using Lemma 4.1, we see that in Conjecture 2.5, (a) and (g) are always equivalent. Now, it follows from Theorem 3.1 that as $\Gamma \in LH\mathfrak{F}_{\phi,A}$, statements (b) to (g) are equivalent. □

Corollary 4.3. *For any commutative ring A of finite global dimension, $LH\mathfrak{F} \cap \mathfrak{F}_{\phi,A}$ is closed under extensions and taking Weyl groups with respect to finite subgroups.*

Proof. Let $1 \to \Gamma_1 \to \Gamma \to \Gamma_2 \to 1$ be a short exact sequence of groups where each Γ_i is of type Φ over A and in $LH\mathfrak{F}$. Noting that $LH\mathfrak{F} \subseteq LH\mathfrak{F}_{\phi,A}$, using Proposition 4.2 we get that $\operatorname{Gcd}_A(\Gamma_i) < \infty$, which implies that $\operatorname{Gcd}_A(\Gamma) < \infty$ by Proposition 2.9 of [13]. $LH\mathfrak{F}$ is extension-closed (Result 2.4 of [18]), so $\Gamma \in LH\mathfrak{F}$ and since $\operatorname{Gcd}_A(\Gamma) < \infty$, we can use Proposition 4.2 to say that Γ is of type Φ over A.

For any finite subgroup $G \leqslant \Gamma$, the Weyl group with respect to G is defined as $W_\Gamma(G) := N_\Gamma(G)/G$. Proposition 2.5 of [13] gives us that $\operatorname{Gcd}_A(W_\Gamma(G)) \leqslant \operatorname{Gcd}_A(\Gamma)$. And $LH\mathfrak{F}$ is Weyl group closed (this follows from the fact that $H\mathfrak{F}$ is Weyl group closed- see Proposition 7.1 of [19]). So, if an $LH\mathfrak{F}$-group is of type Φ over A, from Proposition 4.2, $\operatorname{Gcd}_A(\Gamma) < \infty$, and $W_\Gamma(G)$, for any finite $G \leqslant \Gamma$, which is also in $LH\mathfrak{F}$ has finite Gorenstein cohomological dimension over A and, by Proposition 4.2 again, is of type Φ over A. □

Remark 4.4. We are not in a position to replace $LH\mathfrak{F}$ by $LH\mathfrak{F}_{\phi,A}$ in the statement of Corollary 4.3 because we do not know whether $LH\mathfrak{F}_{\phi,A}$ is closed under extensions or under taking Weyl subgroups, which we do know for $LH\mathfrak{F}$.

Over the ring of integers, Talelli proved in [23], that $(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (f)$ in Conjecture 2.5. Now, when $\Gamma \in H_1\mathfrak{F}$, which is Statement (h) in Conjecture 2.5, it is easy to show that Γ is of type Φ over A for any A of finite global dimension - see Proposition 2.4 of [22], and therefore (h) implies (a) to (g) in Conjecture 2.5 since $H_1\mathfrak{F} \subseteq H\mathfrak{F} \subseteq H\mathfrak{F}_{\phi,A} \subseteq LH\mathfrak{F}_{\phi,A}$. Whether any of the statements (a) to (g) in Conjecture 2.5 implies $\Gamma \in H_1\mathcal{F}$ when $A = \mathbb{Z}$ is an open question.

Note that, in [17], it was shown that for any integer n, there are groups in $H_{n+1}\mathcal{F}$ that are not in $H_n\mathcal{F}$. Can we make a similar claim with $\mathcal{F}_{\phi,A}$ replacing $\mathcal{F}$? The answer is yes and the following result from [17] is the reason why.

Theorem 4.5. *(Theorem 4.1 of [17]) Let $\mathcal{X}$ be a subgroup-closed class of groups containing the class of all finite groups such that there is a countable group in $H_1\mathcal{X}\backslash\mathcal{X}$. Then, $H_\alpha\mathcal{X} \neq H_\beta\mathcal{X}$, for any two distinct countable ordinals α and β.*

With the aid of the following lemma, we can use Theorem 4.5 to obtain that $H_\alpha\mathcal{F}_{\phi,A} \neq H_\beta\mathcal{F}_{\phi,A}$, for any two distinct countable ordinals, where A is a commutative ring of finite global dimension.

Lemma 4.6. *For any commutative ring A of finite global dimension, a free abelian group is of type Φ over A iff it is of finite rank.*

Proof. Let Γ be a free abelian group that is of type Φ over A. Then, since $B(\Gamma, A)$ restricts to a free module over finite subgroups of Γ by Lemma 1.16, $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) < \infty$, and so by Theorem 1.18, $\mathrm{Gcd}_A(\Gamma) < \infty$, i.e. Γ admits complete resolutions (see Theorem 1.8). It has been shown in Corollary 2.10 of [21] that free abelian groups of infinite rank cannot admit complete resolutions over $\mathbb{Z}$, and the exact same proof works for rings of finite global dimension.

Now, let Γ be the free abelian group of rank n, then it has finite cohomological dimension over A, and the group algebra $A\Gamma$ has finite global dimension. It is therefore obvious that Γ is of type Φ over A. □

Using the above two results, we can prove the following distinction of classes in Kropholler's hierarchy with the class of type Φ groups being the base class.

Proposition 4.7. *Let A be a commutative ring of finite global dimension. Then, $H_\alpha\mathcal{F}_{\phi,A} \neq H_\beta\mathcal{F}_{\phi,A}$, for any two distinct countable ordinals α and β.*

Proof. Theorem 7.10 of [11] tells us that $\mathbb{A}_{\aleph_0}$, the free abelian group of rank $\aleph_0$, is in $H_2\mathcal{F}$ and as $H_1\mathcal{F}$-groups are of type Φ over A (as noted before, this follows from Proposition 2.4 of [22]), we have that $\mathbb{A}_{\aleph_0}$ is in $H_1\mathcal{F}_{\phi,A}$ but it is not in $\mathcal{F}_{\phi,A}$ by Lemma 4.6. Note that $\mathcal{F}_{\phi,A}$ is subgroup-closed (this was proved over $A = \mathbb{Z}$ in Proposition 2.3.i of [23], same proof works here). Thus, $\mathcal{F}_{\phi,A}$ satisfies the hypothesis of Theorem 4.5, and we are done. □

Taking the base class to be $H_1\mathcal{F}$ instead of $\mathcal{F}$ is helpful while considering the question as to whether the groups in $H_{n+1}\mathcal{F}\backslash H_n\mathcal{F}$ as constructed in [17] can admit complete resolutions, as we explain in the following remark.

Remark 4.8. A crucial result from [17] that we need here is Proposition 3.7 of [17] which shows that, for any class of groups $\mathcal{X}$ containing all finite groups, if we take any countable $H \in H\mathcal{X}$, then for any integer n, one can choose a group $Q_n \in H\mathcal{X}$ such that Q_n contains a subgroup isomorphic to H and any contractible CW-complex of dimension $\leqslant n$ on which Q_n acts cellularly has a global fixed point. Following the treatment in the proof of Theorem 4.1 of [17](we have quoted this result before - see Theorem 4.5 of this article), we get that if H is a countable group in $H_1\mathcal{X}\backslash\mathcal{X}$, where $\mathcal{X}$ in addition to containing all finite groups is also subgroup closed, then $\ast_n Q_n$, the free product of the Q_n's over all $n \in \mathbb{N}$, with each Q_n as guaranteed by Proposition 3.7 of [17] (note that we can choose each Q_n to be in $H\mathcal{X}$), is in $H_2\mathcal{X}\backslash H_1\mathcal{X}$. Taking $\mathcal{X} = H_1\mathcal{F}$, which contains all finite groups and is subgroup closed as $\mathcal{F}$ is subgroup closed, and H to be the free abelian group of rank $\aleph_0$ (this group is in $H_2\mathcal{F}\backslash H_1\mathcal{F}$, by Theorem 7.10 of [11]), we get, in the notations introduced above, that $\ast_n Q_n \in H_3\mathcal{F}\backslash H_2\mathcal{F}$.

Now let A be a commutative ring of finite global dimension. If $\ast_n Q_n$ admits complete resolutions over A, then so does Q_n, which is not possible as Q_n has a subgroup isomorphic to the free abelian group of rank $\aleph_0$ which does not admit complete resolutions over A. From the same treatment, it follows that if we assume, as an induction hypothesis, that for all $n \leqslant k$, there is a countable group in $H_{k+1}\mathcal{F}\backslash H_k\mathcal{F}$ that does not admit complete resolutions over A, and if we then are to construct a group in $H_{k+2}\mathcal{F}\backslash H_{k+1}\mathcal{F}$ using the method mentioned above (which is the method used in [17]), then that group cannot have complete resolutions over A either, and consequently cannot satisfy any of the $(a) - (g)$ conditions in Conjecture 2.5 in light of Theorem 1.8 and Proposition 4.2. It is noteworthy that there are no known examples of groups in $H_2\mathcal{F}\backslash H_1\mathcal{F}$ that admit complete resolutions over $\mathbb{Z}$.

Remark 4.9. Continuing with the theme of replacing $\mathcal{F}$ with $\mathcal{F}_{\phi,A}$, with A a fixed commutative ring of finite global dimension, it is worth noting that Proposition 2.4 need not be true with $H_1\mathcal{F}_{\phi,A}$-groups because the free abelian group of rank $\aleph_0$ is in $H_1\mathcal{F}_{\phi,A}$ as noted in the proof of Proposition 4.7 above and by Lemma 4.6, it cannot admit complete resolutions over A. Note that this also tells us that the statement of Lemma 1.3 need not be true if we replaced "finite stabilisers" by "type Φ stabilisers".

One can actually show that $H_1\mathcal{F}_{\phi,A} \neq H_2\mathcal{F}_{\phi,A}$ without making any use of Theorem 4.5. We get from Theorem 7.10 of [11] that the free abelian group of rank $\aleph_{\omega_0}$, where ω_0 is the first infinite ordinal, is in $H_3\mathcal{F}$ but not in $H_2\mathcal{F}$ - this straightaway implies that it is in $H_2\mathcal{F}_{\phi,A}$ as $H_1\mathcal{F} \subseteq \mathcal{F}_{\phi,A}$ and it is also easy to see that it cannot be in $H_1\mathcal{F}_{\phi,A}$ because if it were in $H_1\mathcal{F}_{\phi,A}$, then, since all of its $\mathcal{F}_{\phi,A}$-subgroups are free abelian groups of finite rank by Lemma 4.6 and since all such subgroups are in $H_1\mathcal{F}$ (by Theorem 7.10 of [11] again), it would be in $H_2\mathcal{F}$.

We now make a small detour in this section and show in Proposition 4.13 that without using Lemma 3.11 and by making a few changes to a result of Benson, we can prove that $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (g)$ in Conjecture 2.5 with the same extra assumption as before that $\Gamma \in LH\mathfrak{F}_{\phi,A}$ and an extra mild condition on the base ring.

We first state the following result by Benson.

Theorem 4.10. *(Theorem 5.7 of [3]) Let $\Gamma \in LH\mathfrak{F}$ and let R be a commutative ring. Take M to be an RG-module such that over finite subgroups M has projective dimension at most r and* $\operatorname{proj.dim}_{R\Gamma} M \otimes_R B(\Gamma, R) \leqslant r$. *Then,* $\operatorname{proj.dim}_{RG} M \leqslant r$.

Theorem 4.11 below is our variation on Theorem 4.10. It is noteworthy that Theorem 4.11 follows immediately from Theorem 3.1 since the assumption that $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) < \infty$ implies that $\operatorname{Gcd}_A(\Gamma) < \infty$, which in turn implies that $k(A\Gamma) < \infty$, for $\Gamma \in LH\mathfrak{F}_{\phi,A}$. Still, we record Theorem 4.11 separately because the way we align our assumptions with the assumptions of Theorem 4.10 gives us a way of arriving at Lemma 1.9 in a way entirely independent from the approach in [13] (see Remark 4.12).

Theorem 4.11. *Let $\Gamma \in LH\mathfrak{F}_{\phi,A}$ where A is a commutative ring of finite global dimension. Then, Γ is of type Φ over A if* $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) < \infty$.

Proof. Let $\operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) = m < \infty$, let t be the global dimension of A and let M be an $A\Gamma$-module with finite projective dimension over finite subgroups of Γ.

First note that if $\Gamma \in LH\mathfrak{F}$, Theorem 4.11 follows directly from Theorem 4.10. We explain why. Then, $\operatorname{proj.dim}_{AG} M \leqslant t$, for all finite $G \leqslant \Gamma$. Since, $\Omega^t(M)$ is A-projective, we have $\operatorname{proj.dim}_{A\Gamma} \Omega^t(M) \otimes_A B(\Gamma, A) \leqslant m$, and since $B(\Gamma, A)$ is A-free by Lemma 1.16, this gives us $\operatorname{proj.dim}_{A\Gamma} \Omega^t(M \otimes_A B(\Gamma, A)) \leqslant m$, and therefore $\operatorname{proj.dim}_{A\Gamma} M \otimes_A B(\Gamma, A) \leqslant m + t$. So, if we take $r = m + t$ in the hypothesis of Theorem 4.10, we are done.

In [3], Theorem 4.10 is proved by first proving it for $H\mathfrak{F}$-groups (in our language, this means showing that $\operatorname{proj.dim}_{A\Gamma} M < \infty$ if $\Gamma \in H\mathfrak{F}$), and then proving it for $LH\mathfrak{F}$-groups that are not necessarily in $H\mathfrak{F}$, this second part can be replicated with $\mathfrak{F}_{\phi,A}$ replacing $\mathfrak{F}$. Proving Theorem 4.10 for $H\mathfrak{F}$-groups is done by induction on α where $\Gamma \in H_\alpha\mathfrak{F}$. Here again, the inductive step can be replicated with $\mathfrak{F}_{\phi,A}$ replacing $\mathfrak{F}$ (both the steps - the inductive step and the going into $LH\mathfrak{F}_{\phi,A}$ from $H\mathfrak{F}_{\phi,A}$ is similar to the technique shown in the proof of Lemma 3.11; it is the standard technique for such situations). For the base case $\alpha = 0$, note that since M has finite projective dimension over finite subgroups, it has finite projective dimension over $\mathfrak{F}_{\phi,A}$-subgroups as well. □

Remark 4.12. The first paragraph in the proof of Theorem 4.11 gives us that if $\Gamma \in LH\mathfrak{F}_{\phi,A}$ with A of finite global dimension t, then $k(A\Gamma) \leqslant \operatorname{proj.dim}_{A\Gamma} B(\Gamma, A) + t$. Using Lemma 3.8, Lemma 3.9.c., and Proposition 3.3 along with Theorem 3.4 (we are referring to these results separately instead of just referring to Theorem 3.1 because we want to show that we are not using Lemma 1.9 here), this gives us that for $\Gamma \in LH\mathfrak{F}_{\phi,A}$, $\mathrm{silp}(A\Gamma), \mathrm{spli}(A\Gamma) \leqslant \mathrm{Gcd}_A(\Gamma) + t$. We say this "almost" completely gives a new proof of Lemma 1.9 because we believe $\mathrm{Gcd}_A(\Gamma) < \infty$ iff $\Gamma \in \mathfrak{F}_{\phi,A}$ (see (a) and (e) in Conjecture 2.5).

We can now prove the following promised result on Conjecture 2.5. Before that, we note that a commutative ring is called $\aleph_0$-Noetherian (see Section 3 of [12] for this terminology) iff all of its ideals are countably generated (as opposed to finitely generated). A polynomial ring in infinite but countably many variables over a countable field is an example of an $\aleph_0$-Noetherian ring that is not Noetherian.

Proposition 4.13. *Let $\Gamma \in LH\mathfrak{F}_{\phi,A}$ where A is a commutative $\aleph_0$-Noetherian ring with finite global dimension. Then, without using Lemma 3.11, one can show that $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (g)$ in Conjecture 2.5.*

Proof. We have $(g) \Rightarrow (c)$ by Lemma 3.9.c., $(c) \Rightarrow (b)$ by Lemma 3.8, $(b) \Rightarrow (e)$ by Proposition 4.3 of [12] which gives us that $\mathrm{spli}(A\Gamma) \leqslant \mathrm{silp}(A\Gamma)$ with A commutative $\aleph_0$-Noetherian (this is the only instance where we are using the fact that A is $\aleph_0$-Noetherian) and Theorem 1.8, $(e) \Rightarrow (d)$ by Theorem 3.4.a. and Proposition 3.3, $(d) \Rightarrow (a)$ by Theorem 4.11, and $(a) \Leftrightarrow (g)$ by Lemma 4.1. □

Remark 4.14. In the proof of Proposition 4.13 above, to show that $\mathrm{spli}(A\Gamma) \leqslant \mathrm{silp}(A\Gamma)$, we are making no use of the fact that $\Gamma \in LH\mathfrak{F}_{\phi,A}$, instead we are putting an extra condition on A. It is an open question as to whether we can get rid of the $\aleph_0$-Noetherian condition on A and just use a property of Γ to get the same result.

Also, it is noteworthy that although it should follow from Theorem 4.4 of [12] that $\mathrm{silp}(A\Gamma) = \mathrm{spli}(A\Gamma)$ for any group Γ and any commutative $\aleph_0$-Noetherian ring A of finite global dimension, the logic leading up to this result in [12] is not quite correct. That is because, in [12], it is first noted correctly in Proposition 4.3 of [12], that $\mathrm{spli}(A\Gamma) \leqslant \mathrm{silp}(A\Gamma)$ for any group Γ and any commutative $\aleph_0$-Noetherian ring A, and then [12] says that Result 2.4 of [14] (= Remark 1.10 in this article) implies that the converse inequality holds with the extra condition that A has finite self-injective dimension. This is not correct as, in Result 2.4 of [14], A needs to be Noetherian.

We end this section and the article with the remark on a conjecture by Dembegioti and Talelli.

Remark 4.15. It has been conjectured in [9] that for any group Γ, $\mathrm{spli}(\mathbb{Z}\Gamma) = \underline{cd}_{\mathbb{Z}}(\Gamma) + 1$. First, note that by Remark 1.13 (or just directly by Theorem 2.5 of [2]), $\underline{cd}_{\mathbb{Z}}(\Gamma) = \mathrm{Gcd}_{\mathbb{Z}}(\Gamma)$. Now let $\Gamma \in LH\mathfrak{F}_{\phi,\mathbb{Z}}$. Taking $A = \mathbb{Z}$ in Theorem 3.1, it follows that $\mathrm{spli}(\mathbb{Z}\Gamma)$ and $\underline{cd}_{\mathbb{Z}}(\Gamma)$ are finite only when $\mathrm{proj.dim}_{\mathbb{Z}\Gamma} B(\Gamma, \mathbb{Z})$ is finite, and when that is the case, Theorem 3.1 tells us that the conjecture looks like $\mathrm{fin.dim}(\mathbb{Z}\Gamma) = \mathrm{proj.dim}_{\mathbb{Z}\Gamma} B(\Gamma, \mathbb{Z}) + 1$. Again, courtesy of Theorem 3.1, noting the fact the global dimension of $\mathbb{Z}$ is 1, we see that the Dembegioti-Talelli conjecture will be settled for Γ if we can prove that $\operatorname{fin.dim}(\mathbb{Z}\Gamma) \neq \operatorname{proj.dim}_{\mathbb{Z}\Gamma} B(\Gamma,\mathbb{Z})$, i.e. we need to find a $\mathbb{Z}\Gamma$-module whose projective dimension is strictly bigger than that of $B(\Gamma,\mathbb{Z})$ but finite. First, note that we can assume that $\operatorname{proj.dim}_{\mathbb{Z}\Gamma} B(\Gamma,\mathbb{Z}) > 1$ - this is because in Corollary 4.7 of [12], Emmanouil settled the conjecture for the cases where the generalized cohomological dimension is bounded by 1, and as we have seen, $\operatorname{proj.dim}_{\mathbb{Z}\Gamma} B(\Gamma,\mathbb{Z}) < \infty$ implies $\operatorname{proj.dim}_{\mathbb{Z}\Gamma} B(\Gamma,\mathbb{Z}) = \operatorname{Gcd}_{\mathbb{Z}}(\Gamma) = \underline{cd}_{\mathbb{Z}}(\Gamma)$ by Theorem 1.12 and Theorem 1.18. A candidate for a $\mathbb{Z}\Gamma$-module with finite but bigger projective dimension than that of $B(\Gamma,\mathbb{Z})$ can be $B(\Gamma,\mathbb{Z}/p\mathbb{Z})$ for any prime p because we have a short exact sequence $0 \to B(\Gamma,\mathbb{Z}) \xrightarrow{p} B(\Gamma,\mathbb{Z}) \to B(\Gamma,\mathbb{Z}/p\mathbb{Z}) \to 0$ where the first map is multiplication by p.

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