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The Musielak-Orlicz Herz spaces

Baohua Dong, Yu Li and Jingshi Xu

ABSTRACT. Inspired by the fact that the boundedness of certain sublinear operators on $L^p(\mathbb{R}^n)$ implies their boundedness on the Herz spaces, we obtain that the boundedness of certain sublinear operators on Musielak-Orlicz spaces implies their boundedness on Musielak-Orlicz Herz spaces.

CONTENTS

1. Introduction

Herz spaces can be traced back to the work [3] where Beurling studied certain convolution algebra by introduced the space A_q. It is the original version of the non-homogeneous Herz space. The general Herz spaces are introduced in [16] by Herz. Many researchers made their contributions to the theory of Herz spaces at that time, for example, Feichtinger ([7]), Flett ([8]), and Baernstein and Sawyer ([2]), as well as García-Cuerva and Herrero ([11, 12]). Before presenting the definition of Herz spaces, we need the necessary notations. Let $B_k := \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$ and $A_k := B_k \backslash B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\chi_k := \chi_{A_k}$ for $k \in \mathbb{Z}$, $\widetilde{\chi}_k := \chi_k$ if $k \in \mathbb{N}\backslash\{0\}$ and $\widetilde{\chi}_0 := \chi_{B_0}$, where χ_{A_k} is the characteristic function of A_k. Let $\alpha \in \mathbb{R}$, $p, q \in (0, \infty]$. The homogeneous Herz space $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ is defined by

$$\dot{K}_p^{\alpha,q}(\mathbb{R}^n) := \left\{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)} < \infty\right\},$$

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where

$$\|f\|_{\dot{K}_p^{\alpha,q}(\mathbb{R}^n)} := \left\|\left\{2^{k\alpha}\|f\chi_k\|_{L^p}\right\}_{k\in\mathbb{Z}}\right\|_{\ell^q} = \left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|f\chi_k\|_{L^p(\mathbb{R}^n)}^q\right\}^{\frac{1}{q}}.$$

The non-homogeneous Herz space $K_p^{\alpha,q}(\mathbb{R}^n)$ is defined by

$$K_p^{\alpha,q}(\mathbb{R}^n) := \left\{f \in L_{\mathrm{loc}}^p(\mathbb{R}^n) : \|f\|_{K_p^{\alpha,q}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{K_p^{\alpha,q}(\mathbb{R}^n)} := \left\|\left\{2^{k\alpha}\|f\widetilde{\chi}_k\|_{L^p}\right\}_{k\in\mathbb{N}}\right\|_{\ell^q} = \left\{\sum_{k=0}^{\infty} 2^{k\alpha q}\|f\widetilde{\chi}_k\|_{L^p(\mathbb{R}^n)}^q\right\}^{\frac{1}{q}}.$$

The usual modifications are made when $p = \infty$ or $q = \infty$.

Herz spaces are quite useful in harmonic analysis. They can be used to characterize the multipliers on Hardy spaces ([2]). They are natural spaces to consider the problems related to multidimensional θ-summation and θ-means of Fourier transforms ([9, 10, 23]), as in $L^p(\mathbb{R}^n)$ spaces and Wiener amalgam spaces $W(L^p, \ell_\infty)(\mathbb{R}^n)$ ([37, 40]). When $p = q, \alpha = \beta/p, \beta \in \mathbb{R}$, the Herz space $\dot{K}_p^{\beta/p,p}(\mathbb{R}^n)$ is exactly the weighted Lebesgue space $L^p\left(\mathbb{R}^n, |x|^\beta\right)$. One of the motivations to study the boundedness of sublinear operators in Herz spaces is due to the enlightening works of Stein ([36]), Hofmann ([17]), and Soria and Weiss ([38]). These results show that the boundedness on the space $L^p\left(\mathbb{R}^n, |x|^\beta\right)$ can be deduced from its boundedness on $L^p(\mathbb{R}^n)$ for sublinear operators T satisfying the condition (1):

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\mathrm{d}y, f \in L^1(\mathbb{R}^n) \text{ with compact support, } x \notin \operatorname{supp} f.$$

Here and below the notation "$\lesssim$" means "$\leqslant C$" where C is a constant and may change from line to line.

Because Herz spaces are generalizations of weighted Lebesgue spaces, it is natural to have the following question.

Question 1.1. Does "the boundedness of some sublinear operators on $L^p(\mathbb{R}^n)$" imply "the boundedness of the operators on Herz spaces $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ and $K_p^{\alpha,q}(\mathbb{R}^n)$"?

In [30, 32], Li, Yang and their collaborators obtained a positive answer about the question with the help of size condition (I) :

$$\begin{cases} |Tf(x)| \lesssim |x|^{-n}\|f\|_{L^1(\mathbb{R}^n)}, & \text{if } \operatorname{supp} f \subseteq A_k, |x| \geqslant 2^{k+1}, k \in \mathbb{Z}, \\ |Tf(x)| \lesssim 2^{-kn}\|f\|_{L^1(\mathbb{R}^n)}, & \text{if } \operatorname{supp} f \subseteq A_k, |x| \leqslant 2^{k-2}, k \in \mathbb{Z}, \end{cases}$$

and size condition (II) :

$$|Tf(x)| \lesssim |x|^{-n}\|f\|_{L^1(\mathbb{R}^n)},$$

when $\operatorname{supp} f \subseteq B_0$ and $|x| > 2$ or $\operatorname{supp} f \subseteq A_k$ and $|x| \geqslant 2^{k+1}$ with $k \in \mathbb{N}$, and

$$|Tf(x)| \lesssim 2^{-kn}\|f\|_{L^1(\mathbb{R}^n)},$$

when $\operatorname{supp} f \subseteq A_k$ and $|x| \leqslant 2^{k-2}$ with $k \geqslant 2$. The related theorems in [32] are the generalized versions of the results in [17, 36, 38] because the size conditions (I) and (II) are more general conditions than the condition (1).

They also generalized these results in [30, 32] to the cases of fractional integral operators T_σ $(0 < \sigma < n)$ satisfying the size condition $(\mathrm{I})_\sigma$:

$$\begin{cases} \left|T_\sigma(f)(x)\right| \lesssim |x|^{-(n-\sigma)}\|f\|_{L^1(\mathbb{R}^n)}, & \text{if } \operatorname{supp} f \subseteq A_k, |x| \geqslant 2^{k+1}, k \in \mathbb{Z}, \\ \left|T_\sigma(f)(x)\right| \lesssim 2^{-k(n-\sigma)}\|f\|_{L^1(\mathbb{R}^n)}, & \text{if } \operatorname{supp} f \subseteq A_k, |x| \leqslant 2^{k-2}, k \in \mathbb{Z}, \end{cases}$$

and size condition $(\mathrm{II})_\sigma$: let T_σ be a sublinear operator satisfying

$$\left|T_\sigma(f)(x)\right| \lesssim |x|^{-(n-\sigma)}\|f\|_{L^1},$$

when $\operatorname{supp} f \subseteq B_0$ and $|x| \geqslant 2$ or $\operatorname{supp} f \subseteq A_k$ and $|x| \geqslant 2^{k+1}$ with $k \in \mathbb{N}$, and

$$\left|T_\sigma(f)(x)\right| \lesssim 2^{-k(n-\sigma)}\|f\|_{L^1},$$

when $\operatorname{supp} f \subseteq A_k$ and $|x| \leqslant 2^{k-2}$ with $k \geqslant 2$. The size conditions $(\mathrm{I})_\sigma$ and $(\mathrm{II})_\sigma$ contain the classical fractional integral sublinear operators which satisfies

$$\left|T_\sigma(f)(x)\right| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\sigma}} \mathrm{d}y, x \notin \operatorname{supp} f$$

for any integrable function f with compact support.

Besides, there are a lot of very important research works on the theories of sublinear operators in Herz spaces, see [10, 15, 21, 22] and the references therein. After that, many researchers paid their attention on these topics and many function spaces related to Herz space appeared. We just list some but not all the references about these function spaces here, such as Herz spaces with variable exponents [1, 19, 20, 24, 28, 34, 35, 42], weighted Herz spaces and their variable exponents cases [26, 27, 29], Herz type function spaces and their variable exponents generalizations [4, 6, 14, 18, 31, 32, 33, 39, 41, 43, 44], and so on. For more details, please see the above references and references therein.

It is well know that $L^p(\mathbb{R}^n)$ is a particular case of Musielak-Orlicz spaces. In detail, denote by $L^0(\mathbb{R}^n)$ the set of all measurable functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. A function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty]$ is said to be a generalized φ-function if (1) $x \mapsto \varphi(x, |f(x)|)$ is measurable for every $f \in L^0(\mathbb{R}^n)$; (2) $t \mapsto \varphi(x,t)$ is a convex and left-continuous function with $\varphi(x,0) = 0$, $\lim_{t\to 0^+} \varphi(x,t) = 0$ and $\lim_{t\to\infty} \varphi(x,t) = \infty$, for almost every $x \in \mathbb{R}^n$. Denote by $\Phi(\mathbb{R}^n)$ the set of all generalized φ-functions. Let $\varphi \in \Phi(\mathbb{R}^n)$ and $f \in L^0(\mathbb{R}^n)$, the semimodular of f is denoted by

$$\varrho_{L^\varphi}(f) := \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\mathrm{d}x.$$

Then the Musielak-Orlicz space is defined by

$$L^\varphi(\mathbb{R}^n) := \left\{ f \in L^0(\mathbb{R}^n) : \lim_{\lambda\to 0^+} \varrho_{L^\varphi}(\lambda f) = 0 \right\}.$$

The Musielak-Orlicz spaces are Banach spaces when they are equipped with the norm

$$\|f\|_{L^{\varphi}(\mathbb{R}^n)} := \inf\left\{\lambda > 0 : \varrho_{L^{\varphi}}\left(\frac{f}{\lambda}\right) \leqslant 1\right\}.$$

The Musielak-Orlicz space $L^{\varphi}(\mathbb{R}^n)$ is precisely the Lebesgue space $L^p(\mathbb{R}^n)$ if $\varphi(x,t) = t^p, t \in [0,\infty), p \in [1,\infty)$. What is noteworthy is that the boundedness of Calderón-Zygmund singular integrals operators and some fractional integral sublinear operators on $L^{\varphi}(\mathbb{R}^n)$ can be found in [13]. Inspired by above works, in order to consider a similar question like **Question 1.1** in the new setting, we need the Musielak-Orlicz Herz spaces.

Definition 1.1. Let $\varphi \in \Phi(\mathbb{R}^n), \alpha \in \mathbb{R}, p \in (0,\infty]$.
(1) The homogeneous Musielak-Orlicz Herz space $\dot{K}^{\alpha,p}_{\varphi}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{\varphi}(\mathbb{R}^n) = \left\{f \in L^{\varphi}_{\text{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{\varphi}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{\varphi}(\mathbb{R}^n)} = \left\|\left\{2^{k\alpha}\|f\chi_k\|_{L^{\varphi}}\right\}_{k\in\mathbb{Z}}\right\|_{\ell^p}.$$

(2) The non-homogeneous Musielak-Orlicz Herz space $K^{\alpha,p}_{\varphi}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{\varphi}(\mathbb{R}^n) = \left\{f \in L^{\varphi}_{\text{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha,p}_{\varphi}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{K^{\alpha,p}_{\varphi}(\mathbb{R}^n)} = \left\|\left\{2^{k\alpha}\|f\widetilde{\chi}_k\|_{L^{\varphi}}\right\}_{k\in\mathbb{N}}\right\|_{\ell^p}.$$

They are general frameworks of many generalizations of Herz spaces, such as Herz spaces with variable exponents, weighted Herz spaces and their variable exponents cases and so on. And we are interested in the following question.

Question 1.2. Does "the boundedness of some sublinear operators on $L^{\varphi}(\mathbb{R}^n)$" imply "the boundedness of the operators on the Musielak-Orlicz Herz spaces"?

In order to solve the question, one of the difficulties is to calculate

$$\frac{\|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{\varphi}(\mathbb{R}^n)}} \text{ and } \frac{\|\chi_S\|_{L^{\varphi}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}},$$

where $B \subset \mathbb{R}^n$ is a ball and $S \subset B$ is a measurable subset. It is the main reason why we need some restrictions on the index φ (see Section 2 for details). Then we give positive answers to **Question 1.2** respectively by Theorems 1.2 and 1.3 for the sublinear operators satisfying the size conditions (I) and (II).

Theorem 1.2. Let $p \in (0,\infty]$ and $\varphi, \varphi^* \in \mathcal{A}_{\text{loc}} \cap \mathcal{A}_{\infty}$ be proper functions. Let T be a bounded sublinear operator in $L^{\varphi}(\mathbb{R}^n)$ satisfying the size condition (I). Then there exist $\delta_{\varphi}, \delta_{\varphi^*} \in (0,1)$ such that T is bounded on $\dot{K}^{\alpha,p}_{\varphi}(\mathbb{R}^n)$ for all $\alpha \in (-n\delta_{\varphi}, n\delta_{\varphi^*})$.

Theorem 1.3. Let $p \in (0, \infty]$ and $\varphi, \varphi^* \in \mathcal{A}_{\text{loc}} \cap \mathcal{A}_\infty$ be proper functions. Let T be a bounded sublinear operator in $L^\varphi(\mathbb{R}^n)$ satisfying the size condition (II). Then there exist $\delta_\varphi, \delta_{\varphi^*} \in (0,1)$ such that T is bounded on $K_\varphi^{\alpha,p}(\mathbb{R}^n)$ for all $\alpha \in (-n\delta_\varphi, n\delta_{\varphi^*})$.

Many classical operators in harmonic analysis satisfy the condition (1) which can imply size conditions (I) and (II), see [29, 32] and references therein. So the following corollary is due to Theorems 1.2 and 1.3.

Corollary 1.4. Let $p \in (0, \infty]$ and $\varphi, \varphi^* \in \mathcal{A}_{\text{loc}} \cap \mathcal{A}_\infty$ be proper functions. Let T be a bounded sublinear operator in $L^\varphi(\mathbb{R}^n)$ satisfying the condition (1). Then there exist $\delta_\varphi, \delta_{\varphi^*} \in (0,1)$ such that T is bounded respectively on $\dot{K}_\varphi^{\alpha,p}(\mathbb{R}^n)$ and $K_\varphi^{\alpha,p}(\mathbb{R}^n)$ for all $\alpha \in (-n\delta_\varphi, n\delta_{\varphi^*})$.

Next two theorems indicate that **Question 1.2** are also correct respectively for the fractional integral operators under size conditions $(\text{I})_\sigma$ and $(\text{II})_\sigma$.

Theorem 1.5. Let $p \in (0, \infty]$ and $\varphi_2, \varphi_1^* \in \mathcal{A}_{\text{loc}} \cap \mathcal{A}_\infty$ be proper functions. Let $0 < \sigma < n$, φ_1 and φ_2 be σ-connected, T_σ be a bounded sublinear operator from $L^{\varphi_1}(\mathbb{R}^n)$ to $L^{\varphi_2}(\mathbb{R}^n)$ satisfying the size condition $(\text{I})_\sigma$. Then there exist $\delta_{\varphi_2}, \delta_{\varphi_1^*} \in (0,1)$ such that T_σ is bounded from $\dot{K}_{\varphi_1}^{\alpha,p}(\mathbb{R}^n)$ to $\dot{K}_{\varphi_2}^{\alpha,p}(\mathbb{R}^n)$ for all $\alpha \in (-n\delta_{\varphi_2}, n\delta_{\varphi_1^*})$.

Theorem 1.6. Let $p \in (0, \infty]$ and $\varphi_2, \varphi_1^* \in \mathcal{A}_{\text{loc}} \cap \mathcal{A}_\infty$ be proper functions. Let $0 < \sigma < n$, φ_1 and φ_2 be σ-connected, T_σ be a bounded sublinear operator from $L^{\varphi_1}(\mathbb{R}^n)$ to $L^{\varphi_2}(\mathbb{R}^n)$ satisfying the size condition $(\text{II})_\sigma$. Then there exist $\delta_{\varphi_2}, \delta_{\varphi_1^*} \in (0,1)$ such that T_σ is bounded from $K_{\varphi_1}^{\alpha,p}(\mathbb{R}^n)$ to $K_{\varphi_2}^{\alpha,p}(\mathbb{R}^n)$ for all $\alpha \in (-n\delta_{\varphi_2}, n\delta_{\varphi_1^*})$.

The structure of the paper is as follows. Section 2 contain some useful lemmas and the proof of Theorem 1.2. Section 3 is the proof of Theorem 1.5.

2. The proof of Theorems 1.2 and 1.3

The aim of this section is to prove the Theorems 1.2 and 1.3. As mentioned in Section 1, we need to calculate the following two fractions.

$$\frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}{\|\chi_S\|_{L^\varphi(\mathbb{R}^n)}} \text{ and } \frac{\|\chi_S\|_{L^\varphi(\mathbb{R}^n)}}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}},$$

where $B \subset \mathbb{R}^n$ is a ball and $S \subset B$ is a measurable subset. So, we need the following background knowledge.

Let $\mathcal{Q}$ be a family of pairwise disjoint cubes in $\mathbb{R}^n$. Define the averaging operator $T_\mathcal{Q} : L^1_{\text{loc}}(\mathbb{R}^n) \to L^0(\mathbb{R}^n)$ by

$$T_\mathcal{Q} f := \sum_{Q \in \mathcal{Q}} \chi_Q M_Q f = \sum_{Q \in \mathcal{Q}} \chi_Q \cdot \frac{1}{|Q|} \int_Q |f(y)| \mathrm{d}y.$$

The notation $\mathcal{A}$ denotes the set of all $\varphi \in \Phi(\mathbb{R}^n)$ which have the property that the averaging operators $T_{\mathcal{Q}}$ are bounded uniformly in φ for all families $\mathcal{Q}$. In particular, denote $\mathcal{A}_{\text{loc}}$ by the set of all $\varphi \in \Phi(\mathbb{R}^n)$ which have the property that the averaging operators $T_{\mathcal{Q}}$ are bounded uniformly in φ for all families $\mathcal{Q}$ only contained a single cube, that is $\mathcal{Q} = \{Q\}$. So, $\mathcal{A} \subset \mathcal{A}_{\text{loc}}$. For more details, see page 117 in [5].

Let $\varphi \in \Phi(\mathbb{R}^n)$. The associated space of $L^\varphi(\mathbb{R}^n)$ is defined by

$$(L^\varphi(\mathbb{R}^n))' := \left\{g \in L^0(\mathbb{R}^n) : \|g\|_{(L^\varphi(\mathbb{R}^n))'} < \infty\right\},$$

where

$$\|g\|_{(L^\varphi(\mathbb{R}^n))'} := \sup_{f \in L^\varphi : \|f\|_\varphi \leqslant 1} \int_{\mathbb{R}^n} |f(x)||g(x)|\mathrm{d}x.$$

A generalized φ-function is called a proper function if the set of simple functions $S(\mathbb{R}^n)$ satisfies $S(\mathbb{R}^n) \subset L^\varphi(\mathbb{R}^n) \cap (L^\varphi(\mathbb{R}^n))'$.

Let $\varphi \in \Phi(\mathbb{R}^n)$, for every $x \in \mathbb{R}^n$, denote by $\varphi^*(x, \cdot)$ the conjugate function of $\varphi(x, \cdot)$ which is defined by

$$\varphi^*(x,t) = \sup_{\mu \geqslant 0}(\mu t - \varphi(x,\mu))$$

for all $t \in [0, \infty)$. From Corollary 2.7.9 in [5], we know that a generalized φ-function is a proper function if and only if its conjugate function is a proper function. Besides, we have the following two lemmas.

Lemma 2.1 (Lemma 2.6.5 in [5])**.** Let $\varphi \in \Phi(\mathbb{R}^n)$, $f \in L^\varphi(\mathbb{R}^n)$ and $g \in L^{\varphi^*}(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} |f||g|\mathrm{d}x \leqslant 2\|f\|_{L^\varphi(\mathbb{R}^n)}\|g\|_{L^{\varphi^*}(\mathbb{R}^n)}.$$

Lemma 2.2 (Remark 4.5.8 in [5])**.** Let $\varphi \in \Phi(\mathbb{R}^n)$ be a proper function. The following are equivalent.
(1) $\varphi \in \mathcal{A}_{\text{loc}}$.
(2) $\varphi^* \in \mathcal{A}_{\text{loc}}$.
(3) $\left\|\chi_Q\right\|_{L^\varphi(\mathbb{R}^n)} \left\|\chi_Q\right\|_{L^{\varphi^*}(\mathbb{R}^n)} \approx |Q|$, uniformly for all cubes or balls $Q \in \mathbb{R}^n$.

Let $\mathcal{A}_\infty$ be the set of all $\varphi \in \Phi(\mathbb{R}^n)$ which have the following property: for every $\tau \in (0,1)$ there exists $\beta \in (0,1)$ such that if $D \subset \mathbb{R}^n$ (measurable) satisfies

$$|Q \cap D| \leqslant \tau|Q| \text{ for all } Q \in \mathcal{Q},$$

then

$$\left\|\sum_{Q \in \mathcal{Q}} t_Q \chi_{Q \cap D}\right\|_{L^\varphi(\mathbb{R}^n)} \leqslant \beta \left\|\sum_{Q \in \mathcal{Q}} t_Q \chi_Q\right\|_{L^\varphi(\mathbb{R}^n)}$$

for any sequence $\{t_Q\}_{Q \in \mathcal{Q}} \in \mathbb{R}^{\mathcal{Q}}$. Here $\mathbb{R}^{\mathcal{Q}}$ denotes the set of all real number sequences $\{t_Q\}_{Q \in \mathcal{Q}}$. The smallest constant β for $\tau = 1/2$ is called the $\mathcal{A}_\infty$-constant of φ. More detailed information can be found in [5] on pages 151, 159 and 160.

Lemma 2.3 (Lemma 5.4.12 in [5])**.** Let $\varphi \in \mathcal{A}_\infty$ and $\mathcal{Q}$ be a family of pairwise disjoint cubes in $\mathbb{R}^n$. Then there exists $\delta \in (0,1)$ and $K \geqslant 1$ which only depend on the $\mathcal{A}_\infty$-constant of φ such that

$$\left\| \sum_{Q \in \mathcal{Q}} t_Q \left| \frac{f}{M_Q f} \right|^\delta \chi_Q \right\|_{L^\varphi(\mathbb{R}^n)} \leqslant K \left\| \sum_{Q \in \mathcal{Q}} t_Q \chi_Q \right\|_{L^\varphi(\mathbb{R}^n)}$$

for all $\{t_Q\}_{Q \in \mathcal{Q}}$, $t_Q \geqslant 0$, and all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $M_Q f \neq 0$, $Q \in \mathcal{Q}$.

The next main lemma solves the difficulty mentioned at the beginning of the section.

Lemma 2.4. Let $\varphi \in \mathcal{A}_{\mathrm{loc}} \cap \mathcal{A}_\infty$ be a proper function. Then there exists $\delta \in (0,1)$ such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\frac{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}{\|\chi_S\|_{L^\varphi(\mathbb{R}^n)}} \lesssim \frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^\varphi(\mathbb{R}^n)}}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^\delta.$$

Proof. Let B be a ball and $S \subset B$ be a measurable subset arbitrarily, then

$$\|\chi_B\|_{L^\varphi} \frac{|S|}{|B|} = \|\chi_B\|_{L^\varphi} \frac{1}{|B|} \int_{\mathbb{R}^n} \chi_B(y)\chi_S(y)\mathrm{d}y.$$

By Lemma 2.1 and 2.2,

$$\|\chi_B\|_{L^\varphi} \frac{|S|}{|B|} \leqslant \|\chi_B\|_{L^\varphi} \frac{2}{|B|} \|\chi_B\|_{L^{\varphi^*}} \|\chi_S\|_{L^\varphi} \lesssim \|\chi_S\|_{L^\varphi}.$$

Hence

$$\frac{\|\chi_B\|_{L^\varphi}}{\|\chi_S\|_{L^\varphi}} \lesssim \frac{|B|}{|S|}.$$

Next we prove the second inequality of the lemma. We take an open cube Q so that $B \subset Q \subset \sqrt{n}B$. Putting $f = \chi_S$ and $\mathcal{Q} = \{Q\}$ in Lemma 2.3 we have

$$\frac{\|\chi_S\|_{L^\varphi}}{\left\|\chi_Q\right\|_{L^\varphi}} \lesssim \left(\frac{|S|}{|Q|}\right)^\delta.$$

By virtue of $B \subset Q \subset \sqrt{n}B$,

$$\frac{\|\chi_S\|_{L^\varphi}}{\|\chi_B\|_{L^\varphi}} = \frac{\|\chi_S\|_{L^\varphi}}{\left\|\chi_Q\right\|_{L^\varphi}} \cdot \frac{\left\|\chi_Q\right\|_{L^\varphi}}{\|\chi_B\|_{L^\varphi}} \leqslant \frac{\|\chi_S\|_{L^\varphi}}{\left\|\chi_Q\right\|_{L^\varphi}} \cdot \frac{\left\|\chi_{\sqrt{n}B}\right\|_{L^\varphi}}{\|\chi_B\|_{L^\varphi}}.$$

With the help of the first inequality in the lemma that we already proved, we have

$$\frac{\|\chi_S\|_{L^\varphi}}{\|\chi_B\|_{L^\varphi}} \lesssim \left(\frac{|S|}{|Q|}\right)^\delta \cdot \frac{|\sqrt{n}B|}{|B|} \lesssim \left(\frac{|S|}{|B|}\right)^\delta.$$

Thus we finish the proof of Lemma 2.4. □

Now we turn to prove Theorem 1.2. We omit the proof of Theorem 1.3 because it is similar to the proof of Theorem 1.2.

Proof of Theorem 1.2. By the equation $f=\sum_{j\in\mathbb{Z}} f\chi_j$, we have

$$\begin{aligned}
\|Tf\|_{\dot{K}_{\varphi}^{\alpha,p}} &= \left\|\{2^{k\alpha}\|T(f)\chi_k\|_{L^\varphi}\}_{k\in\mathbb{Z}}\right\|_{\ell^p}\\
&\lesssim \left\|\left\{2^{k\alpha}\left\|\left(\sum_{j=-\infty}^{\infty}|T(f\chi_j)|\right)\chi_k\right\|_{L^\varphi}\right\}_{k\in\mathbb{Z}}\right\|_{\ell^p}\\
&\lesssim \left\|\left\{2^{k\alpha}\left\|\left(\sum_{j=-\infty}^{k-2}|T(f\chi_j)|\right)\chi_k\right\|_{L^\varphi}\right\}_{k\in\mathbb{Z}}\right\|_{\ell^p}\\
&\quad+\left\|\left\{2^{k\alpha}\left\|\left(\sum_{j=k-1}^{k+1}|T(f\chi_j)|\right)\chi_k\right\|_{L^\varphi}\right\}_{k\in\mathbb{Z}}\right\|_{\ell^p}\\
&\quad+\left\|\left\{2^{k\alpha}\left\|\left(\sum_{j=k+2}^{\infty}|T(f\chi_j)|\right)\chi_k\right\|_{L^\varphi}\right\}_{k\in\mathbb{Z}}\right\|_{\ell^p}\\
&=: (I_1+I_2+I_3)\,.
\end{aligned}$$

For I_2, using the φ-boundedness of T, we obtain

$$\begin{aligned}
I_2 &\lesssim \left\|\left\{2^{k\alpha}\sum_{j=k-1}^{k+1}\left\|T(f\chi_j)\chi_k\right\|_{L^\varphi}\right\}_{k\in\mathbb{Z}}\right\|_{\ell^p}\\
&\lesssim \left\|\left\{2^{k\alpha}\sum_{j=k-1}^{k+1}\left\|T(f\chi_j)\right\|_{L^\varphi}\right\}_{k\in\mathbb{Z}}\right\|_{\ell^p}\\
&\lesssim \left\|\left\{2^{k\alpha}\sum_{j=k-1}^{k+1}\left\|f\chi_j\right\|_{L^\varphi}\right\}_{k\in\mathbb{Z}}\right\|_{\ell^p}\\
&\lesssim \|f\|_{\dot{K}_{\varphi}^{\alpha,p}}\,.
\end{aligned}$$

For I_1, by the fact that $j\leqslant k-2$, $x\in A_k$, and Lemma 2.1, we have

$$\left|T\left(f\chi_j\right)(x)\right|\lesssim 2^{-kn}\left\|f\chi_j\right\|_{L^1}\lesssim 2^{-kn}\|f\chi_j\|_{L^\varphi}\left\|\chi_{B_j}\right\|_{L^{\varphi^*}}.$$

Then

$$
\begin{aligned}
I_1 &\lesssim \left\| \left\{ 2^{k(\alpha-n)} \left\| \left(\sum_{j=-\infty}^{k-2} \|f\chi_j\|_{L^\varphi} \left\|\chi_{B_j}\right\|_{L^{\varphi^*}} \right) \chi_k \right\|_{L^\varphi} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} \\
&\lesssim \left\| \left\{ 2^{k(\alpha-n)} \|\chi_k\|_{L^\varphi} \left(\sum_{j=-\infty}^{k-2} \|f\chi_j\|_{L^\varphi} \left\|\chi_{B_j}\right\|_{L^{\varphi^*}} \right) \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} \\
&\lesssim \left\| \left\{ \sum_{j=-\infty}^{k-2} \left(2^{k\alpha} \|f\chi_j\|_{L^\varphi} 2^{-kn} \|\chi_{B_j}\|_{L^{\varphi^*}} \|\chi_k\|_{L^\varphi} \right) \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p}.
\end{aligned}
$$

Due to Lemma 2.2 and Lemma 2.4, there exists $\delta_{\varphi^*} \in (0,1)$ such that

$$
2^{-kn} \left\|\chi_{B_j}\right\|_{L^{\varphi^*}} \|\chi_k\|_{L^\varphi} \leqslant 2^{-kn} \left\|\chi_{B_j}\right\|_{L^{\varphi^*}} \left\|\chi_{B_k}\right\|_{L^\varphi} \lesssim \frac{\|\chi_{B_j}\|_{L^{\varphi^*}}}{\|\chi_{B_k}\|_{L^{\varphi^*}}} \lesssim 2^{n\delta_{\varphi^*}(j-k)}.
$$

Then

$$
I_1 \lesssim \left\| \left\{ \sum_{j=-\infty}^{k-2} 2^{j\alpha} \|f\chi_j\|_{L^\varphi} 2^{(n\delta_{\varphi^*}-\alpha)(j-k)} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p}.
$$

If $\alpha \in (-\infty, n\delta_{\varphi^*})$,

$$
I_1 \lesssim \left\| \left\{ 2^{j\alpha} \|f\chi_j\|_{L^\varphi} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^p} = \|f\|_{\dot{K}_\varphi^{\alpha,p}}.
$$

For I_3, we use $j \geqslant k+2$ and Lemma 2.1, we have

$$
\left| T\left(f\chi_j\right)(x) \right| \chi_k(x) \lesssim 2^{-jn} \left\| f\chi_j \right\|_{L^1} \lesssim 2^{-jn} \left\| f\chi_j \right\|_{L^\varphi} \left\| \chi_{B_j} \right\|_{L^{\varphi^*}} \chi_k(x).
$$

It follows that

$$
I_3 \lesssim \left\| \left\{ \sum_{j=k+2}^{\infty} \left(2^{k\alpha-jn} \left\| f\chi_j \right\|_{L^\varphi} \left\| \chi_{B_j} \right\|_{L^{\varphi^*}} \|\chi_k\|_{L^\varphi} \right) \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p}.
$$

By Lemmas 2.2 and 2.4, there exists $\delta_\varphi \in (0,1)$ such that

$$
2^{-jn} \left\|\chi_{B_j}\right\|_{L^{\varphi^*}} \|\chi_k\|_{L^\varphi} \leqslant 2^{-jn} \left\|\chi_{B_j}\right\|_{L^{\varphi^*}} \left\|\chi_{B_k}\right\|_{L^\varphi} \lesssim \frac{\|\chi_{B_k}\|_{L^\varphi}}{\|\chi_{B_j}\|_{L^\varphi}} \lesssim 2^{n\delta_\varphi(k-j)}.
$$

Then

$$
I_3 \lesssim \left\| \left\{ \sum_{j=k+2}^{\infty} 2^{j\alpha} \left\| f\chi_j \right\|_{L^\varphi} 2^{(n\delta_\varphi+\alpha)(k-j)} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p}.
$$

If $\alpha \in (-n\delta_\varphi, \infty)$,

$$I_3 \lesssim \left\| \left\{ 2^{j\alpha} \left\| f\chi_j \right\|_{L^\varphi} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^p} = \|f\|_{\dot{K}_\varphi^{\alpha,p}}.$$

Combining the estimates on I_1, I_2 and I_3, we obtain

$$\|Tf\|_{\dot{K}_\varphi^{\alpha,p}} \lesssim \|f\|_{\dot{K}_\varphi^{\alpha,p}}.$$

This finishes the proof of Theorem 1.2. □

3. The proof of Theorems 1.5 and 1.6

In this section we aim to prove Theorems 1.5 and 1.6. For $0 < \sigma < n$, we say φ_1 and φ_2 are σ-connected if $\varphi_1, \varphi_2 \in \Phi(\mathbb{R}^n)$ and

$$|B|^{\sigma/n} \|\chi_B\|_{L^{\varphi_2}(\mathbb{R}^n)} \lesssim \|\chi_B\|_{L^{\varphi_1}(\mathbb{R}^n)}.$$

The condition is reasonable because it is nearly a necessary and sufficient condition for the boundedness of fractional integral operators between two different Musielak-Orlicz Hardy spaces, for more details please see [25].

Next, we prove Theorem 1.5 only, because the Theorem 1.6 is also similar to the proof of Theorem 1.5.

Proof of Theorem 1.5. First of all we have

$$\begin{aligned}
\|T_\sigma(f)\|_{\dot{K}_{\varphi_2}^{\alpha,p}} &= \left\| \left\{ 2^{k\alpha} \|T_\sigma(f)\chi_k\|_{L^{\varphi_2}} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} \\
&\lesssim \left\| \left\{ 2^{k\alpha} \left\| \left(\sum_{j=-\infty}^{\infty} |T_\sigma(f\chi_j)| \right) \chi_k \right\|_{L^{\varphi_2}} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} \\
&\lesssim \left\| \left\{ 2^{k\alpha} \left\| \left(\sum_{j=-\infty}^{k-2} |T_\sigma(f\chi_j)| \right) \chi_k \right\|_{L^{\varphi_2}} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} \\
&\quad + \left\| \left\{ 2^{k\alpha} \left\| \left(\sum_{j=k-1}^{k+1} |T_\sigma(f\chi_j)| \right) \chi_k \right\|_{L^{\varphi_2}} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} \\
&\quad + \left\| \left\{ 2^{k\alpha} \left\| \left(\sum_{j=k+2}^{\infty} |T_\sigma(f\chi_j)| \right) \chi_k \right\|_{L^{\varphi_2}} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} \\
&=: II_1 + II_2 + II_3.
\end{aligned}$$

For II_2, using the fact that $T_\sigma(f)$ maps $L^{\varphi_1}(\mathbb{R}^n)$ into $L^{\varphi_2}(\mathbb{R}^n)$, we obtain

$$II_2 \lesssim \left\| \left\{ 2^{k\alpha} \sum_{j=k-1}^{k+1} \left\| f\chi_j \right\|_{L^{\varphi_1}} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} \lesssim \|f\|_{\dot{K}^{\alpha,p}_{\varphi_1}}.$$

For II_1, notice that $j \leqslant k-2$, by the condition $(\mathrm{I})_\sigma$, we have

$$\left|T_\sigma\left(f\chi_j\right)(x)\right| \chi_k(x) \lesssim 2^{-k(n-\sigma)} \left\| f\chi_j \right\|_{L^1} \chi_k(x).$$

By Lemma 2.1,

$$\begin{aligned} II_1 &\lesssim \left\| \left\{ 2^{k\alpha} \left\| \left(\sum_{j=-\infty}^{k-2} 2^{-k(n-\sigma)} \|f\chi_j\|_{L^{\varphi_1}} \left\| \chi_{B_j} \right\|_{L^{\varphi_1^*}} \right) \chi_k \right\|_{L^{\varphi_2}} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} \\ &= \left\| \left\{ \sum_{j=-\infty}^{k-2} \left[2^{k(\alpha-n+\sigma)} \|f\chi_j\|_{L^{\varphi_1}} \|\chi_k\|_{L^{\varphi_2}} \left\| \chi_{B_j} \right\|_{L^{\varphi_1^*}} \right] \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} \\ &\lesssim \left\| \left\{ \sum_{j=-\infty}^{k-2} \left(2^{k(\alpha-n)} \|f\chi_j\|_{L^{\varphi_1}} \|\chi_{B_k}\|_{L^{\varphi_1^*}} \|\chi_{B_k}\|_{L^{\varphi_1}} \frac{\|\chi_{B_j}\|_{L^{\varphi_1^*}}}{\|\chi_{B_k}\|_{L^{\varphi_1^*}}} \right) \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p}. \end{aligned}$$

Due to Lemma 2.4, we have

$$\begin{aligned} II_1 &\lesssim \left\| \left\{ \sum_{j=-\infty}^{k-2} 2^{j\alpha} \|f\chi_j\|_{L^{\varphi_1}} 2^{(n\delta_{\varphi_1^*}-\alpha)(j-k)} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} \\ &\lesssim \left\| \left\{ 2^{j\alpha} \left\| f\chi_j \right\|_{L^{\varphi_1}} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} = \|f\|_{\dot{K}^{\alpha,p}_{\varphi_1}}, \end{aligned}$$

where the last inequality is due to $\alpha < n\delta_{\varphi_1^*}$.

For II_3, notice that $j \geqslant k+2$, by the condition $(\mathrm{I})_\sigma$ and Lemma 2.1, we have

$$\left|T_\sigma\left(f\chi_j\right)(x)\right| \chi_k(x) \lesssim 2^{-j(n-\sigma)} \left\| f\chi_j \right\|_{L^1} \lesssim 2^{-j(n-\sigma)} \left\| f\chi_j \right\|_{L^{\varphi_1}} \left\| \chi_{B_j} \right\|_{L^{\varphi_1^*}} \chi_k(x).$$

By Lemma 2.4 and $\alpha > -n\delta_{\varphi_2}$, we deduce that

$$\begin{aligned} II_3 &\lesssim \left\| \left\{ \sum_{j=k+2}^{\infty} \left[2^{k\alpha-j(n-\sigma)} \left\| f\chi_j \right\|_{L^{\varphi_1}} \left\| \chi_{B_j} \right\|_{L^{\varphi_1^*}} \|\chi_k\|_{L^{\varphi_2}} \right] \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} \\ &\lesssim \left\| \left\{ \sum_{j=k+2}^{\infty} \left(2^{k\alpha-jn} \left\| f\chi_j \right\|_{L^{\varphi_1}} \left\| \chi_{B_j} \right\|_{L^{\varphi_1^*}} \left\| \chi_{B_j} \right\|_{L^{\varphi_1}} \frac{\left\| \chi_{B_k} \right\|_{L^{\varphi_2}}}{\left\| \chi_{B_j} \right\|_{L^{\varphi_2}}} \right) \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} \end{aligned}$$

$$\lesssim \left\| \left\{ \sum_{j=k+2}^{\infty} 2^{j\alpha} \left\| f\chi_j \right\|_{L^{\varphi_1}} 2^{(n\delta_{\varphi_2}+\alpha)(k-j)} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p}$$

$$\lesssim \left\| \left\{ 2^{j\alpha} \left\| f\chi_j \right\|_{L^{\varphi_1}} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^p} = \|f\|_{\dot{K}^{\alpha,p}_{\varphi_1}}.$$

Combining II_1, II_2 and II_3, we obtain

$$\|Tf\|_{\dot{K}^{\alpha,p}_{\varphi_2}} \lesssim \|f\|_{\dot{K}^{\alpha,p}_{\varphi_1}}.$$

This finishes the proof of Theorem 1.5. □

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(Baohua Dong) CENTER FOR APPLIED MATHEMATICS OF JIANGSU PROVINCE, DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY OF INFORMATION SCIENCE AND TECHNOLOGY, NANJING 210044, CHINA
baohuadong@nuist.edu.cn

(Yu Li) DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY OF INFORMATION SCIENCE AND TECHNOLOGY, NANJING 210044, CHINA
wlly992021@163.com

(Jingshi Xu) CENTER FOR APPLIED MATHEMATICS OF GUANGXI, GUANGXI COLLEGES AND UNIVERSITIES KEY LABORATORY OF DATA ANALYSIS AND COMPUTATION, SCHOOL OF MATHEMATICS AND COMPUTING SCIENCE, GUILIN UNIVERSITY OF ELECTRONIC TECHNOLOGY, GUILIN 541004, CHINA
jingshixu@126.com

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