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On admissible square roots of non-negative $C^{2,2\alpha}$ functions

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Abstract. We establish a necessary and sufficient condition for $C^{1,\alpha}$ regularity of the admissible square root of a non-negative $C^{2,2\alpha}(\mathbb{R})$ function.

Contents

1. Introduction

The paper concerns the following problem: the regularity of square root of $C^{2,2\alpha}$ non-negative functions. Nirenberg-Trèves' gradient estimate for non-negative $C^{1,1}(\mathbb{R}^n)$ functions [14] implies square roots of these functions are Lipschitz. This estimate plays important roles in analysis of linear and nonlinear PDEs (e.g., [9], [1]). The sum of squares theorem of Fefferman and Phong [4, 5] stated that any non-negative $C^{3,1}$ function in $\mathbb{R}^n$ can be written as a sum of squares of $C^{1,1}$ functions. A detailed proof was given in [7] which was communicated by Fefferman (see also [3],[16]). This decomposition is crucial to obtain C^2 a priori estimates for degenerate real Monge-Ampère equations in [7] and complex Monge-Ampère equation in [15].

For functions of one variable, Glaeser [6] proved that if $0 \le f \in C^2(\mathbb{R})$ is 2-flat on its zeroes (i.e., $f(x) = 0$ implies $f''(x) = 0$), then $f^{1/2} \in C^1(\mathbb{R})$. Mandai [13] proved that for any $0 \le f \in C^2(\mathbb{R})$, f always has an admissible square root $g \in C^1(\mathbb{R})$. In [3], Bony, Broglia, Colombini and Pernazza obtain a necessary and sufficient condition for a non-negative function $f \in C^4(\mathbb{R})$ to have an admissible square root in $C^2(\mathbb{R})$, which is only related to the non-zero

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local minimum points of f. Korobenko-Sawyer [12] consider higher regularity of square root functions under appropriate sufficient conditions.

The main result of this paper is the necessary and sufficient condition for optimal $C^{1,\alpha}$ regularity of square roots of $C^{2,2\alpha}(\mathbb{R})$ non-negative functions. In the rest of this paper, $C^{2,2\alpha}(\mathbb{R})$ indicates $C^{3,2\alpha-1}(\mathbb{R})$ if $1/2 < \alpha \leq 1$. Below is the statement of the main theorem.

Theorem 1.1. *Let* $0 \leq f \in C^{2,2\alpha}(\mathbb{R})$ *with* $||f||_{C^{2,2\alpha}(\mathbb{R})} \leq 1$. $0 < \alpha \leq 1$. *Define the set*

$$\mathcal{A} = \{x_0 \in \mathbb{R} : f(x_0) > 0, f'(x_0) = 0, f''(x_0) > 0\}. \tag{1}$$

Then $f = g^2$ *for some* $g \in C^{1,\alpha}(\mathbb{R})$ *if and only if there is a constant* $M > 0$ *such that*

$$f''(x_0) \leq M \cdot (f(x_0))^{\frac{\alpha}{1+\alpha}}, \quad \forall x_0 \in \mathcal{A}. \tag{2}$$

Moreover, if (2) *is satisfied, then* $||g||_{C^{1,\alpha}(\mathbb{R})} \leq C$ *for some universal* $C > 0$*, depending only on* α *and* M*.*

Remark 1.2. *The condition obtained by Bony, Broglia, Colombini and Pernazza in* [3] *is there is a continuous function* γ *vanishing at every flat points of* f *such that*

$$f''(x_0) \leq \gamma(x_0) \cdot (f(x_0))^{\frac{1}{2}}, \quad \forall x_0 \in \mathcal{A}. \tag{3}$$

Condition (2) *is a* $C^{2,2\alpha}$ *version of* (3)*.*

The main theorem is motivated by regularity problem associated to the isometric embedding problem. Guan and Li [8] showed that if g is a C^4 Riemannian metric on $\mathbb{S}^2$ with Gauss curvature $K_g \geq 0$, then there exists a $C^{1,1}$ isometric embedding $X : (\mathbb{S}^2, g) \to (\mathbb{R}^3, g_{Eucl})$. A natural question is, can the embedding X be improved to $C^{2,1}$? A positive answer was given in Jiang [11] in the graph setting, under the assumption X takes the form $X(x,y) = (x, y, u(x,y))$ in local coordinates. It relies on a square root regularity for square of monotone functions. It is a special case of Theorem 1.1 where $\alpha = 1$ and $\mathcal{A} = \emptyset$, which can be stated as follows.

Corollary 1.3. *Let* $I = [-1/2, 1/2]$*. Assume* $0 \leq f \in C^{3,1}(I)$ *satisfies* $||f||_{C^{3,1}(I)} \leq 1$*. The zero set of* f *in* I *is a closed interval* $N = [x_0', x_0]$ *(possibly* $x_0' = x_0$*).* f *is non-increasing in* $[-1/2, x_0')$ *and* f *is non-decreasing in* $(x_0, 1/2]$*. Then* $\exists g \in C^{1,1}(I)$ *such that* $f = g^2$ *in* I*,* g *is non-decreasing in* I *and* $||g||_{C^{1,1}(I)} \leq C$ *for some universal constant* $C > 0$*.*

2. Fefferman-Phong's Lemma for $C^{2,2\alpha}$ non-negative functions

The following lemma is well known (e.g. [16]). We provide a proof here for completeness.

Lemma 2.1 (Even dominate odd, $C^{2,\alpha}$)**.** *Let* $0 < \alpha \leq 1$*. Let* $f : \mathbb{R} \to \mathbb{R}$ *be a* C^2 *non-negative function such that* $[f]_{C^{2,\alpha}(\mathbb{R})} \leq 1$*. Then*

$$|f'(x)| \leq \frac{3}{2}|f(x)|^{\frac{1+\alpha}{2+\alpha}} + \frac{1}{2}|f''(x)| \cdot f(x)^{\frac{1}{2+\alpha}} + f(x)^{\frac{\alpha}{2+\alpha}} \cdot |f''(x)|^{\frac{1}{\alpha}} \quad \forall x \in \mathbb{R}. \tag{4}$$

Proof. We may assume $f(x) \neq 0$. By Taylor expansion, $\forall x, h \in \mathbb{R}$, $\exists \xi$ between $x, x+h$ such that

$$0 \leq f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{2}\frac{f''(\xi) - f''(x)}{|\xi - x|^\alpha}|\xi - x|^\alpha h^2$$
$$\leq f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{2}|h|^{2+\alpha}.$$

Replacing h with $\pm h$,

$$|f'(x)h| \leq f(x) + \frac{1}{2}|f''(x)|h^2 + \frac{1}{2}|h|^{2+\alpha}. \tag{5}$$

Setting $h = \frac{f(x)^{\frac{2}{2+\alpha}}}{f(x)^{\frac{1}{2+\alpha}} + |f''(x)|^{\frac{1}{\alpha}}}$ in (5) and using $h \leq f(x)^{\frac{1}{2+\alpha}}$, we obtain (4). □

Lemma 2.2 (Even dominate odd, $C^{3,\alpha}$). *Let $0 < \alpha \leq 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^3 non-negative function such that $[f]_{C^{3,\alpha}(\mathbb{R})} \leq 1$. Then*

$$|f'(x)| \leq \frac{13}{6}f(x)^{\frac{2+\alpha}{3+\alpha}} + \frac{3}{2}f(x)^{\frac{1+\alpha}{3+\alpha}} \cdot |f''(x)|^{\frac{1}{1+\alpha}} + f(x)^{\frac{1}{3+\alpha}} \cdot |f''(x)|, \quad \forall x \in \mathbb{R}. \tag{6}$$

$$|f'''(x)| \leq 6f(x)^{\frac{\alpha}{3+\alpha}} + 6|f''(x)|^{\frac{\alpha}{1+\alpha}}, \quad \forall x \in \mathbb{R}. \tag{7}$$

Proof. By Taylor expansion, $\forall x \in \mathbb{R}$,

$$0 \leq f(x+h) \leq f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + \frac{1}{6}|h|^{3+\alpha}. \tag{8}$$

Replacing h with $\pm h$,

$$|f'(x)h + \frac{1}{6}f'''(x)h^3| \leq f(x) + \frac{1}{2}|f''(x)|h^2 + \frac{1}{6}|h|^{3+\alpha} =: A. \tag{9}$$

Replacing h by $2h$ in (9), we have

$$|2 \cdot f'(x)h + 8 \cdot \frac{1}{6}f'''(x)h^3| \leq f(x) + \frac{1}{2}|f''(x)|(2h)^2 + \frac{1}{6}|2h|^{3+\alpha} =: B. \tag{10}$$

Combining (9) and (10),

$$|f'(x)h| \leq \frac{8A + B}{6}, \quad |\frac{1}{6}f'''(x)h^3| \leq \frac{2A + B}{6}. \tag{11}$$

If $f(x) = 0$, then $f'(x) = 0$ since $f \geq 0$. Otherwise, setting $h = \frac{f(x)^{\frac{2}{3+\alpha}}}{f(x)^{\frac{1}{3+\alpha}} + |f''(x)|^{\frac{1}{1+\alpha}}}$ in (11) and using $h \leq f(x)^{\frac{1}{3+\alpha}}$, we have

$$|f'(x)| \leq \frac{1}{6}\left(9 \cdot \frac{f(x)}{h} + 6 \cdot |f''(x)|h + 4 \cdot |h|^{2+\alpha}\right)$$
$$\leq \frac{1}{6}\left(9 \cdot f(x)^{\frac{1+\alpha}{3+\alpha}}(f(x)^{\frac{1}{3+\alpha}} + |f''(x)|^{\frac{1}{1+\alpha}}) + 6 \cdot |f''(x)| \cdot f(x)^{\frac{1}{3+\alpha}} + 4 \cdot f(x)^{\frac{2+\alpha}{3+\alpha}}\right).$$

Thus, (6) holds.

If $f(x) = f''(x) = 0$, then $f'''(x) = 0$ by (8). Otherwise, letting $h = \max\{f(x)^{\frac{1}{3+\alpha}}, |f''(x)|^{\frac{1}{1+\alpha}}\}$ and using $\max\{a,b\} \le a+b$ in (11), and as $(a+b)^\alpha \le a^\alpha + b^\alpha$ for $a, b \ge 0$ and $0 < \alpha \le 1$, we have,

$$\begin{aligned}
|f'''(x)| &\le \frac{3f(x)}{h^3} + \frac{3|f''(x)|}{h} + (\frac{1}{3} + \frac{1}{6} \cdot 2^{3+\alpha}) \cdot |h|^\alpha \\
&\le 3f(x)^{\frac{\alpha}{3+\alpha}} + 3|f''(x)|^{\frac{\alpha}{1+\alpha}} + 3 \cdot \left| f(x)^{\frac{1}{3+\alpha}} + |f''(x)|^{\frac{1}{1+\alpha}} \right|^\alpha \\
&\le 3f(x)^{\frac{\alpha}{3+\alpha}} + 3|f''(x)|^{\frac{\alpha}{1+\alpha}} + 3 \cdot \left(f(x)^{\frac{\alpha}{3+\alpha}} + |f''(x)|^{\frac{\alpha}{1+\alpha}} \right).
\end{aligned}$$

Thus, (7) holds. □

We define some constants which will be used in the rest of the paper.

$$\begin{aligned}
c_0 = 1/10, &\quad C = 1000; \\
N(\alpha) = 2, \text{ if } 0 < \alpha \le 1/2, &\quad N(\alpha) = 3, \text{ if } 1/2 < \alpha \le 1; \\
\epsilon_0 = (\frac{1}{10^5})^{1/(2\alpha)}, \text{ if } 0 < \alpha \le 1/2, &\quad \epsilon_0 = (\frac{1}{10^5})^{1/(2\alpha-1)}, \text{ if } 1/2 < \alpha \le 1; \\
\tilde{c} = \frac{1}{10^3} \cdot (\frac{1}{10^5})^{3/(2\alpha)}, \text{ if } 0 < \alpha \le 1/2, &\quad \tilde{c} = \frac{1}{10^4} \cdot (\frac{1}{10^5})^{4/(2\alpha-1)}, \text{ if } 1/2 < \alpha \le 1.
\end{aligned} \tag{12}$$

Denote the set of flat points of f by

$$\mathcal{F} = \{x \in \mathbb{R} : f(x) = f'(x) = f''(x) = 0\}. \tag{13}$$

We note that if $f \in C^3$ and $f \ge 0$, $x \in \mathcal{F}$ implies $f'''(x) = 0$.

The next lemma is a $C^{2,2\alpha}$-version of Fefferman-Phong's lemma (see [4] and Lemma 18.6.9 of [10]).

Lemma 2.3 (Fefferman-Phong's Lemma). *Let $I = [-1/2, 1/2]$. If $0 \le \phi \in C^{2,2\alpha}(I)$ such that*

$$|\phi^{(k)}(t)| \le C \quad \forall t \in I \text{ for } k = 0, 1, \cdots, N(\alpha), \quad [\phi]_{C^{2,2\alpha}(I)} \le 1 \tag{14}$$

$$\text{and} \quad \max\{\phi(0), |\phi''(0)|\} \ge \tilde{c}, \tag{15}$$

where $N(\alpha)$, C, $\tilde{c}$ are defined in (12). Then there exist universal constants $r_0 > 0$, $\tilde{A} > 0$, $c_2 > 0$ such that, for $t \in (-r_0, r_0)$, either

$$c_2 \le \phi(t) \le C, \quad \|\sqrt{\phi(t)}\|_{C^{1,\alpha}((-r_0,r_0))} \le \tilde{A}; \tag{16}$$

or

$$c_2 \le \phi''(t) \le C, \tag{17}$$

$$\phi(t) = \phi(T) + (t-T)^2 \int_0^1 \phi''(t + s(T-t)) s \, ds, \tag{18}$$

where $t = T$ is the unique strict local minimum point of the function ϕ in $(-r_0, r_0)$.

Moreover, the function

$$g(t) = (t-T)(\int_0^1 \phi''(t+s(T-t))s\,ds)^{1/2} \tag{19}$$

is in $C^{1,\alpha}((-r_0, r_0))$.

Proof. Set $\mu = \min\{\frac{\tilde{c}}{3C}, (\frac{\tilde{c}}{3})^{\frac{1}{2\alpha}}\}$, where $\tilde{c}$ and C are defined in (12).

(i). If $\phi(0) \geq \tilde{c}$, $\forall |t| < \mu$,

$$\phi(t) \geq \frac{1}{3}\tilde{c}, \quad \text{and} \quad |(\sqrt{\phi})'(t_1)| = |\frac{\phi'(t_1)}{2\sqrt{\phi(t_1)}}| \leq \frac{C}{2\sqrt{\frac{1}{3}\tilde{c}}} =: b. \tag{20}$$

By (14), (20), and the mean value theorem, for $|t_1| < \mu$ and $|t_2| < \mu$, $t_1 \neq t_2$,

$$\begin{aligned}
&2|(\sqrt{\phi})'(t_1) - (\sqrt{\phi})'(t_2)|/|t_1-t_2|^\alpha = |\frac{\phi'(t_1)}{\sqrt{\phi(t_1)}} - \frac{\phi'(t_2)}{\sqrt{\phi(t_2)}}|/|t_1-t_2|^\alpha \\
&\leq |\frac{\phi'(t_1)}{\sqrt{\phi(t_1)}} - \frac{\phi'(t_2)}{\sqrt{\phi(t_1)}}|/|t_1-t_2|^\alpha + |\frac{\phi'(t_2)}{\sqrt{\phi(t_1)}} - \frac{\phi'(t_2)}{\sqrt{\phi(t_2)}}|/|t_1-t_2|^\alpha \\
&\leq \frac{1}{\sqrt{\frac{1}{3}\tilde{c}}} \cdot \frac{|\phi''(\xi_1)||t_1-t_2|}{|t_1-t_2|^\alpha} + C \cdot \frac{|\phi'(\xi_2)||t_1-t_2|}{2(\frac{1}{3}\tilde{c})^{3/2}|t_1-t_2|^\alpha} \leq C_1
\end{aligned} \tag{21}$$

where $b, C_1 > 0$ are universal constants, and ξ_1, ξ_2 are some points between t_1, t_2.

(ii). Assume $|\phi''(0)| \geq \tilde{c}$.

(a) If $\phi''(0) \leq -\tilde{c}$, then for $|t| < \mu$, $\phi''(t) \leq -\frac{1}{3}\tilde{c}$. For any $|t_0| < \frac{1}{2}\mu$, expanding ϕ near t_0, we have

$$0 \leq \phi(t_0+h) + \phi(t_0-h) \leq 2 \cdot \left(\phi(t_0) + \frac{1}{2} \cdot (-\frac{1}{3}\tilde{c}) \cdot h^2 + \frac{1}{2}|h|^{2+2\alpha}\right).$$

Letting $h = \frac{1}{2}\mu$, $\forall |t_0| < \frac{1}{2}\mu$, $\phi(t_0) \geq \frac{1}{6}\tilde{c}h^2 - \frac{1}{2}|h|^{2+2\alpha} \geq \frac{1}{24}\mu^2\tilde{c}(1-2^{-2\alpha})$.

Similar to case (i), we have $\sqrt{\phi} \in C^{1,\alpha}((-\mu/2, \mu/2))$.

(b) If $\phi''(0) \geq \tilde{c}$ and $\phi(0) < c_1$, where $c_1 > 0$ is a small and universal constant to be determined, then $|\phi'(0)|$ is also small since $\phi \geq 0$. By expansion of $\phi' \in C^{1,2\alpha}(I)$ near 0,

$$\phi'(t) = \phi'(0) + \phi''(0)t + R(t), \quad \text{where} \quad |R(t)| \leq C|t|^{1+\alpha}. \tag{22}$$

In particular, (22) shows that $\phi'(r) > 0$ and $\phi'(-r) < 0$ if

$$\phi''(0)r > |\phi'(0)| + 2Cr^{1+\alpha}. \tag{23}$$

Fix $r = \min\{\frac{\tilde{c}}{3C}, (\frac{\tilde{c}}{3})^{\frac{1}{2\alpha}}\}$. As $\phi'' \in C^{2\alpha}(I)$,

$$\phi''(t) \geq \frac{1}{3}\tilde{c}, \quad |t| \leq r. \tag{24}$$

This implies $\phi'(t)$ is strictly increasing in $[-r,r]$, thus $\phi'(t)=0$ has a unique solution $t=T$ in $B_r:=(-r,r)$. By Taylor expansion of ϕ near $t=T$, we obtain in B_r,

$$\phi(t)=\phi(T)+(t-T)^2\int_0^1\phi''(t+s(T-t))s\,ds. \tag{25}$$

We note $t=T$ is the unique strict local minimum point of the function ϕ in B_r.

We will estimate Hölder seminorm of g' where g is defined in (19). Assume without loss of generality that $\phi(T)=0$. Then in B_r, $g(t)=\sqrt{\phi(t)}$ if $t\geq T$ and $g(t)=-\sqrt{\phi(t)}$ if $t<T$. By Taylor expansion,

$$\begin{aligned}\lim_{t\to T^+}\frac{g(t)-g(T)}{t-T}&=\lim_{t\to T^+}\frac{\sqrt{\frac{1}{2}\phi''(T)(t-T)^2+O(|t-T|^{2+\min\{1,2\alpha\}})}-\sqrt{0}}{t-T}\\&=\sqrt{\frac{1}{2}\phi''(T)}.\end{aligned}$$

We obtain the same value for the left limit and hence $g'(T)=\sqrt{\frac{1}{2}\phi''(T)}$.

If $t\neq T$, then by Taylor expansion,

$$\phi(t)=\frac{1}{2}\phi''(T)(t-T)^2+A_1, \tag{26}$$

$$\phi'(t)=\phi''(T)(t-T)+A_2, \tag{27}$$

$$\phi''(t)=\phi''(T)+A_3. \tag{28}$$

By (24), (14) and $|t-T|\leq 2r$,

$$\begin{aligned}|A_1|&\leq C_1\cdot|t-T|^{2+\min\{1,2\alpha\}}\leq\frac{1}{3}\phi''(T)(t-T)^2,\\|A_2|&\leq C_2\cdot|t-T|^{1+\min\{1,2\alpha\}}\leq|\phi''(T)(t-T)|,\\|A_3|&\leq C_3\cdot|t-T|^{\min\{1,2\alpha\}}.\end{aligned} \tag{29}$$

Suppose $t>T$. By (26), (27), (29) and $\phi''(t)\sim 1$ in B_r, $\forall t\in B_r$,

$$\begin{aligned}|g'(T)-g'(t)|&=\left|\sqrt{\frac{1}{2}\phi''(T)}-\frac{\phi'(t)}{2\sqrt{\phi(t)}}\right|\\&\leq\left|\sqrt{\frac{1}{2}\phi''(T)}-\frac{\phi'(t)}{2\sqrt{\frac{1}{2}\phi''(T)(t-T)^2}}\right|+\left|\frac{\phi'(t)}{2\sqrt{\frac{1}{2}\phi''(T)(t-T)^2}}-\frac{\phi'(t)}{2\sqrt{\phi(t)}}\right|\\&=\left|\frac{A_2}{\sqrt{2\phi''(T)(t-T)^2}}\right|\\&\quad+\frac{1}{2}|\phi'(t)|\left|\frac{A_1}{\sqrt{\frac{1}{2}\phi''(T)(t-T)^2}\cdot\sqrt{\phi(t)}\cdot\left(\sqrt{\phi(t)}+\sqrt{\frac{1}{2}\phi''(T)(t-T)^2}\right)}\right|\\&\leq b\cdot|T-t|^{\alpha},\end{aligned} \tag{30}$$

where $b > 0$ is a universal constant. Proof is the same for $t < T$.

By (26), (27), (28), and $\phi''(t) \sim 1$ in B_r, there exists a universal $c > 0$ such that, $\forall t \in B_r$ with $t \neq T$,

$$|\phi''(t) \cdot \phi(t) - \frac{1}{2}\phi'(t)^2| = O(|t-T|^{2+\min\{1,2\alpha\}}),$$

$$|g''(t)| = \frac{1}{2}\left|\frac{\phi''(t)\cdot\phi(t) - \frac{1}{2}\phi'(t)^2}{\phi(t)^{3/2}}\right| \leq c \cdot |t-T|^{\min\{0,2\alpha-1\}}. \tag{31}$$

$\forall t_1, t_2 \in B_r$, we want to estimate $|g'(t_1) - g'(t_2)|$. By (30), we only need to deal with

$$T < t_1 < t_2 \quad \text{or} \quad t_2 < t_1 < T, \quad \text{with}$$
$$|t_1 - t_2| \leq |t_1 - T|. \tag{32}$$

We only consider the case $T < t_1 < t_2$ ($t_1 < t_2 < T$ is similar). By assumption (32),

$$|\xi - T| \geq |t_1 - T| \geq |t_1 - t_2|, \quad \forall \xi \in (t_1, t_2).$$

By the mean value theorem, $\exists \xi \in (t_1, t_2)$ such that

$$|g'(t_1) - g'(t_2)| = |g''(\xi)||t_1 - t_2| \leq c \cdot |\xi - T|^{\min\{0,2\alpha-1\}} \cdot |t_1 - t_2| \leq c \cdot |t_1 - t_2|^{\alpha}.$$

(c) If $c_1 \leq \phi(0) < \tilde{c}$, then similar to case (i), we have $\sqrt{\phi} \in C^{1,\alpha}((-\frac{c_1}{3C}, \frac{c_1}{3C}))$.

To summarize, case (i), (ii)(a) and (ii)(c) lead to (16). Case (ii)(b) leads to (17). □

3. A Calderón-Zygmund decomposition

We use the Calderón–Zygmund decomposition, which was originally suggested by Fefferman in [7].

Lemma 3.1. *Let $0 \leq f \in C^{2,2\alpha}(\mathbb{R})$ with $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$. There is a countable collection of intervals $\{Q_\nu\}_{\nu \geq 1}$ taking the form of $(a, b]$, whose interiors are disjoint, such that*

(1) *$\mathbb{R} = \mathcal{F} \cup (\cup_\nu Q_\nu)$ and $\mathcal{F} \cap (\cup_\nu Q_\nu) = \emptyset$, where $\mathcal{F}$ is defined in* (13).

(2) *Let $\delta_\nu = diam(Q_\nu)$. Then for any ν, $\delta_\nu \leq 1$. With $N(\alpha)$ defined in* (12),

$$\inf_{x \in Q_\nu}\left(\sum_{k=0}^{N(\alpha)} \delta_\nu^{k-(2+2\alpha)}|\nabla^k f(x)|\right) > N(\alpha) + 1. \tag{33}$$

Proof. We decompose $\mathbb{R}$ into a (countable) collection of disjoint intervals $(a_n, b_n]$ with the same length, and their common diameter is so large that

$$\inf_{x \in Q'}\left(\sum_{k=0}^{N(\alpha)} (diam(Q'))^{k-(2+2\alpha)}|\nabla^k f(x)|\right) \leq N(\alpha) + 1$$

for every interval Q' in this collection. As $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$, the common diameter can be chosen to be 1.

Let Q' be a fixed interval in this collection. By bisecting, we divide Q' into 2 congruent intervals. Let Q'' be one of these new intervals.

(1) If

$$\inf_{x\in Q''}\left(\sum_{k=0}^{N(\alpha)}(diam(Q''))^{k-(2+2\alpha)}|\nabla^k f(x)|\right) > N(\alpha)+1,$$

then we don't sub-divide Q'' any further, and Q'' is selected as one of the intervals Q_ν.

(2) If

$$\inf_{x\in Q''}\left(\sum_{k=0}^{N(\alpha)}(diam(Q''))^{k-(2+2\alpha)}|\nabla^k f(x)|\right) \leq N(\alpha)+1,$$

then we proceed with the sub-division of Q'', and repeat this process until we are forced to the case (i).

□

Lemma 3.2. *Let* $3Q_\nu$ *be the interval of diameter* $3\delta_\nu$*, with the same center as* Q_ν*, then*

$$\sum_{k=0}^{N(\alpha)}\delta_\nu^{k-(2+2\alpha)}|\nabla^k f(x)| \leq C \quad \forall x\in 3Q_\nu, \tag{34}$$

where C is defined in (12)*.*

Proof. We prove the case where $1/2<\alpha\leq 1$. $0<\alpha\leq 1/2$ is similar.

Let $\tilde{Q}_\nu$ be the step before we get Q_ν. Then $Q_\nu\subset\tilde{Q}_\nu$ and diameter of $\tilde{Q}_\nu$ is $2\delta_\nu$. Since we didn't stop at $\tilde{Q}_\nu$, there is $x_0\in\tilde{Q}_\nu\subset 3Q_\nu$ such that

$$\sum_{k=0}^{3}(2\delta_\nu)^{k-(2+2\alpha)}|\nabla^k f(x_0)|\leq 4.$$

That is

$$|\nabla^k f(x_0)|\leq 4(2\delta_\nu)^{(2+2\alpha)-k}, \quad k=0,1,2,3. \tag{35}$$

Using $||f||_{C^{2,2\alpha}(\mathbb{R})}\leq 1$ and $dist(x,x_0)\leq 3\delta_\nu$, we get $\forall x\in 3Q_\nu$,

$$\begin{aligned} |\nabla^3 f(x)| &\leq |\nabla^3 f(x_0)| + 1\cdot|x-x_0|^{2\alpha-1}\\ &\leq 4(2\delta_\nu)^{2+2\alpha-3}+(3\delta_\nu)^{2\alpha-1}\\ &\leq 11\delta_\nu^{2\alpha-1}. \end{aligned} \tag{36}$$

Using (35) and (36), we get $\forall x\in 3Q_\nu$,

$$\begin{aligned} |\nabla^2 f(x)| &\leq \sup_{3Q_\nu}|\nabla^3 f|\cdot|x-x_0| + |\nabla^2 f(x_0)|\\ &\leq 11\delta_\nu^{2\alpha-1}\cdot 3\delta_\nu + 4(2\delta_\nu)^{(2+2\alpha)-2}\\ &\leq 49\delta_\nu^{2\alpha}. \end{aligned}$$

Going backwards, we get $|\nabla f(x)|\leq 179\delta_\nu^{1+2\alpha}$ and $|f(x)|\leq 601\delta_\nu^{2+2\alpha}$, $\forall x\in 3Q_\nu$.

□

Lemma 3.3. *Let Q_ν^* be the interval of diameter $(1+\epsilon_0)\delta_\nu$, with the same center as Q_ν, then*

$$\inf_{x\in Q_\nu^*}\Big(\sum_{k=0}^{N(\alpha)}\delta_\nu^{k-(2+2\alpha)}|\nabla^k f(x)|\Big)\geq c_0, \tag{37}$$

where c_0, ϵ_0 are defined in (12).

Proof. Let $B=\{x\in\mathbb{R} : dist(x,x_0)\leq \epsilon_0\delta_\nu\}$.

We prove the case where $1/2<\alpha\leq 1$. $0<\alpha\leq 1/2$ is similar.

Assume not, then $\exists x_0\in Q_\nu^*$ such that $\sum_{k=0}^3\delta_\nu^{k-(2+2\alpha)}|\nabla^k f(x_0)|<c_0$.

Using $\|f\|_{C^{2,2\alpha}(\mathbb{R})}\leq 1$ and the mean value theorem, we get

$$|\nabla^3 f(x)|\leq |\nabla^3 f(x_0)|+1\cdot|x-x_0|^{2\alpha-1}\leq (c_0+1)\delta_\nu^{2\alpha-1}\quad \forall x\in B.$$

$$|\nabla^2 f(x)|\leq \sup_B|\nabla^3 f|\cdot|x-x_0|+|\nabla^2 f(x_0)|\leq (2c_0+1)\delta_\nu^{2\alpha}\quad \forall x\in B.$$

Going backwards, we get $|\nabla f(x)|\leq (3c_0+1)\delta_\nu^{1+2\alpha}$ and $|f(x)|\leq[(3c_0+1)\epsilon_0+c_0]\delta_\nu^{2+2\alpha}$. Note $\epsilon_0<\frac{1}{10^5}$, so for any $x\in B$, $\sum_{k=0}^3\delta_\nu^{k-(2+2\alpha)}|\nabla^k f(x)|<4$, contradicting with (33).

□

Lemma 3.4. *Let $\lambda=\epsilon_0/2$. Let Q_ν^+ be the interval of diameter $(1+\lambda)\delta_\nu$, with the same center as Q_ν. Then for $z\in Q_\nu^+$, either*

$$f(z)\geq \tilde{c}\delta_\nu^{2+2\alpha}, \tag{38}$$

or

$$f(z)<\tilde{c}\delta_\nu^{2+2\alpha} \text{ and } |\nabla^2 f(z)|\geq \tilde{c}\delta_\nu^{2\alpha}, \tag{39}$$

where $\tilde{c}$ is defined in (12).

Proof. By translation we assume $z=0$, with

$$f(0)<\tilde{c}\delta_\nu^{2+2\alpha}\quad\text{and}\quad |\nabla^2 f(0)|<\tilde{c}\delta_\nu^{2\alpha}. \tag{40}$$

First, we assume $1/2<\alpha\leq 1$. Let $c>0$ be small such that $2c\delta_\nu<(diam(Q_\nu^*)-diam(Q_\nu^+))/2$. By Taylor expansion, (40) and $\|f\|_{C^{2,2\alpha}(\mathbb{R})}\leq 1$, for any $|x|<2c\delta_\nu$,

$$0\leq f(x)\leq \tilde{c}\delta_\nu^{2+2\alpha}+f'(0)x+\frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}x^2+\frac{1}{6}f'''(0)x^3+\frac{1}{6}|x|^{2+2\alpha}. \tag{41}$$

Taking x and $-x$ in (41), for any $|x|<2c\delta_\nu$,

$$|f'(0)x+\frac{1}{6}f'''(0)x^3|\leq \tilde{c}\delta_\nu^{2+2\alpha}+\frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}x^2+\frac{1}{6}|x|^{2+2\alpha}. \tag{42}$$

In particular, for any $|x|<c\delta_\nu$,

$$|f'(0)x+\frac{1}{6}f'''(0)x^3|\leq \tilde{c}\delta_\nu^{2+2\alpha}+\frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}(c\delta_\nu)^2+\frac{1}{6}|c\delta_\nu|^{2+2\alpha}=:A\delta_\nu^{2+2\alpha}. \tag{43}$$

On the other hand, by substituting x with $2x$ in (42), for any $|x| < c\delta_\nu$,

$$\begin{aligned}|f'(0)(2x) + \frac{1}{6}f'''(0)(2x)^3| &\leq \tilde{c}\delta_\nu^{2+2\alpha} + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}(2x)^2 + \frac{1}{6}|2x|^{2+2\alpha} \\ &\leq \tilde{c}\delta_\nu^{2+2\alpha} + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}(2c\delta_\nu)^2 + \frac{1}{6}|2c\delta_\nu|^{2+2\alpha} =: B\delta_\nu^{2+2\alpha}.\end{aligned} \tag{44}$$

Combining (43) and (44), we obtain for any $|x| < c\delta_\nu$,

$$|f'(0)x| \leq \frac{1}{6}(8A + B)\delta_\nu^{2+2\alpha}, \quad |\frac{1}{6}f'''(0)x^3| \leq \frac{1}{6}(2A + B)\delta_\nu^{2+2\alpha}.$$

Thus, $|f'(0)| \leq \frac{8A+B}{6c}$ and $|f'''(0)| \leq \frac{2A+B}{c^3}$. If $c = \epsilon_0/10$, $\tilde{c} = c^4$, then

$$\sum_{k=0}^{3} \delta_\nu^{k-(2+2\alpha)}|\nabla^k f(0)| \leq \tilde{c} + \frac{8A+B}{6c} + \tilde{c} + \frac{2A+B}{c^3} < 0.01^4 + 0.01 + 0.01^4 + 0.07 < c_0,$$

contradicting with (37).

Next, we deal with the case $0 < \alpha \leq 1/2$.

Let $c > 0$ be small such that $2c\delta_\nu < (diam(Q_\nu^*) - diam(Q_\nu^+))/2$. By Taylor expansion, (40) and $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$, for any $|x| < 2c\delta_\nu$,

$$0 \leq f(x) \leq \tilde{c}\delta_\nu^{2+2\alpha} + f'(0)x + \frac{1}{2}\tilde{c}\delta_\nu^{2\alpha}x^2 + \frac{1}{2}|x|^{2+2\alpha}. \tag{45}$$

If $c = \epsilon_0/10$, $\tilde{c} = c^3$, setting $x = \pm c\delta_\nu$ in (45) yields

$$|f'(0)| \leq (c^2 + \frac{1}{2}c^4 + \frac{1}{2}c^{1+2\alpha})\delta_\nu^{1+2\alpha} < 0.01\delta_\nu^{1+2\alpha}.$$

Hence

$$\sum_{k=0}^{2} \delta_\nu^{k-(2+2\alpha)}|\nabla^k f(0)| \leq \tilde{c} + 0.01 + \tilde{c} < 0.01^3 + 0.01 + 0.01^3 < c_0,$$

contradicting with (37). □

For any $z \in Q_\nu^+$, we apply Fefferman-Phong Lemma 2.3 to the function $\phi(t) = \delta_\nu^{-(2+2\alpha)} \cdot f(z + t\delta_\nu)$.

Corollary 3.5. *Let $C = 1000$. For $z \in Q_\nu^+$, there exist universal constants $r_0 > 0, \tilde{A} > 0,\ c_2 > 0$ such that, for $x \in (z - r_0\delta_\nu, z + r_0\delta_\nu)$,*

either

$$c_2\delta_\nu^{2+2\alpha} \leq f(x) \leq C\delta_\nu^{2+2\alpha}, \tag{46}$$

$$\|\sqrt{f(x)}\|_{C^1((z-r_0\delta_\nu, z+r_0\delta_\nu))} \leq \tilde{A}\delta_\nu^{\alpha},$$

$$\|\sqrt{f(x)}\|_{C^{1,\alpha}((z-r_0\delta_\nu, z+r_0\delta_\nu))} \leq \tilde{A};$$

or

$$c_2\delta_\nu^{2\alpha} \leq f''(x) \leq C\delta_\nu^{2\alpha}, \tag{47}$$

$$f(x) = f(X) + (x - X)^2 \int_0^1 f''(x + t(X - x))t\, dt, \tag{48}$$

where $x = X$ is the unique strict local minimum point of the function f in $(z - r_0\delta_\nu, z + r_0\delta_\nu)$.

Moreover, $g(x) = (x - X)(\int_0^1 f''(x + t(X - x))t\,dt)^{1/2}$ is in $C^{1,\alpha}((z - r_0\delta_\nu, z + r_0\delta_\nu))$ with $C^{1,\alpha}$ norm under control.

4. Proof of Theorem 1.1

Let $0 \le f \in C^{2,2\alpha}(\mathbb{R})$ with $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \le 1$.

4.1. Proof of sufficiency.

4.1.1. Construction of g. We write $\mathbb{R} \setminus \mathcal{F}$ (where $\mathcal{F}$ is defined in (13)) as a countable union of disjoint open intervals, so that $\mathbb{R} \setminus \mathcal{F} = \cup_{k=1}^{\infty} I_k$. Note if $\exists x_0 \in I_k$ with $f(x_0) = 0$, then $f''(x_0) \neq 0$. (If $0 < \alpha \le 1/2$, by Lemma 2.1, $|f(x_0)|$ and $|f''(x_0)|$ dominate $|f'(x_0)|$. If $1/2 < \alpha \le 1$, by Lemma 2.2, $|f(x_0)|$ and $|f''(x_0)|$ dominate $|f'(x_0)|$ and $|f'''(x_0)|$.) For each $m, k \in \mathbb{N}$, define

$$I_{k,m} = \{x \in I_k : dist(x, \mathcal{F}) > \frac{1}{m}\}, \quad B = \{x \in \mathbb{R} : f(x) = 0, f''(x) \neq 0\}.$$

Lemma 4.1. *$I_k \cap B$ is at most countable for each k, and*

$$I_k \cap B = \{\cdots x_{-2} < x_{-1} < x_0 < x_1 < x_2 \cdots\}.$$

Proof. $\forall N > 0$, we claim that $I_{k,m} \cap B \cap [-N, N]$ is finite for each $m, k \in \mathbb{N}$. Assume $I_{k,m} \cap B \cap [-N, N]$ is infinite, then $\exists x_0 \in \mathbb{R}$ such that x_0 is an accumulation point of $I_{k,m} \cap B$. So, there is a sequence $\{x_n\}$ in B such that $\lim_{n\to\infty} x_n = x_0$, and $f(x_0) = \lim_{n\to\infty} f(x_n) = 0$. Note $f \ge 0$, so $f'(x_0) = 0$.

If $f''(x_0) \neq 0$, then $x = x_0$ is a strict local minimum point of f. However, by construction, near x_0 there is a point $x_1 \in B$, so that $f(x_1) = 0$, contradicting with strict local minimality.

If $f''(x_0) = 0$, then $x_0 \in \mathcal{F}$. However, $(x_0 - \frac{1}{2m}, x_0 + \frac{1}{2m}) \cap I_{k,m} = \varnothing$, contradiction.

Now since I_k is an interval and $I_{k,m} \subset I_{k,m+1}$, any point in $I_{k,m+1} \setminus I_{k,m}$ is either on the left or right of $I_{k,m}$. The points in $I_k \cap B \cap [-N, N]$ can be ordered. The lemma follows by letting $N \to \infty$. □

We define the function g as follows. If $x \in \mathcal{F}$, set $g(x) = 0$. For each k, if $I_k \cap B = \varnothing$ in I_k, then define $g(x) = \sqrt{f(x)}$ for $x \in I_k$. Otherwise,

$$I_k \cap B = \{\cdots x_{-2} < x_{-1} < x_0 < x_1 < x_2 \cdots\}.$$

Define $g(x) = (-1)^i \sqrt{f(x)}$ for $x \in [x_{i-1}, x_i]$. Note that g changes sign when crossing each x_i in I_k.

4.1.2. C^1 regularity of g. g is continuous in each $I_k = (a_k, b_k)$. It suffices to discuss the continuity at $x_0 \in \mathcal{F}$. By Taylor expansion of f near x_0, $f(x) = O(|x - x_0|^{2+2\alpha})$, so that $| \pm \sqrt{f(x)}| = O(|x - x_0|^{1+\alpha}) \to 0$ as $x \to x_0$ and $\lim_{x \to x_0} g(x) = 0$.

Lemma 4.2. $g \in C^1(I_k)$ *for each* k.

Proof. If $I_k \cap B = \varnothing$, then $g' = \frac{f'}{2\sqrt{f}} \in C^0(I_k)$. If $I_k \cap B \neq \varnothing$, then for each $x_i \in I_k \cap B$, $x_i \in Q_\nu$ for some $\nu = \nu(x_i)$. By Corollary 3.5, only (47) holds and near x_i, f can locally be written as

$$f(x) = (x - x_i)^2 \int_0^1 f''(x + t(x_i - x))t\,dt,$$

with $\int_0^1 f''(x + t(x_i - x))t\,dt \sim \delta_\nu^{2\alpha}$. By definition of g, near x_i, $g(x) = \pm(x - x_i)(\int_0^1 f''(x + t(x_i - x))t\,dt)^{1/2}$(the sign depends only on the choice of sign of g near x_0), so that g changes sign when crossing x_i. By Corollary 3.5, g' is continuous at x_i. □

The next is a key lemma to obtain uniform estimate for g' under (2).

Lemma 4.3. *Assume condition* (2) *is satisfied. There exists a universal constant* $C_2 > 0$ *such that, for any* $x_0 \in I_k$ *with* $x_0 \in Q_\nu$ *for some* $\nu = \nu(x_0)$*, then*

$$|g'(x_0)| \leq C_2 \delta_\nu^\alpha. \tag{49}$$

Proof. By Corollary 3.5, either (46) holds which implies (49); or

$$f(x) = f(X) + (x - X)^2 \int_0^1 f''(x + t(X - x))t\,dt, \tag{50}$$

where $x = X$ is the unique strict local minimum point of the function f in $(x_0 - r_0\delta_\nu, x_0 + r_0\delta_\nu)$.

If $f(X) = 0$, then $g(x) = \pm(x - X)(\int_0^1 f''(x + t(X - x))t\,dt)^{1/2}$. By (47), local Hölder continuity of g', and $g'(X) = \sqrt{\frac{1}{2} f''(X)}$, there is universal $b > 0$ such that,

$$|g'(x)| \leq |g'(X)| + b|x - X|^\alpha \leq \sqrt{\frac{1}{2} C \delta_\nu^{2\alpha}} + b\delta_\nu^\alpha \leq C_2 \delta_\nu^\alpha, \forall x \in (x_0 - r_0\delta_\nu, x_0 + r_0\delta_\nu).$$

If $f(X) \neq 0$, then by (2) and (47),

$$M \cdot (f(X))^{\frac{\alpha}{1+\alpha}} \geq f''(X) \geq c_2 \delta_\nu^{2\alpha},$$

so that (50) reads

$$f(x) \geq f(X) \geq \left(\frac{c_2}{M}\right)^{\frac{1+\alpha}{\alpha}} \cdot \delta_\nu^{2+2\alpha}.$$

By (34), $f(x) \sim \delta_\nu^{2+2\alpha}$ and the computation is reduced to case (16). □

Corollary 4.4. *Assume $I_k = (a_k, b_k)$, where $b_k < +\infty$. Then*

$$\lim_{x\to b_k^-} g'(x) = 0.$$

Similarly, if $a_k > -\infty$, then $\lim_{x\to a_k^+} g'(x) = 0$.

Proof. By Corollary 3.5, for each $x \in I_k$, $(x - r_0\delta_{\nu(x)}, x + r_0\delta_{\nu(x)}) \subset I_k$. Hence $\lim_{x\to b_k^-} \delta_{\nu(x)} = 0$. By (49), $|g'(x)| \le C_2\delta^{\alpha}_{\nu(x)} \to 0$ as $x \to b_k^-$. □

Corollary 4.5. *For any $x_0 \in \mathcal{F}$, $g'(x)$ is continuous at x_0, with*

$$\lim_{x\to x_0} g'(x) = g'(x_0) = 0.$$

Proof. By Taylor expansion of f near x_0, $f(x) = O(|x - x_0|^{2+2\alpha})$, so that

$$\left|\frac{g(x) - g(x_0)}{x - x_0}\right| = \left|\frac{\pm\sqrt{f(x)}}{x - x_0}\right| = O(|x - x_0|^{\alpha}) \to 0 \text{ as } x \to x_0.$$

If x_0 has a neighbourhood which is contained in $\mathcal{F}$, then the result is trivial. Otherwise, x_0 is the boundary point of some interval $I_k = (a_k, b_k)$. Without loss of generality we assume $x_0 = b_k < +\infty$.

If x_0 is discrete, then x_0 is the boundary point of two consecutive intervals I_k and I_{k+1}, with $a_k < b_k = x_0 = a_{k+1} < b_{k+1}$. By Corollary 4.4,

$$\lim_{x\to b_k^-} g'(x) = \lim_{x\to a_{k+1}^+} g'(x) = 0.$$

Otherwise, $x_0 \in [x_0, a_{k+1}] \subset \mathcal{F}$ for some a_{k+1}. By Corollary 4.4 again,

$$\lim_{x\to b_k^-} g'(x) = \lim_{x\to x_0^+} g'(x) = 0.$$

□

To summarize, $g \in C^1(\mathbb{R})$, with $|g'(x)| \le C_2$, $\forall x \in \mathbb{R}$, since $\delta_\nu \le 1$.

4.1.3. Global Hölder estimate. Let $x, y \in \mathbb{R}$ with $x \neq y$.

(1) If $\exists z \in \mathbb{R} \setminus \mathcal{F}$ such that x and y are both contained in $(z - r_0\delta_{\nu(z)}, z + r_0\delta_{\nu(z)})$, then by Corollary 3.5, the Hölder estimate is trivial if (46) holds or (47) holds with $f(X) = 0$. If case (47) holds with $f(X) \neq 0$, then by (2),

$$M \cdot (f(X))^{\frac{\alpha}{1+\alpha}} \ge f''(X) \ge c_2\delta_\nu^{2\alpha},$$

so that (48) reads

$$f(x) \ge f(X) \ge \Big(\frac{c_2}{M}\Big)^{\frac{1+\alpha}{\alpha}} \cdot \delta_\nu^{2+2\alpha} \text{ and } f(y) \ge \Big(\frac{c_2}{M}\Big)^{\frac{1+\alpha}{\alpha}} \cdot \delta_\nu^{2+2\alpha}. \tag{51}$$

The computation is reduced to case (16), and $|g'(x) - g'(y)|/|x - y|^{\alpha}$ is bounded by a constant depending only on M and α.

(2) Assume $\nexists z \in \mathbb{R} \setminus \mathcal{F}$ such that x and y are both contained in $(z - r_0\delta_{\nu(z)}, z + r_0\delta_{\nu(z)})$.

(a) If $x \in \mathcal{F}$ and $y \in \mathcal{F}$, then by Corollary 4.5, $|g'(x) - g'(y)| = |0 - 0| = 0$.

(b) If $x \notin \mathcal{F}$ and $y \in \mathcal{F}$, then $x \in Q_\nu$ for some $\nu = \nu(x)$ and $|x - y| \geq r_0\delta_\nu$. By (49) and Corollary 4.5,

$$|g'(x) - g'(y)| = |g'(x)| \leq C_2\delta_\nu^\alpha \leq \frac{C_2}{r_0^\alpha} \cdot |x - y|^\alpha.$$

(c) If $x \notin \mathcal{F}$ and $y \notin \mathcal{F}$, then $x \in Q_{\nu(x)}$ and $x \in Q_{\nu(y)}$, with $|x - y| \geq r_0\delta_{\nu(x)}$ and $|x - y| \geq r_0\delta_{\nu(y)}$. By (49),

$$|g'(x) - g'(y)| \leq |g'(x)| + |g'(y)| \leq C_2\delta_{\nu(x)}^\alpha + C_2\delta_{\nu(y)}^\alpha \leq \frac{2C_2}{r_0^\alpha} \cdot |x - y|^\alpha.$$

Remark 4.6. *$C^{1,\alpha}$ estimate of g doesn't depend on the choice of sign of g in each interval I_k.*

4.2. Proof of necessity. Assume (2) doesn't hold and $f = g^2$ for some $g \in C^{1,\alpha}(\mathbb{R})$, then there is a sequence x_n in $\mathcal{A}$ such that

$$f''(x_n) \geq n f^{\frac{\alpha}{1+\alpha}}(x_n), \quad \forall n \in \mathbb{N}. \tag{52}$$

$f(x_n) > 0$, so $x_n \in Q_\nu$ for some $\nu = \nu(n)$.

In case (i) of Lemma 3.4, $f(x_n) \geq \tilde{c}\delta_\nu^{2+2\alpha}$ and $f''(x_n) < C\delta_\nu^{2\alpha}$. By (52),

$$C\delta_\nu^{2\alpha} \geq n(\tilde{c}\delta_\nu^{2+2\alpha})^{\frac{\alpha}{1+\alpha}}, \tag{53}$$

so that δ_ν get cancelled. Letting $n \to \infty$ in (53), contradiction.

In case (ii) of Lemma 3.4, $f(x_n) < \tilde{c}\delta_\nu^{2+2\alpha}$ and $f''(x_n) \geq \tilde{c}\delta_\nu^{2\alpha}$. Define $s_n = \sqrt{\frac{f(x_n)}{f''(x_n)}}$. By (52) and $\delta_\nu \leq 1$,

$$s_n = \sqrt{\frac{f(x_n)}{f''(x_n)}} \leq \sqrt{\frac{f(x_n)}{n f^{\frac{\alpha}{1+\alpha}}(x_n)}} = \frac{f^{\frac{1}{2+2\alpha}}(x_n)}{\sqrt{n}} \leq \tilde{c}^{\frac{1}{2+2\alpha}} \cdot \frac{\delta_{\nu(n)}}{\sqrt{n}} \to 0 \text{ as } n \to \infty. \tag{54}$$

If $1/2 < \alpha \leq 1$, by Taylor expansion and $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$,

$$f(x_n + s_n) \geq f(x_n) + \frac{1}{2}f''(x_n)s_n^2 - \frac{1}{6}|f'''(x_n)|s_n^3 - \frac{1}{6}s_n^{2+2\alpha}.$$

$$f(x_n + s_n) \leq f(x_n) + \frac{1}{2}f''(x_n)s_n^2 + \frac{1}{6}|f'''(x_n)|s_n^3 + \frac{1}{6}s_n^{2+2\alpha}.$$

By (34) and (54), for large n,

$$|f'''(x_n)|s_n^3 = |f'''(x_n)|s_n \cdot \frac{f(x_n)}{f''(x_n)} \leq C\delta_\nu^{2+2\alpha-3} \cdot \tilde{c}^{\frac{1}{2+2\alpha}} \cdot \frac{\delta_\nu}{\sqrt{n}} \cdot \frac{f(x_n)}{\tilde{c}\delta_\nu^{2\alpha}} \leq \frac{1}{2}f(x_n),$$

$$s_n^{2+2\alpha} = s_n^{2\alpha} \cdot \frac{f(x_n)}{f''(x_n)} \leq (\tilde{c}^{\frac{1}{2+2\alpha}} \cdot \frac{\delta_\nu}{\sqrt{n}})^{2\alpha} \cdot \frac{f(x_n)}{\tilde{c}\delta_\nu^{2\alpha}} \leq \frac{1}{2}f(x_n).$$

So, $4f(x_n) \geq f(x_n+s_n) \geq f(x_n) > 0$ for large n. By the mean value theorem,

$$\begin{aligned} 2|g'(x_n+s_n)-g'(x_n)| &= \Big| \pm \frac{f'(x_n+s_n)}{\sqrt{f(x_n+s_n)}} - (\pm \frac{f'(x_n)}{\sqrt{f(x_n)}})\Big| \\ &= \Big|\frac{f'(x_n+s_n)-f'(x_n)}{\sqrt{f(x_n+s_n)}}\Big| = \frac{|f''(\xi_n)|\cdot s_n}{\sqrt{f(x_n+s_n)}}, \end{aligned}$$

where $\xi_n \in (x_n, x_n+s_n)$. By Taylor expansion of f'', for large n,

$$f''(\xi_n) \geq f''(x_n) - |f'''(x_n)|s_n - s_n^{2\alpha} \geq \frac{1}{2}f''(x_n).$$

If $0 < \alpha \leq 1/2$, by expansion to the second order, we also have for large n, $4f(x_n) \geq f(x_n+s_n) \geq f(x_n) > 0$ and $f''(\xi_n) \geq \frac{1}{2}f''(x_n)$.

Therefore, for any $0 < \alpha \leq 1$, by (52),

$$\begin{aligned} 2|g'(x_n+s_n)-g'(x_n)| &= \frac{|f''(\xi_n)|\cdot s_n}{\sqrt{f(x_n+s_n)}} \geq \frac{\frac{1}{2}f''(x_n)\cdot s_n}{\sqrt{4f(x_n)}} = \frac{1}{4}\sqrt{f''(x_n)} \\ &= \frac{1}{4}s_n^\alpha \cdot \frac{f''(x_n)^{1/2}}{s_n^\alpha} = \frac{1}{4}s_n^\alpha \cdot \left(\frac{f''(x_n)}{f(x_n)^\alpha}\cdot f''(x_n)^\alpha\right)^{1/2} \\ &= \frac{1}{4}s_n^\alpha \cdot \left(\frac{f''(x_n)^{1+\alpha}}{f(x_n)^\alpha}\right)^{1/2} \geq \frac{1}{4}s_n^\alpha\cdot\sqrt{n}. \end{aligned}$$

Hence,

$$|g'(x_n+s_n)-g'(x_n)|/s_n^\alpha \geq \frac{1}{8}\sqrt{n} \to \infty \text{ as } n \to \infty.$$

Contradiction.

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