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# On homogeneous operators in the Cowen-Douglas class over polydisc

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ABSTRACT. We classify all tuples of operators in the Cowen-Douglas class of rank 1 and 2 homogeneous with respect to Möb$^n$ using Wilkins' classification of homogeneous hermitian holomorphic vector bundles over bounded symmetric domains. It is also observed that these homogeneous tuples can be realised as the adjoint of the tuples of multiplication operators on the quotient Hilbert modules obtained from certain submodules of the weighted Bergman modules on the open unit polydisc.

CONTENTS



## 1. Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be open unit disc and Möb denote the biholomorphic automorphism group of $\mathbb{D}$. A bounded linear operator on a complex separable Hilbert space $\mathcal{H}$ is said to be homogeneous with respect to Möb if the spectrum $\sigma(T)$ of $T$ is contained in $\overline{\mathbb{D}}$ and, for every $g \in$ Möb, $g(T)$ is unitarily equivalent to $T$. Indeed, since $g \in$ Möb is a rational function $e^{i\theta}\frac{z-a}{1-\bar{a}z}$, for $a \in \mathbb{D}$ and $\theta \in [0, 2\pi)$, with pole outside the closed disc $\overline{\mathbb{D}}$, it follows that

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$g(T) := e^{i\theta}(T - aI)(I - \overline{a}T)^{-1}$ ($I$ is the identity operator on $\mathcal{H}$) is well defined whenever the spectrum $\sigma(T)$ of $T$ is contained in $\overline{\mathbb{D}}$.

An example of homogeneous operator is the multiplication operator $M_z$ by the co-ordinate function on the weighted Bergman space $\mathcal{H}^{(\alpha)}$ over open unit disc $\mathbb{D}$ which is a reproducing kernel Hilbert space of holomorphic functions on $\mathbb{D}$ with the reproducing kernel $K^{(\alpha)}(z,w) = (1 - z\overline{w})^{-\alpha}$ for $\alpha > 0$. It is well known that $M_z^*$ is in the Cowen-Douglas class $\mathrm{B}_1(\mathbb{D})$ over $\mathbb{D}$ of rank 1.

In general, following the ideas of [1], it is easy to establish a one to one correspondence between the unitary equivalence class of commuting $n$ - tuples $\boldsymbol{T}$ in $\mathrm{B}_r(\mathbb{D}^n)$ (for the definition of the Cowen-Douglas class, we refer the reader to Section 3) and equivalence class of the associated hermitian holomorphic vector bundles $E_{\boldsymbol{T}}$ over $\mathbb{D}^n$ determined by $\boldsymbol{T}$. This correspondence extends to homogeneous operators and homogeneous vector bundles as well. Moreover, hermitian holomorphic line bundles are determined up to isomorphism by the curvature forms of the hermitian connections on the bundles. This observation was used by Misra in [7] to classify homogeneous operators in $\mathrm{B}_1(\mathbb{D})$. An operator $T \in \mathrm{B}_1(\mathbb{D})$ is homogeneous with respect to Möb if and only if its curvature form is a positive scalar multiple of the Kähler form $-(1-|z|^2)^{-2} dz \wedge d\overline{z}$ of the Poincaré metric on $\mathbb{D}$. Consequently, it turns out that $M_z^*$ on $\mathcal{H}^{(\alpha)}$ with $\alpha > 0$ are the only homogeneous operators in $\mathrm{B}_1(\mathbb{D})$.

Later, Wilkins extended this result to $\mathrm{B}_2(\mathbb{D})$ in [13] where he first obtained an algebraic classification of all hermitian holomorphic vector bundles over any bounded symmetric domain $\Omega$ homogeneous with respect to the identity component of the group of bi-holomorphic automorphisms $\Omega$, assuming that the group of bi-holomorphic automorphisms of $\Omega$ is semi-simple (cf. Theorem 2.5). This result was then used to determine all irreducible operators $T$ (that is, there is no reducing subspace of $T$ in $\mathcal{H}$) in $\mathrm{B}_2(\mathbb{D})$ homogeneous with respect to the group Möb. The classification, in general, of irreducible homogeneous operators in the Cowen-Douglas class over $\mathbb{D}$ has been completed (cf. [6]) recently by first obtaining an explicit description of all the homogeneous hermitian holomorphic vector bundles over $\mathbb{D}$ and then deciding which ones of these give rise to bounded operators in the Cowen-Douglas class.

The notion of a homogeneous operators has a natural generalization to commuting tuples of operators. A commuting $n$ - tuple of operators $(T_1, T_2, \dots, T_n)$ is said to be homogeneous with respect to a subgroup $G$ of the group of bi-holomorphic automorphisms $\mathrm{Aut}(\mathbb{D}^n)$ of $\mathbb{D}^n$, if the Taylor joint spectrum of $(T_1, T_2, \dots, T_n)$ lies in $\overline{\mathbb{D}}^n$ and $g(T_1, T_2, \dots, T_n)$ (which can be defined by the holomorphic functional calculus) is unitarily equivalent with $(T_1, T_2, \dots, T_n)$ for all $g \in G$. In this generality, the focus has been on the study of commuting tuples of homogeneous operators in the Cowen-Douglas class.

In the following section, we describe all hermitian holomorphic vector bundles over $\mathbb{D}^n$ of rank at most 2 homogeneous with respect to $\text{Möb}^n$ in terms of an $n$ - tuple of pairs of skew-symmetric matrices satisfying certain algebraic identity (cf. Theorem 2.7). Thus, Theorem 2.7 extends Theorem 1.1 in [13] to the case of homogeneous bundles of rank at most 2 over unit poly-disc. Unfortunately, this is not the case when the rank of the bundle is strictly greater than 2 as explained in Remark 3.2. In general, it turns out that the homogeneity forces the components $\mathcal{K}_{ij}$ with $i \neq j$ of the curvature $\mathcal{K} = \sum_{i,j=1}^{n} \mathcal{K}_{ij} dz_i \wedge d\bar{z}_j$ of these bundles at $(0, \ldots, 0) \in \mathbb{D}^n$ to be a nilpotent matrix as pointed out in Lemma 2.1. The main crux of the proof of Theorem 2.7 is that these nilpotent matrices turn out to be the zero matrix for rank 2 homogeneous bundles which also makes the case of rank 2 homogeneous vector bundles over $\mathbb{D}^n$ different from other higher rank homogeneous bundles over $\mathbb{D}^n$. We show this in Theorem 2.6 with the help of the identities in Theorem 2.5.

In Section 3, we describe all irreducible tuples of operators (that is, there is no common reducing subspaces) in $\mathrm{B}_r(\mathbb{D}^n)$, $r = 1, 2$, homogeneous with respect to the group $\text{Möb}^n$ with the help of Theorem 2.7. Also, it is shown that all homogeneous irreducible tuples of operators in $\mathrm{B}_2(\mathbb{D}^n)$ can be realized as the adjoint of the multiplication operators on the quotient space obtained from certain submodules of the Hilbert modules $\mathcal{H}^{(\lambda)} := \mathcal{H}^{(\lambda_1)} \otimes \cdots \otimes \mathcal{H}^{(\lambda_{n+1})}$ over the holomorphic function algebra $\mathcal{A}(\mathbb{D}^{n+1})$.

We point out that these results were obtained differently in [3] for $n$ - tuples of operators in the Cowen-Douglas class over $\mathbb{D}^n$ of rank at most 3 by classifying all irreducible representations of Lie algebra of the Lie subgroup $(K^{\mathbb{C}}P^-)^n$ of the Lie group $SU(1,1)$. The novelty here is in obtaining a characterization of hermitian holomorphic homogeneous vector bundles of rank at most 2 over $\mathbb{D}^n$ in terms of an algebraic identity as well as in identifying these homogeneous tuples as compression of the tuple of multiplication operators by co-ordinate functions on $\mathcal{H}^{(\lambda)}$ onto certain quotient space. It would be nice if we could extend these techniques to classify homogeneous operators in $\mathrm{B}_r(\mathbb{D}^n)$ that would lead to the classification of all indecomposable finite dimensional representations of $(K^{\mathbb{C}}P^-)^n$ as well. In case it is not, it would still be interesting to investigate the subclass of homogeneous operators in $\mathrm{B}_r(\mathbb{D}^n)$ obtained from Wilkins' technique.

In Section 4, another application of Theorem 2.5 is provided. In Theorem 4.1, we show that the curvature forms of the line bundles over the classical bounded symmetric domains $D$ homogeneous with respect to the identity component of the group of all bi-holomorphic automorphisms of $D$ turn out to be a scalar multiple of the $(1,1)$ form $\sum_{i,j=1}^{n} dz_i \wedge d\bar{z}_j$. Note that this result was previously obtained in [8, Theorem 2.4] for the homogeneous hermitian holomorphic line bundles over classical bounded symmetric domains associated to tuples of operators in $\mathrm{B}_1(D)$ homogeneous with respect to the maximal compact subgroup of the group of all bi-holomorphic automorphisms of $D$. However, we reproduce this result by a completely different method; namely, using Theorem 2.5, for any hermitian holomorphic line bundles over $D$ homogeneous with respect to the identity component of the group of all bi-holomorphic automorphisms of $D$.

## 2. Hermitian holomorphic homogeneous vector bundles of rank at most 2

In this section, we study hermitian holomorphic vector bundles over the unit poly-disc $\mathbb{D}^n$ homogeneous with respect to the identity component $\text{Möb}^n$ of the group of all bi-holomorphic automorphisms of $\mathbb{D}^n$ as well as provide a complete description of these vector bundles of rank at most 2 following the techniques developed in [13]. In particular, one of the main results (Theorem 2.7) extends Theorem 1.1 in [13] to the case of open unit poly-disc in $\mathbb{C}^n$.

We begin by recalling the definition of homogeneous vector bundle. An automorphism of a vector bundle $\pi : E \to \Omega$ is a diffeomorphism $\hat{g} : E \to E$ such that $\pi\circ\hat{g} = g\circ\pi$ for some automorphism $g : \Omega \to \Omega$. The bundle map $\hat{g}$ is a lift of $g$. We say that a hermitian holomorphic vector bundle $E$ is homogeneous with respect to a subgroup $G$ of the bi-holomorphic automorphism group of $\Omega$ if the action of $G$ on $\Omega$ is transitive and the lift of this action is isometric.

Let $E \to \mathbb{D}^n$ be a hermitian holomorphic vector bundle of rank $r$. Since $\mathbb{D}^n$ is a contractible domain, Grauert's theorem (cf. [**?**]) yields a global holomorphic frame $\{\gamma_1, \dots, \gamma_r\}$ for $E$ over $\mathbb{D}^n$. Consequently, the Hermitian metric $H(\boldsymbol{z})$ on the fibre $E_{\boldsymbol{z}}$ over $\boldsymbol{z} \in \mathbb{D}^n$ turns out to be the Grammian $((\langle\gamma_i(\boldsymbol{z}), \gamma_j(\boldsymbol{z})\rangle))_{i,j=1}^r$ of this frame with respect to the given hermitian structure on $E$. Moreover, if $E \to \mathbb{D}^n$ is homogeneous with respect to the group $G = \text{Möb}^n$, it follows that the Hermitian metric $H(\boldsymbol{z})$ for $\boldsymbol{z} \in \mathbb{D}^n$ satisfies the identity

$$H(\boldsymbol{z}) \;=\; J(g, \boldsymbol{z})H(g(\boldsymbol{z}))J(g, \boldsymbol{z})^*,\ \boldsymbol{z}, \in \mathbb{D}^n,\ g \in \text{Möb}^n$$

for some continuous function $J : \text{Möb}^n \times \mathbb{D}^n \to GL_r(\mathbb{C})$ which is holomorphic in the second variable. The following lemma reveals some properties of the curvature of hermitian holomorphic vector bundles over $\mathbb{D}^n$ homogeneous with respect to $\text{Möb}^n$. The following lemma has been proved in [3] (cf. [3, Lemma 5.11]) when the hermitian structure of $E \to \mathbb{D}^n$ is induced by a reproducing kernel. However, since the same proof also works for our case, we omit the proof.

**Lemma 2.1.** *Let $E \to \mathbb{D}^n$ be a hermitian holomorphic vector bundle of rank $r$ which is homogeneous with respect to the group $\text{Möb}^n$. Then, for $1 \le j \ne k \le n$, $\bar{\partial}_j(H^{-1}\partial_k H)(0,0)$ is a nilpotent matrix where $H(\boldsymbol{z}) = ((\langle\gamma_i(\boldsymbol{z}), \gamma_j(\boldsymbol{z})\rangle))_{i,j=1}^r$ is the hermitian metric on $E_{\boldsymbol{z}}$, $\boldsymbol{z} \in \mathbb{D}^n$ with respect to a global holomorphic frame $\{\gamma_1, \dots, \gamma_r\}$ for $E \to \mathbb{D}^n$ and $\partial_k = \frac{\partial}{\partial z_k}$, $\bar{\partial}_j = \frac{\partial}{\partial \bar{z}_j}$.*

In fact, for an irreducible hermitian holomorphic vector bundle of rank at most 2 over $\mathbb{D}^n$, the matrices $\bar{\partial}_j(H^{-1}\partial_k H)(0,0)$ for $1 \le j \ne k \le n$ turn out to be a zero matrix which is one of the main results in this section. In order to accomplish this, we need the following lemmas.

**Lemma 2.2.** *Let $\{(P_j, Q_j) : j = 1, \dots, n\}$ be the collection of pairs of $2\times 2$ complex matrices satisfying the following identities*

$$[P_j, P_k] = 0,\ [Q_j, Q_k] = 0,\ \textit{and}\ [P_j, Q_k] = -\delta_{jk}Q_j,\ \textit{for}\ j, k \in \{1, \dots, n\}.$$

*Further, we assume that $P_j$'s are hermitian matrices. Then either all $Q_j$ are zero matrices, or, there exists $j_0 \in \{1, \dots, n\}$ such that*

$$P_{j_0} = \begin{pmatrix} (\lambda + \frac{1}{2}) & 0 \\ 0 & (\lambda - \frac{1}{2}) \end{pmatrix} \text{ and } Q_{j_0} = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}$$

*for some $\lambda, \mu \in \mathbb{R}$ with $\mu > 0$ and, for all $j \in \{1, \dots, n\}$ with $j \neq j_0$,*

$$P_j = \lambda_j I_{2\times 2}, \text{ and } Q_j = 0,$$

*for some constant $\lambda_j \in \mathbb{R}$.*

**Proof.** We begin by pointing out that since the matrices $\{P_j\}_{j=1}^n$ are hermitian and any two of them commute, there exists $\{v_1, v_2\} \subset \mathbb{C}^2$ such that

$$P_j v_i = \lambda_i^j v_i, \; i = 1, 2, \; j = 1, \dots, n, \tag{2.1}$$

where $\lambda_1^j, \lambda_2^j$ are eigenvalues of $P_j$, $j = 1, \dots, n$. Consequently, for each $j = 1, \dots, n$, the equation $[P_j, Q_j] = -Q_j$ yields that

$$P_j Q_j v_i = (\lambda_i^j - 1) Q_j v_i, \text{ for } i = 1, 2 \text{ and } j \in \{1, \dots, n\}. \tag{2.2}$$

Therefore, either all $Q_j$'s are zero matrices, or, there exists $j_0 \in \{1, \dots, n\}$ such that $Q_{j_0} v_i$ is an eigenvector of $P_{j_0}$ for some $i = 1, 2$. We note that $Q_{j_0} v_i$ cannot be $\alpha v_i$ for some non-zero $\alpha$ as, in that case, we would have $\lambda_i^{j_0} = \lambda_i^{j_0} - 1$. Furthermore, we must have that there exists $i \in \{1, 2\}$ such that $Q_{j_0} v_i = 0$,. So without loss of generality, we assume that $Q_{j_0} v_1 = \alpha v_2$, for some non-zero constant $\alpha \in \mathbb{C}$ and $Q_{j_0} v_2 = 0$. Also, it verifies that if $\lambda_{j_0}$ is an eigenvalue for $P_{j_0}$ then $\lambda_{j_0} - 1$ is an eigenvalue of $P_{j_0}$. Thus, the difference of two eigenvalues of $P_{j_0}$ is $\pm 1$ which concludes the first part of the statement.

Now, for $j \neq j_0$, we have

$$[P_j, Q_{j_0}] v_1 = P_j Q_{j_0} v_1 = \lambda_1^{j_0} Q_{j_0} v_1 = 0$$

which implies that $\lambda_1^j = \lambda_2^j$ as $Q_{j_0} v_1 = \alpha v_2$. Therefore, letting $\lambda_j = \lambda_1^j = \lambda_2^j$, for $j \in \{1, \dots, n\}$ with $j \neq j_0$, we are done with the proof. □

**Lemma 2.3.** *Let $P$ and $Q$ be two skew-hermitian $2 \times 2$ non-zero matrices satisfying the identity $[P, [P, Q]] = -Q$. Then there exists a unitary transformation $U : \mathbb{C}^2 \to \mathbb{C}^2$ such that*

$$UPU^* = \begin{pmatrix} i(\lambda + \frac{1}{2}) & 0 \\ 0 & i(\lambda - \frac{1}{2}) \end{pmatrix} \text{ and } UQU^* = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} \tag{2.3}$$

*for some $\lambda, \mu \in \mathbb{R}$ with $\mu > 0$.*

**Proof.** Let $P$ and $Q$ be two matrices as given satisfying $[P, [P, Q]] = -Q$. Let us consider the matrices $P' = iP$ and $Q' = Q - i[P, Q]$. Then we note that $P'$ is hermitian and $[P', Q'] = -Q'$. Therefore, it follows from Lemma 2.2 that

$$P' = \begin{pmatrix} (\lambda + \frac{1}{2}) & 0 \\ 0 & (\lambda - \frac{1}{2}) \end{pmatrix} \text{ and } Q' = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$$

for some non-zero $\alpha \in \mathbb{C}$. Then the proof follows by observing that $Q = \frac{1}{2}(Q' - Q'^*)$ and taking $\mu = |\alpha|$. □

**Corollary 2.4.** *Let $\{(P_j, Q_j) : j = 1, \ldots, n\}$ be the collection of pairs of non-zero $2 \times 2$ skew-hermitian complex matrices satisfying the following identities*

$$[P_j, P_k] = 0,\ [Q_j, Q_k] = 0,\ \text{and } [P_j, [P_j, Q_k]] = -\delta_{jk} Q_j,\ \text{for } j, k \in \{1, \ldots, n\}.$$

*Then either all $Q_j$ are zero matrices, or, there exists $j_0 \in \{1, \ldots, n\}$ and a unitary transformation $U : \mathbb{C}^2 \to \mathbb{C}^2$ such that*

$$UP_{j_0}U^* = \begin{pmatrix} i(\lambda + \frac{1}{2}) & 0 \\ 0 & i(\lambda - \frac{1}{2}) \end{pmatrix} \text{ and } UQ_{j_0}U^* = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}$$

*for some $\lambda, \mu \in \mathbb{R}$ with $\mu > 0$ and, for all $j \in \{1, \ldots, n\}$ with $j \neq j_0$,*

$$P_j = -i\lambda_j I_{2\times 2},\ \text{and } Q_j = 0,$$

*for some constant $\lambda_j \in \mathbb{R}$.*

**Proof.** Define $P'_j = iP_j$ and $Q'_j = Q_j - i[P_j, Q_j]$ for $j = 1, \ldots, n$. Then note that pair of matrices $(P'_j, Q'_j)$, for $j = 1, \ldots, n$, satisfy the hypothesis of Lemma 2.2. Therefore, the proof follows from Lemma 2.2 and Lemma 2.3. □

We now prove that the matrices $\bar{\partial}_j(H^{-1}\partial_k H)(0,0)$ in Lemma 2.1 are zero matrix for any $1 \leq j \neq k \leq n$ whenever $r = 2$, with the help of Theorem 3.4 in [13] along with the lemmas above. In this regard, we need some elementary properties of a bounded symmetric domain.

Following [13, Section 3] a *bounded symmetric domain* $\Omega$ is a bounded open subset of some $\mathbb{C}^n$ with the property that each point $p \in \Omega$ is an isolated fixed point of some bi-holomorphic mapping $s_p : \Omega \to \Omega$ such that $s_p^2 = \mathrm{I}d$. Cartan's Uniqueness Theorem ensures that this symmetry is uniquely determined [12, Proposition 2.3]. Any bounded symmetric domain $\Omega$ admits a Kähler metric invariant under the symmetries of $\Omega$. One such metric is the Bergman metric on $\Omega$ [5, Chapter VIII, Section 3] by which we mean the following. Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain with the Bergman kernel $K : \Omega \times \Omega \to \mathbb{C}$. Then the Bergman metric on $\Omega$ is a Kähler metric with Kähler potential

$$\rho(z) = -\log K(z, z) \text{ for } z \in \Omega.$$

We can, therefore, consider any bounded symmetric domain as a hermitian symmetric space. The group generated by the symmetries of $\Omega$ is a semi-simple Lie group acting smoothly on $\Omega$ and is the identity component of the group of bi-holomorphic automorphisms of $\Omega$. We now state one of the main theorems in [13] which will be used in the rest of this article. From now on, we only consider hermitian holomorphic vector bundles $E \to \Omega$ over a bounded symmetric domain $\Omega$ homogeneous with respect to the group generated by all symmetries of $\Omega$. In other words, the vector bundles $E \to \Omega$ are homogeneous under the action of the identity component of the group of bi-holomorphic automorphisms of $\Omega$.

**Theorem 2.5.** [13, Theorem 3.4] *Let $\Omega$ be a complex bounded symmteric domain and let $p$ be a point in $\Omega$. Then there is a one to one correspondence between isomorphism classes of homogeneous hermitian holomorphic vector bundles of rank $r$ over $\Omega$ and unitary equivalence classes of pairs $(\mathcal{K}_0, \tau_0)$ satisfying following equations*

$$\begin{aligned}
[\mathcal{K}_0(V,W), \tau_0(Y)] &= [[\tau_0(V), \tau_0(W)], \tau_0(Y)] + \tau_0(R(V,W)Y),\\
[\mathcal{K}_0(V,W), \mathcal{K}_0(Y,Z)] &= [[\tau_0(V), \tau_0(W)], \mathcal{K}_0(Y,Z)] + \mathcal{K}_0(R(V,W)Y,Z)\\
&\quad + \mathcal{K}_0(Y, R(V,W)Z),\\
\mathcal{K}_0(JY, JZ) &= \mathcal{K}_0(Y,Z),
\end{aligned}$$

*for all $V, W, Y, Z \in T_p\Omega$, where $\tau_0 : T_p\Omega \to \mathfrak{u}(r)$ is a linear transformation and $\mathcal{K}_0 : T_p\Omega \times T_p\Omega \to \mathfrak{u}(r)$ is a skew-symmetric bilinear form with values in the Lie algebra $\mathfrak{u}(r)$ of skew-hermitian $r \times r$ matrices, and $J$ is the complex structure on $\Omega$. Moreover, if $(\mathcal{K}_0, \tau_0)$ is the pair representing some homogeneous holomorphic vector bundle $E$, then $\mathcal{K}_0$ represents the curvature of the Chern connection on $E$ at the point $p$.*

We make use of this theorem in our case where the bounded symmetric domain $\Omega$ is the polydisc. The identity component of the group of bi-holomorphic automorphisms of $\mathbb{D}^n$ is Möb$^n$ which is a semi-simple Lie group. The Bergman metric on $\mathbb{D}^n$ is, by definition, given by the Gramian matrix $((h_{jk}(\boldsymbol{p})))_{j,k=1}^n$, for $\boldsymbol{p} \in \mathbb{D}^n$, where

$$h_{jk}(\boldsymbol{p}) = -\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K^{(2)}(\boldsymbol{z}, \boldsymbol{z})|_{\boldsymbol{p}} \tag{2.4}$$

where $K^{(2)}(\boldsymbol{z}, \boldsymbol{w}) = (1 - z_1\overline{w}_1)^{-2} \cdots (1 - z_n\overline{w}_n)^{-2}$. Note that $h_{jj}(\boldsymbol{p}) = 2(1 - |z_j|^2)^{-2}$ and $h_{jk}(\boldsymbol{p}) = 0$, for $j \neq k$ verifying that the Bergman metric on $\mathbb{D}^n$ is the product metric obtained from the Bergman metric on $\mathbb{D}$.

**Theorem 2.6.** *Let $E \to \mathbb{D}^n$ be a hermitian holomorphic vector bundle of rank 2 which is homogeneous with respect to the group Möb$^n$. Then, for $1 \leq j \neq k \leq n$,*

$$\bar{\partial}_k(H^{-1}\partial_j H)(0,0) = 0,$$

*where $\partial_j = \frac{\partial}{\partial z_j}$ and $\bar{\partial}_k = \frac{\partial}{\partial \bar{z}_k}$. where $H(\boldsymbol{z}) = ((\langle \gamma_i(\boldsymbol{z}), \gamma_j(\boldsymbol{z})\rangle))_{i,j=1}^r$ is the hermitian metric on $E_{\boldsymbol{z}}$, $\boldsymbol{z} \in \mathbb{D}^n$ with respect to a global holomorphic frame $\{\gamma_1, \dots, \gamma_r\}$ for $E \to \mathbb{D}^n$ and $\partial_k = \frac{\partial}{\partial z_k}$, $\bar{\partial}_j = \frac{\partial}{\partial \bar{z}_j}$.*

**Proof.** Let $\mathbb{D}^n$ be equipped with the Bergman metric $\rho_n$. It follows from the discussion above that the hermitian manifold $(\mathbb{D}^n, \rho_n)$ is the manifold $(\mathbb{D}, \rho_1)^n := (\mathbb{D}, \rho_1) \times \cdots \times (\mathbb{D}, \rho_1)$ with the product hermitian structure. Note that the curvature tensors of this product Riemannian metric turn out to be

$$R(\partial_j, \bar{\partial}_j)\partial_j = 4\bar{\partial}_j, \text{ and } R(\partial_j, \bar{\partial}_j)\bar{\partial}_j = -4\partial_j, , \ j = 1, \dots, n.$$

Define for each $j = 1, \dots, n$,

$$F_j = \frac{1}{4}\mathcal{K}_0(\partial_j, \bar{\partial}_j) - \frac{1}{4}[\tau_0(\partial_j), \tau_0(\bar{\partial}_j)], \text{ and } A_j = \frac{1}{2}\tau_0(\partial_j)$$

where $\mathcal{K}_0$ and $\tau_0$ as in Theorem 2.5 satisfying identities therein. It can be seen, for $j = 1, \dots, n$, that

$$[F_j, A_j] = \frac{1}{2}\tau_0(\bar{\partial}_j) \text{ and } [F_j, [F_j, A_j]] = -A_j.$$

We note that all $A_j$'s cannot be zero simultaneously, because it would imply that the bundle $E$ were reducible contradicting the hypothesis. It therefore follows from Corollary 2.4 that there exists $j_0 \in \{1, \dots, n\}$ such that, for all $j \in \{1, \dots, n\}$ with $j \neq j_0$,

$$F_j = \theta_j I_{2\times 2} \text{ and } A_j = 0, \ \theta_j = i\lambda_j, \ \lambda_j > 0, \text{ and } j = 1, \dots, n.$$

Thus, we are left with a pair of skew-hermitian $2\times 2$ matrices $(F_{j_0}, A_{j_0})$ and $n-1$ scalars $\{\theta_j \ : \ j \in \{1, \dots, n\} \setminus \{j_0\}\}$. Since $E$ is an irreducible vector bundle, $A_{j_0}$ cannot be zero. Following Corollary 2.4, we can choose an orthonormal basis for $\mathbb{C}^2$ with respect to which the matrices $F_{j_0}$ and $A_{j_0}$ take the form

$$F_{j_0} = \begin{pmatrix} i(\lambda + \frac{1}{2}) & 0 \\ 0 & i(\lambda - \frac{1}{2}) \end{pmatrix} \text{ and } A_{j_0} = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}$$

with $\mu > 0$ and hence the real numbers $\lambda, \mu$ determines the pair $(F_{j_0}, A_{j_0})$. Recalling the definition of $F_{j_0}$ and $A_{j_0}$, it can be seen that

$$\mathcal{K}_0(\partial_{j_0}, \bar{\partial}_{j_0}) = 2i(F_{j_0} + [A_{j_0}, [F_{j_0}, A_{j_0}]]).$$

Therefore, $k_1^{j_0}$ and $k_2^{j_0}$ are equal to $-2\lambda \pm (1+4\mu^2)$. Also, note that $|k_1^{j_0} - k_2^{j_0}| > 2$.

Now putting $V = \frac{\partial}{\partial z_j}, W = \frac{\partial}{\partial \bar{z}_k}, Y = \frac{\partial}{\partial z_{j_0}}$ and $Z = \frac{\partial}{\partial \bar{z}_{j_0}}$ in the second equation in Theorem 2.5, we have, for $j \neq k$, that

$$[\mathcal{K}_0(\partial_j, \bar{\partial}_k), \mathcal{K}_0(\partial_{j_0}, \bar{\partial}_{j_0})] \ = \ 0$$

since $A_j = \tau_0(\partial_j) = 0$, for all $j \neq j_0$, and all of $R(\partial_j, \partial_k)$, $R(\bar{\partial}_j, \bar{\partial}_k)$ as well as $R(\partial_j, \bar{\partial}_k)$ are the zero linear operators whenever $j \neq k$. Thus, it completes the proof since Lemma 2.1 yields that $\mathcal{K}_0(\partial_j, \bar{\partial}_k)$ is a nilpotent matrix and, on the other hand, we have shown above that $\mathcal{K}_0(\partial_{j_0}, \bar{\partial}_{j_0})$ is a diagonal matrix with distinct eigenvalues $k_1^{j_0}$ and $k_2^{j_0}$. □

We are now in the position to prove the following theorem which generalizes Theorem 1.1 in [13].

**Theorem 2.7.** *Let $E \to \mathbb{D}^n$ be an irreducible hermitian holomorphic vector bundle of rank 2 over $\mathbb{D}^n$ which is homogeneous with respect to Möb$^n$. Then*

(i) *the isomorphism classes of homogeneous line bundles $E \to \mathbb{D}^n$ are completely determined by n real numbers $\{\alpha_1, \dots, \alpha_n\}$; and*

(ii) *there is a one to one correspondence between isomorphism classes of such vector bundles $E \to \mathbb{D}^n$ of rank 2 and unitary equivalence classes of tuples of pairs $((F_1, A_1), \dots, (F_n, A_n))$ where $F_j$ and $A_j$ are skew-hermitian $2 \times 2$ matrices satisfying $[F_j, [F_j, A_j]] = -A_j$, $j = 1, \dots, n$.*

**Proof.** (i) Let $E \overset{\pi}{\to} \mathbb{D}^n$ be a hermitian holomorphic line bundle which is homogeneous under the action of the group Möb$^n$. Let $\mathcal{K}_0$ be the curvature of $E$ at 0. It then follows from Lemma 2.1 that $\mathcal{K}_0(\partial_j, \bar{\partial}_k) = 0$ for $j \neq k$. Furthermore, since $E$ has rank 1 it turns out from Theorem 2.5 that $\tau_0 = 0$. Thus $\alpha_j = \mathcal{K}_0(\partial_j, \bar{\partial}_j)$, $j = 1, \dots, n$ completely determine $\mathcal{K}_0$ and $\tau_0$.

(ii) For $r = 2$ and $j = 1, \dots, n$, we define

$$F_j = \frac{1}{4}\mathcal{K}_0(\partial_j, \bar{\partial}_j) - \frac{1}{4}\left[\tau_0(\partial_j), \tau_0(\bar{\partial}_j)\right], \text{ and } A_j = \frac{1}{2}\tau_0(\partial_j)$$

where $\mathcal{K}_0$ and $\tau_0$ as in Theorem 2.5 satisfying the identities therein. Note that, for $j = 1, \dots, n$,

$$[F_j, A_j] = \frac{1}{2}\tau_0(\bar{\partial}_j) \text{ and } [F_j, [F_j, A_j]] = -A_j.$$

Moreover, Theorem 2.6 yields that only non-zero terms of the curvature tensor $\mathcal{K}_0$ at origin are $\mathcal{K}_0(\partial_j, \bar{\partial}_j)$, $j = 1, \dots, n$. Thus $\{(F_j, A_j) : j = 1, \dots, n\}$ completely determine $\mathcal{K}_0$ and $\tau_0$. Hence the desired result follows from Theorem 2.5. □

It is well known that the curvature of a hermitian holomorphic line bundle over a simply connected domain is a unitary invariant of such bundles. Unfortunately, this, in general, is not the case for the bundles with higher rank. However, using the theorem above as in [13], it can be seen that the curvature itself determines the equivalence classes of *irreducible* hermitian holomorphic vector bundles of rank 2 over $\mathbb{D}^n$ which are *homogeneous* under the action of Möb$^n$ on $\mathbb{D}^n$ as demonstrated in the following theorem.

**Corollary 2.8.** *Let $E \to \mathbb{D}^n$ be an irreducible hermitian holomorphic vector bundle of rank 2 over $\mathbb{D}^n$ which is homogeneous with respect to Möb$^n$. Let $k_1^j, k_2^j$ be the eigenvalues of $\mathcal{K}_0(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j})$, $j = 1, \dots, n$. Then there exists $j_0 \in \{1, \dots, n\}$ such that $|k_1^{j_0} - k_2^{j_0}| > 2$ and for all $j \neq j_0$, $k_1^j = k_2^j = \theta_j = i\lambda_j$, $\lambda_j \in \mathbb{R}$. Moreover, $k_1^{j_0}, k_2^{j_0}$ and $\{\lambda_j : j \in \{1, \dots, n\} \setminus \{j_0\}\}$ determine the isomorphism class of the bundle E.*

**Proof.** Let $\{(F_j, A_j) : j = 1, \dots, n\}$ be a collection of pairs of skew-hermitian matrices defined as in Theorem 2.7. In other words, for $j = 1, \dots, n$, we have

$$F_j = \frac{1}{4}\mathcal{K}_0(\partial_j, \bar{\partial}_j) - \frac{1}{4}\left[\tau_0(\partial_j), \tau_0(\bar{\partial}_j)\right], \text{ and } A_j = \frac{1}{2}\tau_0(\partial_j)$$

where $\mathcal{K}_0$ and $\tau_0$ as in Theorem 2.5 satisfying the equations therein.

Therefore, from the proof of Theorem 2.6 we conclude that $|k_1^{j_0} - k_2^{j_0}| > 2$ and, for $j \in \{1, \dots, n\}$ with $j \neq j_0$, $k_1^j = k_2^j = \theta_j = i\lambda_j$ for some $\lambda_j \in \mathbb{R}$. Thus, following Theorem 2.7 and Theorem 2.5, we have that $k_1^{j_0}, k_2^{j_0}$ and $\{\lambda_j : j \neq j_0, j \in \{1, \dots, n\}\}$ determine the isomorphism class of the bundle $E \to \mathbb{D}^n$. $\square$

**Remark 2.9.** Note that Theorem 2.7 and Corollary 2.8 would be false if the bundles were not irreducible. For instance, even for $n = 1$ and a bundle of rank 2 as given in the corollary above, the eigenvalues of the curvature matrix $\{k_1, k_2\}$ at origin is same as those of the bundle obtained by taking direct sum of two homogeneous line bundles $L_1$ and $L_2$ over $\mathbb{D}$ such that $k_i$ is the curvature of $L_i$, $i = 1, 2$, respectively at the origin. For a detailed discussion on these issues we refer the reader to [11, Chapter 4].

## 3. Homogeneous tuples in $\mathrm{B}_1(\mathbb{D}^n)$ and $\mathrm{B}_2(\mathbb{D}^n)$

In this section, we describe all tuples of operators in $\mathrm{B}_1(\mathbb{D}^n)$ and in $\mathrm{B}_2(\mathbb{D}^n)$ homogeneous with respect to the group $\text{Möb}^n$ with the help of the classification of hermitian holomorphic vector bundles over $\mathbb{D}^n$ homogeneous with respect to $\text{Möb}^n$ obtained in the previous section. Let us, therefore, recall following [1, 2] that the Cowen-Douglas class $\mathrm{B}_r(\mathbb{D}^n)$ over $\mathbb{D}^n$ of rank $r$ consists of tuple $\boldsymbol{T} = (T_1, \dots, T_n)$ of commuting bounded linear operators on a Hilbert space $\mathcal{H}$ such that every $\boldsymbol{z} \in \mathbb{D}^n$ is a joint eigenvalue of $\boldsymbol{T}$ with $r$ dimensional eigenspace, the linear span of the eigenspaces is dense in $\mathcal{H}$ and the range of the operator $(T_1 - z_1 I, \dots, T_n - z_n I)$ is closed in $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. It has been proved in [2] that the corresponding $n$ - tuple of operators $\boldsymbol{T}$ is simultaneously unitarily equivalent to the adjoint of the $n$ - tuple of multiplication operators $\boldsymbol{M} = (M_{z_1}, \dots, M_{z_n})$ by the co-ordinate functions on a Hilbert space $\mathcal{H}_K$ of holomorphic functions on $\Omega^* := \{\overline{\boldsymbol{z}} : \boldsymbol{z} \in \Omega\}$ possessing a reproducing kernel $K$. Also, it can be shown that any $n-$tuple $\boldsymbol{T} = (T_1, \dots, T_n)$ of operators in $\mathrm{B}_r(\mathbb{D}^n)$ gives rise to a hermitian holomorphic vector bundle $E_{\boldsymbol{T}} \to \mathbb{D}^n$ of rank $r$ (cf. [1]). It then turns out that the hermitian metric $H(\boldsymbol{z})$ on the fibre $E_{\boldsymbol{T}}|_{\boldsymbol{z}}$ of $E$ over $\boldsymbol{z} \in \mathbb{D}^n$ is determined by the reproducing kernel $K$, namely, $H(\boldsymbol{z}) = K(\overline{\boldsymbol{z}}, \overline{\boldsymbol{z}})$, $\boldsymbol{z} \in \mathbb{D}^n$. Since as a set $\mathbb{D}^n$ is same as $(\mathbb{D}^n)^*$ and the complex conjugate of holomorphic functions on $\mathbb{D}^n$ turns out to be holomorphic on $(\mathbb{D}^n)^*$, writing $K(\boldsymbol{z}, \boldsymbol{w})$ instead of $K(\overline{\boldsymbol{z}}, \overline{\boldsymbol{w}})$ does not effect the results in the present section. Therefore, from now on we consider the reproducing kernel associated to $\boldsymbol{T} \in \mathrm{B}_r(\mathbb{D}^n)$ as $K(\boldsymbol{z}, \boldsymbol{w})$ on $\mathbb{D}^n$.

It is easy to see that an $n-$tuple $\boldsymbol{T}$ operators in $\mathrm{B}_r(\mathbb{D}^n)$ is homogeneous with respect to the group $\text{Möb}^n$ if and only if the bundle $E_{\boldsymbol{T}} \to \mathbb{D}^n$ associated to it, is homogeneous with respect to the group $\text{Möb}^n$ as a hermitian holomorphic vector bundle. Moreover, the homogeneity of such a hermitian holomorphic vector bundle $E_{\boldsymbol{T}} \to \mathbb{D}^n$ of rank $r$ under the action of the group $\text{Möb}^n$ can be characterised by the fact that $K$ satisfies the identity

$$K(\boldsymbol{z}, \boldsymbol{w}) \;=\; J(g, \boldsymbol{z}) K(g(\boldsymbol{z}), g(\boldsymbol{w})) J(g, w)^*, \;\; \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n, \; g \in \text{Möb}^n \quad (3.1)$$

for some continuous function $J : \text{Möb}^n \times \mathbb{D}^n \to GL_r(\mathbb{C})$ which is holomorphic in the second variable. Following [3, equation (4.1), Page 7] a reproducing kernel $K : \mathbb{D}^n \times \mathbb{D}^n \to M_r(\mathbb{C})$ satisfying this equation (3.1) is said to be *quasi-invariant* with respect to the group $\text{Möb}^n$.

We realize these homogeneous tuples of operators as adjoint of the multiplication operators by the co-ordinate functions on some quotient Hilbert modules. For a bounded domain $\Omega \subset \mathbb{C}^n$, let $\mathcal{A}(\Omega)$ denote the unital Banach algebra obtained by taking norm closure (with respect to the supremum norm on the closure $\overline{\Omega}$ of $\Omega$) of the space of all functions holomorphic in a neighbourhood of $\overline{\Omega}$. Now a complex separable Hilbert space $\mathcal{H}$ is said to be a Hilbert module over the unital Banach algebra $\mathcal{A}(\Omega)$ with the module map $\mathcal{A}(\Omega) \times \mathcal{H} \overset{\pi}{\to} \mathcal{H}$ defined by the point-wise multiplication if the module action $\mathcal{A}(\Omega) \times \mathcal{H} \overset{\pi}{\to} \mathcal{H}$ is norm continuous.

We first recall some basic facts about quotient modules following [4]. Let $\lambda = (\lambda_1, \dots, \lambda_{n+1}) \in \mathbb{R}^{n+1}$ with $\lambda_j > 0$ for $j = 1, \dots, n+1$ and $\mathcal{H}^{(\lambda)}$ be the reproducing kernel Hilbert module over $\mathcal{A}(\mathbb{D}^{n+1})$ with the reproducing kernel

$$K^{(\lambda)}(\boldsymbol{z}, \boldsymbol{w}) = \prod_{j=1}^{n+1} (1 - z_j \overline{w}_j)^{-\lambda_j}.$$

Suppose that $\mathcal{H}_0^{(\lambda)}$ is the submodule of $\mathcal{H}^{(\lambda)}$ consisting of holomorphic functions in $\mathcal{H}^{(\lambda)}$ which vanish of order 2 along the submanifold $\mathcal{Z} := \{z \in \mathbb{D}^{n+1} : z_n = z_{n+1}\}$. Following the equation (1.5) in Section 1 in [4], the submodule $\mathcal{H}_0^{(\lambda)}$ consists of functions $f \in \mathcal{H}^{(\lambda)}$ such that $\partial_{n+1} f$ vanishes identically on $\mathcal{Z}$. Now consider the vector space $J\mathcal{H}^{(\lambda)} := \{(h, \partial_{n+1} h)^{\text{tr}} : h \in \mathcal{H}^{(\lambda)}\}$ and note that $J\mathcal{H}^{(\lambda)}$, as a vector space, is isomorphic to $\mathcal{H}^{(\lambda)}$ via the map $h \overset{J}{\mapsto} (h, \partial_{n+1} h)^{\text{tr}}$, where $\partial_{n+1} = \frac{\partial}{\partial z_{n+1}}$ Moreover, the inner product on $J\mathcal{H}^{(\lambda)}$ induced from the inner product on $\mathcal{H}^{(\lambda)}$ via $J$ turns $J\mathcal{H}^{(\lambda)}$ to be a reproducing kernel Hilbert space (cf. [4, Proposition 2.3]) with the reproducing kernel

$$JK^{(\lambda)}(z, w) = \left(\!\!\left( \partial_{n+1}^i \bar{\partial}_{n+1}^j K^{(\lambda)}(z, w) \right)\!\!\right)_{i,j=0}^{1}.$$

Finally, the module action of $\mathcal{A}(\mathbb{D}^n)$ on $J\mathcal{H}^{(\lambda)}$ by the formula $f \cdot (h, \partial_{n+1} h)^{\text{tr}} = (fh, \partial_{n+1} f h + f \partial_{n+1} h)^{\text{tr}}$ makes $J\mathcal{H}^{(\lambda)}(\mathbb{D}^n)$ a Hilbert module over $\mathcal{A}(\mathbb{D}^n)$ and the map $J$ to be a module isomorphism.

It follows from Theorem 3.3 in [4] that the quotient module $\mathcal{H}_q^{(\lambda)} := \mathcal{H}^{(\lambda)} \ominus \mathcal{H}_0^{(\lambda)}$, as a Hilbert module over $\mathcal{A}(\mathbb{D}^n)$, is isomorphic to the Hilbert module $J\mathcal{H}^{(\lambda)}|_{\mathcal{Z}}$. Consequently, $\mathcal{H}_q^{(\lambda)}$ is a reproducing kernel Hilbert module with the reproducing kernel

$$\begin{aligned} K_q^{(\lambda)}(\boldsymbol{z},\boldsymbol{w}) &= \prod_{j=1}^{n-1}(1-z_j\overline{w}_j)^{-\lambda_j}\begin{pmatrix} (1-z_n\overline{w}_n)^2 & \lambda_{n+1}z_n(1-z_n\overline{w}_n) \\ \lambda_{n+1}\overline{w}_n(1-z_n\overline{w}_n) & \lambda_{n+1}(1+\lambda_{n+1}z_n\overline{w}_n) \end{pmatrix} \\ &\times \ (1-z_n\overline{w}_n)^{-\lambda_n-\lambda_{n+1}-2}. \end{aligned}$$

Note that the restriction of the module action described above to $\mathcal{A}(\mathcal{Z})$ turns out to be the scalar action on the quotient module $\mathcal{H}_q^{(\lambda)}$. Consequently, the compression of tuple of multiplication operators $(M_{z_1},\dots,M_{z_n})$ – denoted by $\boldsymbol{M}^{(2)}$ – onto the quotient space $\mathcal{H}_q^{(\lambda)}$ turns out to be the ordinary co-ordinate multiplications. Moreover, it turns out that the adjoint of $\boldsymbol{M}^{(2)}$ is in $\mathrm{B}_2(\mathbb{D}^n)$ (cf. [4, Proposition 3.6]) and is homogeneous with respect to the group Möb$^n$ and irreducible (cf. [9, Theorem 4.2, Theorem 5.1]), that is, there is no non-trivial reducing subspace of $\boldsymbol{M}^{(2)}$. In the following theorem, we show that these are all irreducible operator tuples in $\mathrm{B}_2(\mathbb{D}^n)$ homogeneous with respect to the group Möb$^n$.

**Theorem 3.1.** (i) *$\boldsymbol{T} = (T_1,\dots,T_n) \in \mathrm{B}_1(\mathbb{D}^n)$ is homogeneous with respect to Möb$^n$ if and only if $\boldsymbol{T}$ is unitarily equivalent to $\boldsymbol{M}^*$ on $\mathcal{H}^{(\lambda)}(\mathbb{D}^n)$ for some $\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{R}_{>0}$.*

(ii) *An irreducible tuple of operators $\boldsymbol{T} = (T_1,\dots,T_n) \in \mathrm{B}_2(\mathbb{D}^n)$ is homogeneous with respect to Möb$^n$ if and only if $\boldsymbol{T}$ is unitarily equivalent to ${\boldsymbol{M}^{(2)}}^*$ on $\mathcal{H}_q^{(\lambda)}$ with the reproducing kernel $K_q^{(\lambda)}$ for some choice of $\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{R}_{>0}$.*

**Proof.** (i) Let us begin with a tuple of operators $\boldsymbol{T} = (T_1,\dots,T_n)$ in $\mathrm{B}_1(\mathbb{D}^n)$ homogeneous with respect to Möb$^n$. Then $\boldsymbol{T} = (T_1,\dots,T_n)$ is unitarily equivalent to the adjoint of the multiplication operators $\boldsymbol{M} = (M_{z_1},\dots,M_{z_n})$ on some reproducing kernel Hilbert space $\mathcal{H}_K$ with the reproducing kernel $K$ on $\mathbb{D}^n$. It turns out that the hermitian holomorphic line bundle $E_{\boldsymbol{M}^*} \to \mathbb{D}^n$ associated to $\boldsymbol{M}^* = (M_{z_1}^*,\dots,M_{z_n}^*)$ is homogeneous with respect to Möb$^n$. Therefore, from the first part of Theorem 2.7 we have that $\alpha_1,\dots,\alpha_n \in \mathbb{R}$ together determine the isomorphism class of $E_{\boldsymbol{M}^*} \to \mathbb{D}^n$. Moreover, it follows that $\alpha_j = \mathcal{K}_0(\partial_j,\bar{\partial}_j)$, $j = 1,\dots,n$. Since the curvature matrix of a line bundle remains unchanged under a change of variable, it follows that $\alpha_j = \bar{\partial}_j(\hat{K}_0^{-1}\partial_j\hat{K}_0)(0,0)$, for $j = 1,\dots,n$, where $\hat{K}_0(\boldsymbol{z},\boldsymbol{w})$ is the normalized kernel at the origin associated to $K$ defined as

$$\hat{K}_0(\boldsymbol{z},\boldsymbol{w}) = K_0(\boldsymbol{z},0)^{-1}K_0(\boldsymbol{z},\boldsymbol{w})K_0(0,\boldsymbol{w})^{-1},$$

and $K_0(\boldsymbol{z},\boldsymbol{w}) = K(0,0)^{-\frac{1}{2}}K(\boldsymbol{z},\boldsymbol{w})K(0,0)^{-\frac{1}{2}}$. Thus, it implies that $\alpha_j > 0$ for $j = 1,\dots,n$. Now consider the reproducing kernel Hilbert space $\mathcal{H}_{K^{(\lambda)}}$ with the reproducing kernel $(1-z_1\overline{w}_1)^{-\lambda_1}\cdots(1-z_n\overline{w}_n)^{-\lambda_n}$ with $\lambda_j = \alpha_j$, $j = 1,\dots,n$ and observe that $K$ is equivalent to $K^{(\lambda)}$. The converse is well-known. Thus, it completes the proof of part (i).

(ii) Let $\boldsymbol{M}^{(2)}$ be the compression of the $n$-tuple of multiplication operators $(M_{z_1}, \dots, M_{z_n})$ onto the quotient space $\mathcal{H}_q^{(\lambda)}$. It follows from [9, Theorem 5.1, page no. 188] that the adjoint $\boldsymbol{M}^* = (M_{z_1}^*, \dots, M_{z_n}^*)$ of the multiplication operators on $\mathcal{H}_q$ is homogeneous with respect to the identity component Möb$^n$ of the automorphism group of $\mathbb{D}^n$. Further, $\boldsymbol{M}^*$ is irreducible ([9, Theorem 4.2, page no. 180]).

Conversely, let $\boldsymbol{T} = (T_1, \dots, T_n) \in \mathrm{B}_2(\mathbb{D}^n)$ such that $\boldsymbol{T}$ is irreducible and $\boldsymbol{T}$ and $\mathrm{g}(\boldsymbol{T})$ are simultaneously unitarily equivalent for all $g = (g_1, \dots, g_n) \in$ Möb$^n$. We then show that $\boldsymbol{T}$ can be realized as $\boldsymbol{M}^{(2)^*}$ on the quotient space $\mathcal{H}_q^{(\lambda)}$ for some $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_j > 0$ for $j = 1, \dots, n$.

Suppose that $E_{\boldsymbol{T}} \to \mathbb{D}^n$ is the hermitian holomorphic vector bundle of rank 2 corresponding to $\boldsymbol{T} \in \mathrm{B}_2(\mathbb{D}^n)$. Since $\boldsymbol{T}$ is irreducible and homogeneous with respect to Möb$^n$, so is the hermitian holomorphic vector bundle $E_{\boldsymbol{T}}$ over $\mathbb{D}^n$. Therefore, from Corollary 2.8 we have that there exists $j_0 \in \{1, \dots, n\}$ such that $|k_1^{j_0} - k_2^{j_0}| > 2$ and for all $j \neq j_0$, $k_1^j = k_2^j = \theta_j$ where $k_1^j, k_2^j$ are the eigenvalues of $\mathcal{K}_0(\partial_j, \bar{\partial}_j)$, $j = 1, \dots, n$. Moreover, $k_1^{j_0}$, $k_2^{j_0}$ and $\{\theta_j : j \in \{1, \dots, n\} \setminus \{j_0\}\}$ determine the isomorphism class of the bundle $E_{\boldsymbol{T}}$ and hence they determine the unitary equivalence class of $\boldsymbol{T} \in \mathrm{B}_2(\mathbb{D}^n)$. It, therefore, suffices to show that $k_1^{j_0}$, $k_2^{j_0}$ and $\{\theta_j : j \in \{1, \dots, n\} \setminus \{j_0\}\}$ can be realized as the eigenvalues of the curvature matrices $\mathcal{K}_0^{(\lambda)}(\partial_j, \bar{\partial}_j)$, for $j = 1, \dots, n$, at $0 \in \mathbb{D}^n$ of the vector bundle $E_{\boldsymbol{M}^{(2)^*}} \to \mathbb{D}^n$ corresponding to the operator tuple $\boldsymbol{M}^{(2)^*}$ on the quotient space $\mathcal{H}_q^{(\lambda)} := \mathcal{H}^{(\lambda)} \ominus \mathcal{H}_0^{(\lambda)}$ for some suitable choice of $\lambda = (\lambda_1, \dots, \lambda_{n+1})$ with $\lambda_j > 0$, $j = 1, \dots, n+1$. Note that, for $1 \leq j \leq n-1$, $\mathcal{K}_0^{(\lambda)}(\partial_j, \bar{\partial}_j) = \lambda_j I_2$ and $\mathcal{K}_0^{(\lambda)}(\partial_n, \bar{\partial}_n)$ is the diagonal matrix with $\lambda_n$ and $\lambda_n + 2\lambda_{n+1} + 2$ as diagonal entries. Therefore, following Corollary 2.8 the required $\lambda = (\lambda_1, \dots, \lambda_{n+1})$ is given by $\lambda_j = \mu_j = -i\theta_j$ for $j = 1, \dots, n-1$, $\lambda_n = \min\{k_1^n, k_2^n\}$, and $\lambda_{n+1} = \frac{\max\{k_1^n, k_2^n\} - \min\{k_1^n, k_2^n\} - 2}{2}$. Finally, a similar argument as in part (i) shows that $\lambda_j > 0$ for $j = 1, \dots, n-1$. □

**Remark 3.2.** Note that the key result in obtaining the classification of irreducible tuples of operators in $\mathrm{B}_r(\mathbb{D}^n)$ homogeneous with respect to Möb$^n$ with $r = 1, 2$ using the techniques developed by Wilkins in [13] is that the curvature matrix of the corresponding vector bundles at the origin is a block diagonal matrix. But in general, for higher rank bundles, this might not be true. For instance, it follows from Theorem 7.5 and Theorem 7.7 in [3] that the adjoint of the multiplication operators $(M_{z_1}, M_{z_2})$ by co-ordinate functions on the reproducing kernel Hilbert space $\mathcal{H}_{\mathrm{K}}$ is irreducible and homogeneous with respect to the group Möb$^2$ with the reproducing kernel $\mathrm{K}$ obtained in Proposition 7.1 in [3]. Also, it can be seen that the adjoint of $(M_{z_1}, M_{z_2})$ is in $\mathrm{B}_3(\mathbb{D}^2)$ and hence, gives rise to a homogeneous vector bundle of rank 3 over $\mathbb{D}^2$. It turns out that the curvature matrix of this vector bundle at the origin is not a block diagonal matrix. In particular, we have that $\bar\partial_2(\mathrm{K}^{-1}\partial_1\mathrm{K})(0,0)$ is a non-zero matrix whose all entries are zero except the 32-th entry which is $\sqrt{(\lambda_1^{-2}+\mu_1^2)(\lambda_2^{-2}+\mu_2^2)^{-1}}$.

So in order to apply these techniques, it is important to understand the 1-form $\tau_0$ in terms of the reproducing kernel which induces the hermitian structure of the bundle. It would be nice if we could extend these techniques to classify homogeneous operators in $\mathrm{B}_r(\mathbb{D}^n)$ that would lead to the classification of all indecomposable finite dimensional representations of $\widetilde{\mathrm{SU}(1,1)}^n$ as well. In case it is not, it would still be interesting to investigate the subclass of homogeneous operators in $\mathrm{B}_r(\mathbb{D}^n)$ obtained from Wilkins' technique.

## 4. Hermitian holomorphic homogeneous line bundles over bounded symmetric domains

The goal of this section is to provide a different proof of the well-known result in [8, Theorem 2.4] which states that the curvature matrix of a homogeneous hermitian holomorphic line bundle associated to a tuple of operators in $\mathrm{B}_1(\Omega)$ over a classical bounded symmetric domain $\Omega$ at the origin is a scalar times the identity matrix. Here, we only consider the first four domains of classical type. However, our result holds for any hermitian holomorphic line bundles over irreducible bounded symmetric domains homogeneous with respect to the identity component of the group of all bi-holomorphic automorphisms of such domains.

We begin by recalling from [10, Chapter 4] that for any classical bounded symmetric domain $\Omega \hookrightarrow \mathbb{C}^N$ the Euclidean co-ordinates $(z_1,\dots,z_N)$ are complex geodesic co-ordinates for $(\Omega,\omega)$ where $\omega$ is the Kählar form of the Bergman metric $g_0 = 2\mathrm{Re}(\sum_{i,j=1}^N g_{i\bar j} dz_i \otimes d\bar z_j)$ on $\Omega$ normalized at origin so that $g_{i\bar j}(0) = \delta_{ij}, 1 \le i,j \le N$. In fact, since $\sigma_0 : \Omega \to \Omega$ defined by $\sigma_0(z) = -z$ is a symmetry at origin the functions $g_{i\bar j}$ are even. So, $dg_{i\bar j}(0) = 0$ for $1 \le i,j \le N$. It then follows from the definition of curvature tensors that $R_{i\bar j k\bar l}(0) = -\partial_i\bar\partial_j g_{k\bar l}(0)$. In fact, for $1 \le i,j,k,l \le N$ and a complex geodesic co-ordinates $(z_1,\dots,z_N)$, we observe from the definition ([10, Chapter 2, Section 2]) of curvature tensors that

$$R_{i,\bar j,k,\bar l}(0) = R^l_{i,\bar j,k}(0)$$

and from the expression of curvature in terms of Christoffel symbols of a Kähler manifold we have that

$$R^l_{i,\bar j,k}(0) = -\frac{\partial \Gamma^l_{i,k}}{\partial \bar z_j}(0) = -\frac{\partial^2 g_{k\bar l}}{\partial z_i \partial \bar z_j}(0).$$

Furthermore, we also note, for $1 \le i,j \le N$ and a complex geodesic co-ordinates $(z_1,\dots,z_N)$, that

$$R_{i,\bar j,\bar k,l} = -\overline{R_{j,\bar i,k,\bar l}}. \tag{4.1}$$

Let $\Omega = D^I_{p,q}$ be the bounded symmetric domain of type $I$. It is well known ([10, Page 84]) that the curvature tensors of $D^I_{p,q}$ at the origin with respect to a complex geodesic co-ordinates take the form

$$R_{ij,\overline{kl},rs,\overline{tu}} \quad = \quad -\delta_{ik}\delta_{rt}\delta_{ju}\delta_{ls} - \delta_{it}\delta_{rk}\delta_{jl}\delta_{su}, \tag{4.2}$$

for $1 \leq i, k, r, t \leq p$ and $1 \leq j, l, s, u \leq q$. We now present the main theorem in this section.

**Theorem 4.1.** *Let $\Omega$ be an irreducible bounded symmetric domain and $L \xrightarrow{\pi} \Omega$ be a hermitian holomorphic line bundle over $\Omega$. Suppose that $L \xrightarrow{\pi} \Omega$ is homogeneous under the action of the identity component of the automorphism group of $\Omega$. Then*

(i) *for any $p, q \in \mathbb{N}$ with $\Omega = D^I_{p,q}$ and for any $n \in \mathbb{N}$ with $\Omega = D^{II}_n$, or, $\Omega = D^{III}_n$, the curvature matrix of $L \xrightarrow{\pi} \Omega$ at the origin is constant times the identity matrix;*

(ii) *for any $n \in \mathbb{N}$ with $n > 2$ and $\Omega = D^{IV}_n$, the curvature matrix of $L \xrightarrow{\pi} \Omega$ at the origin is constant times the identity matrix.*

**Proof.** We begin by pointing out that the second equation in Theorem 2.5, for the homogeneous line bundle $L \xrightarrow{\pi} \Omega$, turns out to be

$$\mathcal{K}_0(R(V,W)Y,Z) + \mathcal{K}_0(Y,R(V,W)Z) = 0. \tag{4.3}$$

(i) Let $\Omega = D^I_{p,q}$ and $L \xrightarrow{\pi} \Omega$ be a hermitian holomorphic line bundle which is homogeneous with respect to the identity component of the automorphism group of $\Omega$. Let $\mathcal{K}$ be the curvature matrix of $L \to \Omega$, that is, by definition

$$\mathcal{K}(z) = \sum_{1\leq i,k\leq p, 1\leq j,l\leq q} \mathcal{K}_{ij,\overline{kl}}(z) dz_{ij} \wedge d\overline{z}_{kl}.$$

Let us consider the matrix $\mathsf{K}(z) = ((\mathcal{K}_{ij,\overline{kl}}(z)))_{1\leq i,k\leq p, 1\leq j,l\leq q}$. Following the equation (4.3), for $1 \leq i \leq p$ and $1 \leq j \leq q-1$, note that

$$\mathcal{K}(\partial_{ij+1}, R(\partial_{ij}, \bar{\partial}_{ij+1})\bar{\partial}_{ij}) = -\mathcal{K}(R(\partial_{ij}, \bar{\partial}_{ij+1})\partial_{ij+1}, \bar{\partial}_{ij}),$$

where $\partial_{ij} = \frac{\partial}{\partial Z_{ij}}$ and $\bar{\partial}_{ij} = \frac{\partial}{\partial \overline{Z}_{ij}}$. Consequently, it follows from the equation (4.2) that

$$\mathsf{K}(0)_{ij,\overline{ij}} = \mathcal{K}(\partial_{ij}, \bar{\partial}_{ij}) = \mathcal{K}(\partial_{ij+1}, \bar{\partial}_{ij+1}) = \mathsf{K}(0)_{ij+1,\overline{i,j+1}}.$$

In a similar way, we can also show that $\mathsf{K}(0)_{ij,\overline{ij}} = \mathsf{K}(0)_{i+1j,\overline{i+1j}}$ for $1 \leq i \leq p-1$ and $1 \leq j \leq q$. Furthermore, for $1 \leq j \neq l \leq q$,it can be seen from the equations (4.2) and (4.3) that

$$\mathsf{K}(0)_{ij,\overline{kl}} = -\mathcal{K}(R(\partial_{ij}, \bar{\partial}_{ij+1})\partial_{ij+1}, \bar{\partial}_{kl}) = \mathcal{K}(\partial_{ij+1}, R(\partial_{ij}, \bar{\partial}_{ij+1})\bar{\partial}_{kl}) = 0.$$

A similar argument also yields that $\mathsf{K}(0)_{ij,\overline{kl}} = 0$ whenever $1 \leq i \neq k \leq p$. Thus it shows that the curvature matrix of the line bundle $L \xrightarrow{\pi} \Omega$ is a scalar times the identity matrix, for $\Omega = D^I_{p,q}$.

For other two domains, we point out that $D^{II}_n$ and $D^{III}_n$ are totally geodesic submanifolds of $D^I_{n,n}$ ([10, Lemma 1, page 85]). So, the curvature tensors of $D^{II}_n$ and $D^{III}_n$ at origin is obtained by restricting the curvature tensors of $D^I_{n,n}$ to the respective tangent subspaces at origin. Recall that

$$D^{II}_n = \{Z \in D^I_{n,n} : Z^{tr} = -Z\}, \text{ and } D^{III}_n = \{Z \in D^I_{n,n} : Z^{tr} = Z\}.$$

Therefore, it can be seen from the definition of complex geodesic co-ordinates that $\{X_{ij} : i < j\}$ and $\{Y_{ij} : i \leq j\}$ form a geodesic co-ordinates of $D^{II}_n$ and $D^{III}_n$, respectively, where

$$X_{ij} = \frac{1}{\sqrt{2}}(Z_{ij} - Z_{ji}) \text{ and } Y_{ij} = \frac{1}{\sqrt{2}}(Z_{ij} + Z_{ji}), \text{ for } i < j, \text{ and } Y_{ii} = Z_{ii}.$$

Thus, it can be seen with the help of the equation (4.2) with these new co-ordinates that an analogous way of reasoning verifies that the curvature matrix of the line bundle $L \xrightarrow{\pi} \Omega$ is a scalar times identity for $\Omega = D^{II}_n$ and $\Omega = D^{III}_n$. So it completes the proof (i).

(ii) Let $n \in \mathbb{N}$ with $n > 2$ and $\Omega = D^{IV}_n$. As in part (i), we want to use Theorem 2.5 to verify that the curvature matrix of the line bundle $L \xrightarrow{\pi} D^{IV}_n$ with $n > 2$ is scalar times the identity matrix.

Recall from [10, Page 87] that the curvature tensor of $D^{IV}_n$ at the origin with respect to the complex geodesic co-ordinates $(z_1, \ldots, z_n)$ takes the form

$$R_{i\bar{j}k\bar{l}}(0) = -(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \tag{4.4}$$

It follows from the above formula that whenever $i \neq j \neq k$ as well as $i \neq k$ we have that

$$R_{i\bar{j}k\bar{l}}(0) = 0 \text{ for any } 1 \leq l \leq n$$

which, in other words, reads that

$$R(\partial_i, \bar{\partial}_j)\partial_k = 0, \text{ for } i \neq j \neq k \text{ and } i \neq k,$$

where $\partial_i = \frac{\partial}{\partial Z_i}$ and $\bar{\partial}_i = \frac{\partial}{\partial \bar{Z}_i}$. For $i \neq j$, the equation (4.4) yields that $R_{i\bar{j}i\bar{l}} = \delta_{jl}$ and $R_{j\bar{i}j\bar{l}} = \delta_{il}$. In other words,

$$R(\partial_i, \bar{\partial}_j)\partial_i = \partial_j \text{ and } R(\partial_i, \bar{\partial}_j)\bar{\partial}_j = -\bar{\partial}_i \tag{4.5}$$

for $1 \leq i \neq j \leq n$. Now using the equation (4.3) we have, for $1 \leq i \neq j \leq n$, that

$$\mathcal{K}(R(\partial_j, \bar{\partial}_i)\partial_j, \bar{\partial}_i) = -\mathcal{K}(\partial_j, R(\partial_j, \bar{\partial}_i)\bar{\partial}_i),$$

and consequently, by (4.5) it turns out that

$$\mathsf{K}(0)_{i\bar{i}} = \mathcal{K}(\partial_i, \bar{\partial}_i) = \mathcal{K}(\partial_j, \bar{\partial}_j) = \mathsf{K}(0)_{j\bar{j}}.$$

Since dimension of $\Omega = \mathbb{D}_n^{IV}$ is at least 3, for given $1 \leq i \neq j \leq n$, there exists $1 \leq k \leq n$ such that $i \neq k$ and $j \neq k$. It can then be seen from the previous calculation that $R(\partial_i, \bar{\partial}_j)\partial_k = 0$. Therefore, a similar computation as above using the equations (4.3) and (4.5) shows that $\mathsf{K}(0)_{i\bar{j}} = 0$ for all $1 \leq i \neq j \leq n$ completing the proof of (ii). □

**Remark 4.2.** For $n = 2$, it turns out that that the domain $D_2^{IV}$ is bi-holomorphic to the unit bi-disc $\mathbb{D}^2$ (cf. [10, Lemma 1, Page 76]). Thus, this case has been already considered in Section 2.

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## References


[1] COWEN, MICHAEL J.; DOUGLAS, RONALD G. Operators possessing an open set of eigenvalues. *Functions, series, operators, Vol. I, II* (Budapest, 1980), 323–341. Colloq. Math. Soc. János Bolyai, 35, *North-Holland, Amsterdam*, 1983. MR0751007 (85k:47033), Zbl 0584.47002. 403, 411

[2] CURTO, RAÚL E.; SALINAS, NORBERTO. Generalized Bergman kernels and the Cowen–Douglas theory. *Amer. J. Math.* **106** (1984), no. 2, 447–488. MR0737780 (85e:47042), Zbl 0583.47037, doi: 10.2307/2374310. 411

[3] DEB, PRAHLLAD; HAZRA, SOMNATH. Homogeneous hermitian holomorphic vector bundles and operators in the Cowen–Douglas class over the poly-disc. Preprint, 2020. arXiv:2007.14105. 404, 405, 412, 414

[4] DOUGLAS, RONALD G.; MISRA, GADADHAR; VARUGHESE, CHERIAN. On quotient modules—the case of arbitrary multiplicity. *J. Funct. Anal.* **174** (2000), no. 2, 364–398. MR1768979 (2001f:47012), Zbl 0979.46034, doi: 10.1006/jfan.2000.3594. 412, 413

[5] HELGASON, SIGURDUR. Differential geometry, Lie groups, and symmetric spaces. Pure and Applied Mathematics, 80. *Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London*, 1978. xv+628 pp. ISBN: 0-12-338460-5. MR0514561 (80k:53081), Zbl 0451.53038. 407

[6] KORÁNYI ADAM; MISRA, GADADHAR. A classification of homogeneous operators in the Cowen–Douglas class. *Adv. Math.* **226** (2011), no. 6, 5338–5360. MR2775904 (2012g:47074), Zbl 1220.47026, doi: 10.1016/j.aim.2011.01.012. 403

[7] MISRA, GADADHAR. Curvature and the backward shift operators. *Proc. Amer. Math. Soc.* **91** (1984), no. 1, 105–107. MR0735574 (86b:47048), Zbl 0548.47015, doi: 10.1090/S0002-9939-1984-0735574-5. 403

[8] MISRA, GADADHAR; NARSIMHA SASTRY, N. S.. Homogeneous tuples of operators and representations of some classical groups. *J. Operator Theory* **24** (1990), no. 1, 23–32. MR1086542 (92g:47039), Zbl 0776.47006. 404, 415

[9] MISRA GADADHAR; ROY, SUBRATA SHYAM. On the irreducibility of a class of homogeneous operators. *System theory, the Schur algorithm and multidimensional analysis*, 165–198. Oper. Theory Adv. Appl., 176, *Birkhäuser, Basel,* 2007. MR2342900 (2008i:47044), Zbl 1141.47020, arXiv:math/0611631. 413, 414

[10] MOK, NGAIMING. Metric rigidity theorems on Hermitian locally symmetric manifolds. Series in Pure Mathematics, 6. *World Scientific Publishing Co., Inc., Teaneck, NJ*, 1989. xiv+278 pp. ISBN: 9971-50-800-1; 9971-50-802-8. MR1081948 (92d:32046), Zbl 0912.32026, doi: 10.1142/0773. 415, 416, 417, 418

[11] ROY, SUBRATA SHYAM. Homogeneous operators in the Cowen–Douglas class. *Ph.D. Thesis, Indian Statistical Institute*, 2008. 411

[12] UPMEIER, HARALD. Jordan algebras in analysis, operator theory, and quantum mechanics. CBMS Regional Conference Series in Mathematics, 67. *Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI*, (1987). viii+85 pp. ISBN: 0-8218-0717-X. MR0874756 (88h:17032), Zbl 0608.17013, doi: 10.1090/cbms/067. 407

[13] WILKINS, DAVID R. Homogeneous vector bundles and Cowen–Douglas operators. *Internat. J. Math.* **4** (1993), no. 3, 503–520. MR1228586 (95b:47034), Zbl 0805.32015, doi: 10.1142/S0129167X93000261. 403, 405, 407, 408, 409, 410, 414

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