

Construction of the motivic cellular spectrum \mathbf{KO}^{geo} over $Spec(\mathbb{Z})$

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ABSTRACT. We construct a periodic motivic spectrum over $Spec(\mathbb{Z})$ which when pulled back to any scheme S with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$ is the HP^1 -spectrum constructed by Panin and Walter. This spectrum \mathbf{KO}^{geo} is constructed using closed subschemes of the Grassmannians $Gr(r, n)$. Using this we show that \mathbf{KO}^{geo} is cellular.

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1. Introduction

Throughout this paper, all schemes we consider are separated and quasi-compact. Given any scheme S , we let Sch_S be a small category equivalent to the category of S -schemes of finite type. After fixing Sch_S , the small categories Sch_S^{aff} , Sm_S and Sm_S^{aff} are the full subcategories of Sch_S generated by (globally) affine, smooth and smooth affine S -schemes respectively. Let *ind-scheme* refer to any presheaf on the category Sm_S which is a directed colimit of representable presheaves.

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Panin and Walter in [PW18] construct an HP^1 -spectrum \mathbf{BO} over any regular Noetherian finite-dimensional scheme S containing $\frac{1}{2}$ and show that it is isomorphic to Hornbostel's hermitian K -theory spectrum \mathbf{KO} in the stable motivic homotopy category $SH(S)$. Here $HP^1 \simeq S^{4,2}$ is the quaternionic projective line. The main advantage of their construction is that $\mathbf{BO}_{2i} = \mathbb{Z} \times RGr$ and $\mathbf{BO}_{2i+1} = \mathbb{Z} \times HGr$ are ind-schemes $\mathbb{Z} \times RGr = \text{colim}_n RGr(n, 2n)$ and $\mathbb{Z} \times HGr = \text{colim}_n HGr(2n, 4n)$. Here $RGr(r, n)$ and $HGr(2r, 2n)$ are open subschemes of a Grassmannian scheme of appropriate degree. This makes it easier to prove many properties. For example, Røndigs and Østvær in [RØ16] use this model to compute the slice spectral sequence of hermitian K -theory.

In this paper, we remove the $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$ condition and extend the construction of \mathbf{BO} to arbitrary schemes. We will denote this spectrum by \mathbf{KO}^{geo} to be more in line with standard notations. Firstly for any S -scheme X , we define $KSp^\perp(X)$ to be the K -theory space associated to the symmetric monoidal category of unimodular alternating forms over X . This gives us a functor $KSp^\perp : Sm_S^{op} \rightarrow \mathbf{sSet}$ (cf. Def. 2.4). Using the presheaf KSp^\perp , we can extend [PW18, Thm. 8.2] to get isomorphisms

$$\mathbb{Z} \times HGr \cong \mathbb{Z} \times BSp_\infty \cong R\Omega_S^1 B\left(\coprod_n BSp_n\right) \cong KSp^\perp$$

in the unstable motivic homotopy category $H_*(S)$ over any scheme S . As the \mathbb{A}^1 -invariance of orthogonal and symplectic K -theories are only known when 2 is invertible, we still cannot extend the representability of hermitian K -theory [PW18, Thm. 5.1] to $\text{Spec}(\mathbb{Z})$. We, however, prove a weaker result for KSp^\perp in section 3 using results from [AHW18].

Theorem. *Let S be ind-smooth over a Dedekind ring with perfect residue fields. Then, for any affine $\text{Spec}(R) \cong X \in Sm_S^{aff}$ there are isomorphisms*

$$[S^n \wedge X_+, \mathbb{Z} \times HGr]_{\mathbb{A}^1} \cong \pi_n \text{Sing}^{\mathbb{A}^1}(KSp^\perp(X))$$

for all $n \in \mathbb{N}$. In particular, this holds over $\text{Spec}(\mathbb{Z})$.

Stably, we are able to extend several results about \mathbf{BO} to $\text{Spec}(\mathbb{Z})$. Collecting all the results from sections 5 and 6 we get the following theorem.

Theorem. *For any scheme S , there exists a motivic cellular HP^1 -spectrum*

$$\mathbf{KO}_S^{geo} = (\mathbf{KO}_0^{geo}, \mathbf{KO}_1^{geo}, \dots) \in SH(S)_{HP^1} \cong SH(S)$$

such that,

- (1) $\mathbf{KO}_{2n}^{geo} \cong \mathbb{Z} \times RGr$ and $\mathbf{KO}_{2n+1}^{geo} \cong \mathbb{Z} \times HGr \cong KSp^\perp$ in $H_*(S)$;
- (2) $\Omega_{HP^1}^2 \mathbf{KO}^{geo} \cong \mathbf{KO}^{geo}$ in $SH(S)_{HP^1}$ and hence $\Omega_T^4 \mathbf{KO}^{geo} \cong \mathbf{KO}^{geo}$ as objects in $SH(S)$;
- (3) for any morphism of schemes $f : S_1 \rightarrow S_2$, there exists a canonical isomorphism $Lf^* \mathbf{KO}_{S_2}^{geo} \xrightarrow{\sim} \mathbf{KO}_{S_1}^{geo}$ in $SH(S_1)$;

(4) if $f : S \rightarrow \text{Spec}(\mathbb{Z})$ is any scheme with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$, $Lf^*\mathbf{KO}^{geo}$ is isomorphic to the motivic spectrum \mathbf{BO} given in [PW18].

In particular when S is regular Noetherian of finite dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$, \mathbf{KO}^{geo} represents hermitian K-theory.

We do not know what cohomology theory \mathbf{KO}^{geo} represents over $\text{Spec}(\mathbb{Z})$. The hope is that it represents some version of hermitian K-theory. In a recent paper, Schlichting [Sch19] introduced the notion of K-theory of forms which generalises the K-theory of spaces with duality. In this formalism $\mathbf{Symb}(X)$ becomes the category of quadratic spaces for a suitable choice of category with forms structure on vector bundles $Vect(X)$. If this theory satisfies Nisnevich excision and \mathbb{A}^1 -invariance then KSp will represent it in the unstable homotopy category. It is still unknown if this is true. However, a recent paper by Bachmann and Wickelgren ([BW21]) suggests that there is a version of the hermitian K-theory ring spectrum which can be defined over arbitrary schemes (although it might not represent hermitian K-theory any more). It is unknown if this spectrum is stably equivalent to ours when 2 is not invertible.

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2. Hermitian K-theory

Hermitian K-theory evolved out of the study of bilinear forms over rings and more generally schemes. Given a scheme X and a quasicoherent \mathcal{O}_X -module \mathcal{E} , recall that a bilinear form on \mathcal{E} is a morphism of \mathcal{O}_X -modules $\mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{O}_X$. We call a bilinear form *unimodular* (sometimes referred to as non-degenerate) if the adjoint map $\mathcal{E} \rightarrow \mathcal{E}^*$ is an isomorphism. When \mathcal{E} is isomorphic to a trivial vector bundle of rank n , each bilinear form can be represented by an element of $GL_n(\Gamma(X, \mathcal{O}_X))$. We will be interested in two classes of bilinear forms in particular. A *symmetric bilinear form* is a bilinear form ψ such that $\psi\tau = \psi$ where $\tau : \mathcal{E} \otimes \mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes \mathcal{E}$ is the switch map, an *alternating bilinear form* on \mathcal{E} is a bilinear form ϕ , such that $\phi \circ \Delta = 0$, where Δ is the diagonal map of sheaves $\Delta : \mathcal{E}_X \rightarrow \mathcal{E} \otimes \mathcal{E}$. We call a vector bundle equipped with a unimodular alternating form (resp. unimodular symmetric form) a symplectic space (resp. symmetric space). Over any scheme X , $H_+ = (\mathcal{O}_X^{\oplus 2}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ and $H_- = (\mathcal{O}_X^{\oplus 2}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$ are the hyperbolic symmetric and symplectic spaces respectively. Let $\mathbf{Symb}(X)$ and $\mathbf{Symp}(X)$ denote the categories of symmetric and symplectic spaces over a scheme X respectively, where a morphism $f : (V, \phi) \rightarrow (W, \psi)$ is a morphism

of vector bundles $f : V \rightarrow W$ such that $f^*\psi f = \phi$. The orthogonal sum \perp turns $\mathbf{Symm}(X)$ and $\mathbf{Symp}(X)$ into (essentially small) symmetric monoidal categories. For objects $\mathcal{E} \in \mathbf{Symm}(X)$ and $\mathcal{F} \in \mathbf{Symp}(X)$ we denote their corresponding isomorphism groupoids by $O(\mathcal{E})$ and $Sp(\mathcal{F})$ respectively.

Definition 2.1. Let R be a commutative ring.

- (1) The *symplectic K-theory space* $KSp^\perp(R)$ is the space $K^\perp(i\mathbf{Symm}(\mathrm{Spec}(R)))$ and $KSp_n(R) = \pi_n KSp^\perp(R)$ are the symplectic K-groups.
- (2) The *orthogonal K-theory space* $KO^\perp(R)$ is the space $K^\perp(i\mathbf{Symm}(\mathrm{Spec}(R)))$ and $KO_n^\perp(R) = \pi_n KO^\perp(R)$ the orthogonal K-groups.

Here $K^\perp(-)$ is the K-theory space of a symmetric monoidal category [Wei13, Def. 4.3]. These were defined in [Kar73] as ${}_1K^h(R)$ and ${}_{-1}K^h(R)$ respectively for rings where 2 is invertible. The zeroth orthogonal K-group $KO_0^\perp(R)$ is equal to the classical Grothendieck-Witt group $GW(R)$ (called the *Witt-Grothendieck group* in [Lam05, Def. 1.1]).

There exist monoidal functors

$$\coprod_n O(H_+^n(R)) \rightarrow \mathbf{Symm}(\mathrm{Spec}(R)) \quad \text{and} \quad \coprod_n Sp(H_-^n(R)) \rightarrow \mathbf{Symp}(\mathrm{Spec}(R))$$

given on objects by $n \mapsto H_+^n$ and $n \mapsto H_-^n$ respectively. These then induce maps

$$\Omega_s B\left(\coprod_n BO(H_+^n)\right)(R) \rightarrow KO^\perp(R)$$

$$\Omega_s B\left(\coprod_n BSp(H_-^n)\right)(R) \rightarrow KSp^\perp(R)$$

of group completions. When $\frac{1}{2} \in R$, [Lam06, Lem. 1.5] implies that $\coprod_n O(H_+^n)$ is a cofinal monoidal subcategory of $i\mathbf{Symm}(\mathrm{Spec}(R))$. We prove an analogous result for symplectic spaces below.

Lemma 2.2. *Let R be any ring. Every symplectic space over R is isometric to a subspace of H_-^n for some n .*

Proof. First note that H_-^n is isometric to $(R^{2n}, \begin{pmatrix} 0 & -I_{2n} \\ I_{2n} & 0 \end{pmatrix})$. Let (P, ϕ) be a symplectic space over R . As P is projective, there exists Q such that $P \oplus Q \cong R^m$ for some m . The symplectic space $P \perp P \perp H_-(Q)$, where $H_-(Q) = (Q \oplus Q^*, \begin{pmatrix} 0 & I_Q \\ -I_Q & 0 \end{pmatrix})$, then has an underlying space isomorphic to R^{2m} . Thus we have reduced to the case when the underlying module is free. Let (R^{2m}, S) be a symplectic space. As S is an alternating invertible matrix, we have $S^{-1} = L - L^T$ for some strictly lower triangular matrix L . We will show that $(R^{4m}, \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix})$ is isometric to $(R^{4m}, \begin{pmatrix} 0 & -I_{2m} \\ I_{2m} & 0 \end{pmatrix})$. Consider the matrix $\begin{pmatrix} L & I_{2m} \\ L^T & I_{2m} \end{pmatrix}$. Its transpose

is $\begin{pmatrix} L^T & L \\ I_{2m} & I_{2m} \end{pmatrix}$ and hence we have

$$\begin{aligned} \begin{pmatrix} L^T & L \\ I_{2m} & I_{2m} \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix} \begin{pmatrix} L & I_{2m} \\ L^T & I_{2m} \end{pmatrix} &= \begin{pmatrix} L^T & L \\ I_{2m} & I_{2m} \end{pmatrix} \begin{pmatrix} SL & S \\ -SL^T & -S \end{pmatrix} \\ &= \begin{pmatrix} L^T SL - LSL^T & -I_{2m} \\ I_{2m} & 0 \end{pmatrix} \end{aligned} \quad (2.1)$$

As $L^T - L$ is the two sided inverse of S , we have

$$L^T SL - LSL^T = L^T(I_{2m} - SL^T) - LSL^T = L^T - L^T SL^T - LSL^T = L^T - L^T = 0$$

proving that $(R^{4m}, \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix}) \cong (R^{4m}, \begin{pmatrix} 0 & -I_{2m} \\ I_{2m} & 0 \end{pmatrix})$. \square

From this, we have the following theorem.

Theorem 2.3. *The morphism $\Omega_s B(\coprod_n BS p(H^n))(R) \rightarrow KSp^\perp(R)$ induces isomorphisms $\pi_n \Omega_s B(\coprod_n BS p(H^n))(R) \xrightarrow{\sim} \pi_n KSp^\perp(R)$ for all $n \geq 2$.*

Proof. This follows from the Cofinality Theorem [Gra76] and Lemma 2.2 above. \square

There are several ways to extend the constructions of $KO^\perp(R)$ and $KSp^\perp(R)$ to arbitrary schemes. Firstly, we note that for any scheme S , KO^\perp and KSp^\perp define lax-functors $(Sch_S^{aff})^{op} \rightarrow \mathbf{sSet}$ from affine S -schemes to simplicial sets. The naive way to extend this to arbitrary schemes is to use $\mathbf{Sym}(X)$ for non-affine schemes as well. For any $X \in Sm_S$, we define the categories of big symmetric and big symplectic spaces, $\mathbf{Sym}_{Sm_S}(X)$ and $\mathbf{Symm}_{Sm_S}(X)$ respectively, along the same lines as the category of big vector bundles (i.e. we fix a choice of pullback for each form)[Wei13, Sec. 10.5]. These are equivalent as symmetric monoidal categories to $\mathbf{Sym}(X)$ and $\mathbf{Symm}(X)$ respectively. We can then define simplicial presheaves

$$KO^\perp : Sm_S^{op} \rightarrow \mathbf{sSet}$$

$$KSp^\perp : Sm_S^{op} \rightarrow \mathbf{sSet}$$

extending $KO^\perp(R)$ and $KSp(R)^\perp$ respectively.

Definition 2.4. Let S be any scheme.

- (1) $KO^\perp : Sm_S^{op} \rightarrow \mathbf{sSet}$ is the simplicial presheaf given by $KO^\perp(X) = K^\perp(i\mathbf{Symm}_{Sm_S}(X))$.
- (2) $KSp^\perp : Sm_S^{op} \rightarrow \mathbf{sSet}$ is the simplicial presheaf given by $KSp^\perp(X) = K^\perp(i\mathbf{Sym}_{Sm_S}(X))$.

For simplicity, we will denote $Sp(H^n)$ by Sp_{2n} from now on.

Theorem 2.5. *The morphism $\coprod_n Sp_{2n}(-) \rightarrow i\mathbf{Symp}_{Sm_S^{op}}(-)$ in $\mathbf{Fun}(Sm_S^{op}, \mathbf{Cat})$ given by $n \mapsto H_-^n$ induces a local weak equivalence of simplicial presheaves*

$$\coprod_n BSp_{2n} \rightarrow B(i\mathbf{Symp}_{Sm_S^{op}})$$

with respect to the Zariski topology on Sm_S . In particular they are isomorphic as objects in $H_*(S)$.

Proof. Over a local ring R , every symplectic space is isometric to some H_-^n [Lam06, Thm. 5.8]. This implies that $\coprod_n Sp_{2n}(R) \rightarrow i\mathbf{Symp}(R)$ is an equivalence of groupoids and hence induces a weak equivalence of simplicial sets

$$\coprod_n BSp_{2n}(R) \rightarrow Bi\mathbf{Symp}(R).$$

Since BSp_{2n} is a degreewise representable simplicial sheaf, the stalks are just $\coprod_n BSp_{2n}(\mathcal{O}_{U,u})$, the evaluations at $\mathrm{Spec}(\mathcal{O}_{U,u})$. The isomorphism of simplicial sets, $\mathrm{colim}_{x \in U} Bi\mathbf{Symp}(U) \xrightarrow{\sim} Bi\mathbf{Symp}(\mathcal{O}_{U,u})$, then induces a weak equivalence $\coprod_n BSp_{2n}(\mathcal{O}_{U,u}) \rightarrow Bi\mathbf{Symp}(\mathcal{O}_{U,u})$ of stalks in the Zariski topology and so we are done. \square

This result implies that the induced map of objectwise group completions is also a weak equivalence.

Corollary 2.6. *For any scheme S , there are isomorphisms*

$$\Omega_S^1 B(\coprod_n BSp_{2n}) \xrightarrow{\sim} K^\perp(\coprod_n Sp_n) \xrightarrow{\sim} KSp^\perp$$

in $H_*(S)$.

Proof. Theorem 2.5 implies $KSp^\perp(\coprod_n Sp_n(\mathcal{O}_{U,u})) \xrightarrow{\sim} KSp^\perp(\mathcal{O}_{U,u})$ for any point (U, u) . Therefore, it is enough to show that these spaces are weakly equivalent to the corresponding stalks. This follows from the analogous result for the classifying spaces and the construction of group completions given in [Gra76]. \square

Remark 2.7. Currently there is some ambiguity regarding the correct definition of symplectic K-theory over an arbitrary scheme. In general the hermitian K-theory for categories with duality gives us Grothendieck-Witt spaces $GW(X)$ and $GW^-(X)$ ([Sch10a]) for any scheme X . These spaces give us all the desired properties for regular Noetherian schemes with $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. In the general case, a recent paper [CDH⁺23] constructs multiple models of Grothendieck-Witt spaces (actually Ω -spectra) which are equivalent when 2 is invertible.

Let X be a scheme, $i : U \hookrightarrow X$ be an open embedding and $n \in \mathbb{N}$. We denote by $KO^{[n]}(X)$, $KSp^{[n]}(X)$, $KO^{[n]}(X, U)$ and $KSp^{[n]}(X, U)$, the Grothendieck-Witt

spaces as in [Sch10b]. By [Sch10b, Sec. 8 Cor. 1] we have homotopy equivalences

$$KSp^{[n]}(X) \xrightarrow{\sim} KO^{[n+4k+2]}(X) \quad \text{and} \quad KSp^{[n]}(X, U) \xrightarrow{\sim} KO^{[n+4k+2]}(X, U) \tag{2.2}$$

for all $n, k \in \mathbb{Z}$ and for any open embedding $i : U \hookrightarrow X$.

Theorem 2.8 ([PW18, Thm. 5.1]). *Let S be any regular Noetherian separated scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. For any $X \in Sm_S$ and any $n \geq 0$, there is an isomorphism of groups*

$$KO_i^{[n]}(X) = \pi_i(KO^{[n]}(X)) \cong [S^i \wedge X_+, KO^{[n]}]_{\mathbb{A}^1} \tag{2.3}$$

In the case of affine schemes where two is invertible, we get back KSp^\perp and KO^\perp .

Theorem 2.9. *Let $X \cong \text{Spec}(R)$ and $\frac{1}{2} \in R$, we then have homotopy equivalences*

$$KO^\perp(X) \xrightarrow{\sim} KO^{[0]}(X) \tag{2.4}$$

$$KSp^\perp(X) \xrightarrow{\sim} KSp^{[0]}(X). \tag{2.5}$$

Proof. When $\frac{1}{2} \in R$, every skew-symmetric form is alternating and hence $KO^\perp(X)$ and $KSp^\perp(X)$ are equal to ${}_1K^h(R)$ and ${}_{-1}K^h(R)$ of [Kar73] respectively. The results [Sch04, Cor. 4.6] and [Sch10b, Prop. 6] supply the desired homotopy equivalences. □

3. Unstable representability

The main reference for this section is [PW18, Sec. 8]. We fix a noetherian scheme S of finite Krull dimension. The functor $X \mapsto Sp_{2n}(\Gamma(X, \mathcal{O}_X))$ is representable by a group scheme, in fact a closed subgroup scheme of GL_{2n} , which we also denote by Sp_{2n} . There are morphisms of schemes $Sp_{2n} \rightarrow Sp_{2n+2}$ given by $A \mapsto A \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for each n . Let $Sp_\infty = \text{colim}_n Sp_{2n}$ denote the colimit which is a group object in the category of motivic spaces. Let $HGr(2r, 2n)$ be the quaternionic Grassmannian scheme which classifies rank $2r$ subbundles of H^{2n} and HGr is the ind-scheme $\text{colim}_n HGr(2n, 4n)$.

Theorem 3.1. *Over any noetherian scheme S of finite Krull dimension we have a sequence of isomorphisms*

$$\mathbb{Z} \times HGr \cong \mathbb{Z} \times BSp_\infty \cong R\Omega_s^1 B\left(\prod_n Sp_{2n}\right) \cong KSp^\perp$$

as objects in $H_*(S)$.

This is equivalent to [PW18, Thm. 8.2] if 2 is invertible. We need the following lemma which is essentially [MV99, Thm. 4.1] but tweaked to correct a mistake initially pointed out in [ST15]. Recall that a pointed graded simplicial sheaves of monoids is a quadruple $(M, +, \alpha, f)$ where $(M, +)$ is a sheaf of

simplicial monoids and $\alpha : \mathbb{N} \rightarrow M$, $f : M \rightarrow \mathbb{N}$ are morphisms of sheaf of monoids with $f\alpha = Id$. In the lemma below, by sheaf we mean sheaf over Sm_S with the Nisnevich topology.

Lemma 3.2. *Let $(M, +, \alpha, f)$ be a pointed graded simplicial sheaf of monoids over the with $M_\infty = \text{colim}_n f^{-1}(n)$. Assume the following three conditions hold.*

- (1) *The map $\underline{\pi}_0^{\mathbb{A}^1}(f) : \underline{\pi}_0^{\mathbb{A}^1}(M) \rightarrow \mathbb{N}$ is an isomorphism of constant sheaves.*
- (2) *The monoid $(M, +)$ is commutative in $H_*(S)$ under the induced monoidal structure.*
- (3) *The diagram*

$$\begin{array}{ccc} M_n \times M_n & \xrightarrow{+} & M_{2n} \\ \downarrow & & \downarrow \alpha(2)+ \\ M_{n+1} \times M_{n+1} & \longrightarrow & M_{2n+2} \end{array}$$

commutes in $H_(S)$.*

Then the canonical morphism $M_\infty \times \mathbb{Z} \rightarrow R\Omega_S^1 BM$ is an \mathbb{A}^1 -weak equivalence.

Here $\underline{\pi}_0^{\mathbb{A}^1}(\mathcal{X})$ is just the sheafification of the presheaf of sets

$$U \mapsto [U, \mathcal{X}]_{\mathbb{A}^1}.$$

Proof. By [MV99, Lem. 4.1.1], we can replace M with a term wise free simplicial monoid and hence assume $M^+ \cong R\Omega_S^1 BM$. By [MV99, Lem. 4.1.7], there is an \mathbb{A}^1 -fibrant replacement functor $M \rightarrow Ex_{\mathbb{A}^1}(M)$ taking monoid objects to monoid objects. As $\underline{\pi}_0^{\mathbb{A}^1}(f) : \underline{\pi}_0^{\mathbb{A}^1}(M) \rightarrow \mathbb{N}$ is an isomorphism, $Ex_{\mathbb{A}^1}(M)$ is also graded as

$$\underline{\pi}_0^{\mathbb{A}^1}(Ex_{\mathbb{A}^1}(M))(U) \cong \pi_0(Ex_{\mathbb{A}^1}(M))(U).$$

The morphism $M \rightarrow Ex_{\mathbb{A}^1}(M)$ induces an \mathbb{A}^1 -weak equivalence of each graded component as they are disjoint and hence \mathbb{A}^1 -weak equivalences $R\Omega_S^1 BM \xrightarrow{\sim} R\Omega_S^1 BEx_{\mathbb{A}^1} M$ and $M_\infty \xrightarrow{\sim} Ex_{\mathbb{A}^1}(M)_\infty$ of homotopy colimits. Therefore, we can replace M and M_∞ with $Ex_{\mathbb{A}^1}(M)$ and $(Ex_{\mathbb{A}^1}(M))_\infty$ respectively and reduce to the situation where M is \mathbb{A}^1 -fibrant. Hence, we can assume $(M, +, 0)$ is commutative in the simplicial homotopy category $H^s(Sm_S)$ and the diagram in (3) commutes up to simplicial homotopy. Now we need to show that $M_\infty \times \mathbb{Z} \rightarrow M^+$ is a Nisnevich local weak equivalence of simplicial sheaves. As the Nisnevich site has enough points, we use the stalk functors to reduce to the case where all objects are Kan complexes. The first two conditions then imply that the map $M_\infty \times \mathbb{Z} \rightarrow M^+$, where M^+ is the group completion of the simplicial monoid M , is a homology isomorphism. Condition (3) implies that M_∞ is an H-space and therefore $\pi_1(M_\infty)$ is abelian and acts trivially on all higher homotopy groups. The map is then a weak equivalence by Whitehead's theorem. \square

Remark 3.3. Condition 3 was not part of [MV99, Thm. 4.1.10]. As the counterexample in [ST15, Remark 8.5] shows, this additional condition is necessary.

Proof of Theorem 3.1. Firstly by Corollary 2.6 we have $R\Omega_s^1 B(\coprod_n BSp_{2n}) \xrightarrow{\sim} KSp^1$ in $H_*(S)$. We will show that the graded sheaf of monoids $\coprod_n BSp_{2n}$ satisfies the conditions of Lemma 3.2. Condition (1) is clear. For (2), we need $\coprod_n BSp_{2n}$ to be commutative in $H_*(S)$. Let ΔR be the simplicial ring with $\Delta R_n = R[t_0, \dots, t_n]/(t_0 + \dots + t_n - 1)$ and structure maps same as the topological simplex. Taking stalks we can reduce to the case of showing the simplicial monoid $\coprod_n \text{Sing}^{\text{A}^1}(BSp_{2n})(R) = \coprod_n BSp_{2n}(\Delta R)$ is homotopy commutative for any ring R . Fixing $n, m \in \mathbb{N}$, there exists a permutation matrix $P_{n,m} \in GL_{2n+2m}(R)$ such that,

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = P_{n,m} \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} P_{n,m}^{-1}$$

in $Sp_{2n+2m}(R)$ for any $A \in Sp_{2n}(R)$ and $B \in Sp_{2m}(R)$. From [Weil13, III.1.2.1] we see that $P_{n,m}$ is a product of elementary matrices $P_{n,m} = E_1 \dots E_k$. As these block matrices are of even rank, we can choose $E_i \in Sp_{2n+2m}(R)$. Each of these E_i can be written as the sum of the identity matrix and a nilpotent matrix $E_i = I + N_i$. The matrix $F = I + tN_i$ is then an element of $Sp_{2n+2m}(R[t])$ and gives us a path between E_i and I in $Sp_{2n+2m}(\Delta R)$. Therefore, there is a simplicial homotopy between the maps $(A, B) \mapsto A \oplus B$ and $(A, B) \mapsto B \oplus A$ as functors $Sp_{2n}(\Delta R) \times Sp_{2m}(\Delta R) \rightarrow Sp_{2n+2m}(\Delta R)$. For (3), taking stalks again we need to show that for any ring R and any n the maps $Sp_{2n}(\Delta R) \times Sp_{2n}(\Delta R) \rightarrow Sp_{4n+4}(\Delta R)$ given by

$$(A, B) \mapsto A \oplus B \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$(A, B) \mapsto A \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus B \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

are homotopic. Like (2) these maps are equal up to conjugation by a permutation matrix and are simplicially homotopic. Hence, we get an isomorphism $\mathbb{Z} \times BSp_\infty \xrightarrow{\sim} R\Omega_s^1 B(\coprod_n BSp_{2n})$ in $H_*(S)$. Finally, we have $HGr(2n, \infty) \xrightarrow{\sim} BSp_{2n}$ from [PW18, Sec 8.] inducing $\mathbb{Z} \times HGr \xrightarrow{\sim} \mathbb{Z} \times BSp_\infty$ in $H_*(S)$. Note that Panin and Walter assume the underlying scheme is regular with 2 invertible but the proof works for arbitrary schemes. \square

The above proof also shows that $\mathbb{Z} \times HGr$ is a unital commutative monoid object in $H_*(S)$. Using results from [AHW18], we can say more in the case of affine schemes.

Theorem 3.4. *Let S be ind-smooth over a Dedekind ring with perfect residue fields. Then, for any $X \in Sm_S^{\text{aff}}$ there is an isomorphism,*

$$[X_+, \mathbb{Z} \times HGr]_{\mathbb{A}^1} \cong KSp_0^1(X)$$

of groups.

Proof. By [AHW18, Ex. 2.3.4] we have for each n , bijections of sets

$$[X_+, BSp_{2n}]_{\mathbb{A}^1} \cong \pi_0(B_{Nis}Sp_{2n})(X) \cong \text{Sym}p_n(X)$$

where $\text{Sym}p_n(X)$ is the set of isometry classes of rank $2n$ symplectic spaces over X and $B_{Nis}Sp_{2n}$ is a Nisnevich fibrant replacement of BSp_{2n} . As the filtered colimit $BSp_\infty = \text{colim}_n BSp_{2n}$ is also the homotopy colimit, the sheaf $\underline{\mathbb{Z}} \times \text{colim}_n B_{Nis}Sp_{2n}$ is a Nisnevich fibrant replacement of $\mathbb{Z} \times BSp_\infty$ giving,

$$[X_+, \mathbb{Z} \times BSp_\infty]_{\mathbb{A}^1} \cong \underline{\mathbb{Z}}(X) \times \text{colim}_n \pi_0(B_{Nis}Sp_{2n}(X)) \cong \underline{\mathbb{Z}}(X) \times \text{colim}_n \text{Sym}p_n(X)$$

as sets. There is a map $\underline{\mathbb{Z}}(X) \times \text{colim}_n \text{Sym}p_n(X) \rightarrow KSp_0^\perp(X)$ which is given by

$$(i, [A]) \mapsto [A] - \left(\frac{\text{rank}(A)}{2} - i\right)[H_-]$$

on each connected open subscheme of X . As the monoid structure on $\mathbb{Z} \times BSp_\infty$ is induced by \perp , this is a monoid homomorphism. We will show that this map is a bijection. First, we note that it is enough to prove this in the case when X is connected as all our schemes are locally connected. Given any $[A] - [B] \in KSp_0^\perp(X)$, by Theorem 2.3, there exists a symplectic space C such that $B \perp C \cong H_-^k$ for some k . Then, $[A] - [B] = [A] + [C] - (k[H_-]) = [A \perp C] - k[H_-]$ and hence the map is surjective. Now suppose $[A] - k[H_-] = [B] - j[H_-]$ in $KSp^\perp(X)$. Then, $A \perp H_-^{j+p} \cong B \perp H_-^{k+p}$ for some p . Hence, they have the same preimage in $\mathbb{Z} \times \text{colim}_n \text{Sym}p_n(X)$ and this is a group isomorphism. \square

Theorem 3.5. *Let S be ind-smooth over a Dedekind ring with perfect residue fields. Then, for any affine $\text{Spec}(R) \cong X \in \text{Sm}_S^{\text{aff}}$ there are isomorphisms*

$$[S^n \wedge X_+, \mathbb{Z} \times HGr]_{\mathbb{A}^1} \cong \pi_n \text{Sing}^{\mathbb{A}^1}(KSp^\perp(X))$$

for all $n \geq 0$. In particular, this holds over $\text{Spec}(\mathbb{Z})$.

Proof. By Theorem 4.1.2 of [AHW18], $B_{Nis}Sp_{2k}$ is \mathbb{A}^1 -naive, and by Proposition 4.1.16 of [MV99], the map $BSp_{2k} \rightarrow B_{Nis}Sp_{2k}$ induces an isomorphism of groups $\pi_n BSp_{2k}(U) \xrightarrow{\sim} \pi_n B_{Nis}Sp_{2k}(U)$ for all $U \in \text{Sm}_S$ and all $n \geq 1$. Therefore, the morphism of colimits $BSp_\infty \rightarrow \text{colim}_k B_{Nis}Sp_{2k}$ also induces an isomorphism on all higher homotopy groups. By Theorem 3.4, we have a group isomorphism $\pi_0 \underline{\mathbb{Z}} \times \text{colim}_k B_{Nis}Sp_{2k}(X) \cong KSp_0(X)$ for every $X \in \text{Sm}_S^{\text{aff}}$. As $\pi_0 BG = *$ for any group, there is a weak equivalence $KSp_0^\perp(X) \times BSp_\infty(X) \rightarrow \underline{\mathbb{Z}}(X) \times \text{colim}_k B_{Nis}Sp_{2k}(X)$. We then have bijections,

$$\begin{aligned} [S^n \wedge X_+, \mathbb{Z} \times BSp_\infty]_{\mathbb{A}^1} &\cong [S^n \wedge X_+, \underline{\mathbb{Z}} \times \text{colim}_k B_{Nis}Sp_{2k}]_{\mathbb{A}^1} \\ &\cong \pi_n \text{Sing}^{\mathbb{A}^1}(\underline{\mathbb{Z}}(X) \times \text{colim}_k B_{Nis}Sp_{2k}(X)) \end{aligned}$$

for all $n \geq 0$. The map of spaces $KSp_0^\perp(X) \times BSp_\infty(X) \rightarrow KSp^\perp(X)$ is a topological group completion, when X is affine, by Theorem 2.3. Further, from the proof of Theorem 3.1 we have that $\text{Sing}^{\mathbb{A}^1} BSp_\infty(X)$ is a grouplike H-space.

Putting these together, we get that the map $Sing^{\mathbb{A}^1}(KSp_0^{\perp}(X) \times BSp_{\infty}(X)) \rightarrow Sing^{\mathbb{A}^1}(KSp^{\perp}(X))$ is a levelwise weak equivalence. Therefore,

$$\pi_n Sing^{\mathbb{A}^1}(\underline{\mathbb{Z}} \times \operatorname{colim}_k B_{Nis} Sp_{2k}) \cong \pi_n Sing^{\mathbb{A}^1}(KSp^{\perp}(X))$$

and hence we have

$$[S^n \wedge X_+, \mathbb{Z} \times BSp_{\infty}]_{\mathbb{A}^1} \cong \pi_n Sing^{\mathbb{A}^1}(KSp^{\perp}(X))$$

thus completing the proof. \square

We have a stronger result in the case when $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$.

Theorem 3.6. *Let S be a regular Noetherian scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. For any $X \in Sm_S$ and for all $n \in \mathbb{N}$,*

$$[S^n \wedge X_+, \mathbb{Z} \times HGr]_{\mathbb{A}^1} \cong KSp_n^{[0]}(X).$$

Proof. The statement follows from Theorems 2.8 and 3.1. \square

4. Hermitian K-theory spectrum

Recall from Section 2 that for any scheme S we have simplicial presheaves $KO^{[n]} : Sm_S^{op} \rightarrow \mathbf{sSet}$. When S is a regular Noetherian scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(\mathcal{O}_S, S)$, Panin and Walter showed that there is a motivic T -spectrum $\mathbf{KO} = (KO^{[0]}, KO^{[1]}, KO^{[2]}, \dots)$, where $T = \mathbb{A}^1/\mathbb{A}^1 - 0$ [PW18, Sec. 7]. We will recall this construction below. First, we need the fact that for any exact category with strict duality $(\mathcal{E}, *, \eta)$ we can define a family of exact category with weak equivalence and duality structures on the category of bounded chain complexes $Ch^b(\mathcal{E})$.

Definition 4.1 (Shifted dualities). For $n \in \mathbb{Z}$, we define the functor $\eta^n : Ch^b(\mathcal{E}) \rightarrow Ch^b(\mathcal{E})^{op}$ to be given by $E \mapsto (E.)^*[n]$. Where $(E.)^*$ is the chain complex $(E.)^*_i = E_{-i}^*$. That is,

$$(E^*, d^*) : \dots E_{-i+1}^* \xrightarrow{d_{-i}^*} E_{-i}^* \xrightarrow{d_{-i-1}^*} E_{-i-1}^* \rightarrow \dots$$

$[n]$ is the usual shift functor $E[n]_i = E_{i-n}$. In particular $(d^{*n})_i = (d_{-i-1-n})^*$. Let $\eta^n : (-) \Rightarrow (-)^{*n}$ be the natural transformation given by

$$(\eta^n)_i = (-1)^{\frac{n(n-1)}{2}} \eta_{E_i} \quad E \in Ch^b(\mathcal{E})$$

The pairs $(*^n, \eta^n)$, give us exact categories with weak equivalences and duality

$$(Ch^b(\mathcal{E}), *^n, \eta^n, q)$$

for each $n \in \mathbb{Z}$. We will apply this construction to $\mathcal{E} = Vect(X)$ and $*^n = \vee^n$ given by $\vee^n : E \mapsto E^{\vee}[n]$.

Definition 4.2 (Koszul complex). Let $p : E \rightarrow X$ be a vector bundle of rank n . The pullback $p^*E = E \times_X E \rightarrow E$ is a vector bundle over E (as a scheme) and has a section $s : E \rightarrow E \times_X E$ given by the diagonal map. The *Koszul complex* $\kappa(E)$ is the chain complex of vector bundles over E given by

$$\kappa(E) : (0 \rightarrow \Lambda^n p^*E^\vee \rightarrow \Lambda^{n-1} p^*E^\vee \rightarrow \dots \rightarrow \Lambda^2 p^*E^\vee \rightarrow p^*E^\vee \rightarrow \mathcal{O}_E \rightarrow 0)$$

with grading $\kappa(E)_i = \Lambda^{n-i} E^\vee$ and differentials $d : \Lambda^{k+1} p^*E^\vee \rightarrow \Lambda^k p^*E^\vee$ given by

$$d(x_0 \wedge x_1 \wedge \dots \wedge x_k) = \sum_{i=0}^k (-1)^i s^*(x_i) x_0 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_k$$

where s^* is the dual of the section $s : \mathcal{O}_E \rightarrow p^*E$.

The canonical isomorphism $\Lambda^r p^*E \xrightarrow{\sim} (\Lambda^{n-r} p^*E)^\vee \otimes \Lambda^n p^*E$ induces an isomorphism of chain complexes $\theta(E) : \kappa(E) \xrightarrow{\sim} \kappa(E)^\vee \otimes \Lambda^n p^*E[n]$. Given an isomorphism $\lambda : \det E = \Lambda^n E \xrightarrow{\sim} \mathcal{O}_X$, this gives us a non-degenerate symmetric form in $Ch^b(Vect(E), \vee^n, \eta^n, q)$,

$$\kappa(E, \lambda) : \kappa(E) \xrightarrow{\sim} \kappa(E)^\vee \otimes \Lambda^n p^*E[n] \xrightarrow{\sim} \kappa(E)^\vee[n]$$

where we choose the sign of the isomorphisms $\Lambda^r p^*E \xrightarrow{\sim} (\Lambda^{n-r} p^*E)^\vee \otimes \Lambda^n p^*E$ to be compatible with the natural transformation η^n . Let $Vect(E)^{q_{E-X}}$ be the full subcategory of $Vect(E)$ generated by complexes which are acyclic when restricted to the open subscheme $E - X$. It follows that $\kappa(E, \lambda)$ is an object of $Vect(E)^{q_{E-X}}$. Given a pair (E, λ) , where $p : E \rightarrow X$ is a rank n vector bundle and $\lambda : \det E = \Lambda^n E \xrightarrow{\sim} \mathcal{O}_X$ an isomorphism between the determinant bundle and the trivial bundle, the Thom class $th(E, \lambda) = [(\kappa(E, \lambda))]$ is the corresponding element in $KO_0^{[n]}(E, E - X)$. From this we see that for every pair (E, λ) , where E is a vector bundle and $\lambda : \Lambda^n E \xrightarrow{\sim} \mathcal{O}_X$, we have a functor

$$Ch^b(Vect(X), \vee^m, \eta^n, q) \rightarrow Ch^b(Vect(E)^{q_{E-X}}, \vee^{m+n}, \eta^n, q)$$

which on objects is given by tensoring with $\kappa(E)$,

$$C. \mapsto p^*C. \otimes \kappa(E).$$

This induces a map of spaces $KO^{[m]}(X) \rightarrow KO^{[m+n]}(E, E - X)$ which we denote by $\otimes th(E, \lambda)$.

Theorem 4.3. *Let $X \in Sm_S$ with S a regular Noetherian scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. For any pair (E, λ) described above, the map $\otimes th(E, \lambda) : KO^{[m]}(X) \rightarrow KO^{[m+n]}(E, E - X)$ is a weak equivalence of spaces.*

Proof. This is a corollary of [PW18, Thm. 5.1], which contains the hypotheses on regularity and invertibility of 2. \square

For any scheme X , let $E = \mathcal{O}_X$ be the trivial bundle with $\lambda = id$, $\kappa(E, \lambda)$ is then given by,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{A}^1_X} & \xrightarrow{t} & \mathcal{O}_{\mathbb{A}^1_X} & \longrightarrow & 0 \\ & & \downarrow^{-1} & & \downarrow^1 & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{A}^1_X} & \xrightarrow{-t} & \mathcal{O}_{\mathbb{A}^1_X} & \longrightarrow & 0 \end{array}$$

where t is the variable in $\mathbb{A}^1_{\text{Spec}(R)} \cong \text{Spec}(R[t])$. We then have an induced equivalence of simplicial sets,

$$\otimes th(\mathcal{O}, id) : KO^{[n]}(X) \xrightarrow{\sim} KO^{[n+1]}(\mathbb{A}^1 \times X, (\mathbb{A}^1 \setminus \{0\}) \times X)$$

which is functorial on X . Therefore, there is a levelwise weak equivalence of simplicial presheaves $KO^{[n]}(-) \xrightarrow{\sim} KO^{[n+1]}(\mathbb{A}^1 \times -, (\mathbb{A}^1 \setminus \{0\}) \times -)$. If $KO_f^{[n]}$ is an \mathbb{A}^1 -fibrant replacement of $KO^{[n]}$, we have a zigzag of \mathbb{A}^1 -weak equivalences

$$\begin{aligned} \mathbf{Hom}_{sPSh.(S)}(- \wedge T, KO_f^{[n]}) &\xrightarrow{\sim} \mathbf{Hom}_{sPSh.(S)}(- \wedge T, KO^{[n]}) \\ &\xrightarrow{\sim} KO^{[n]}(\mathbb{A}^1 \times -, (\mathbb{A}^1 \setminus \{0\}) \times -) \end{aligned}$$

for each n . As $\mathbf{Hom}_{sPSh.(S)}(- \wedge T, KO_f^{[n]})$ is fibrant, the zigzag lifts to a weak equivalence of simplicial presheaves

$$KO_f^{[n]}(-) \xrightarrow{\sim} \mathbf{Hom}_{sPSh.(S)}(- \wedge T, KO_f^{[n+1]}).$$

Taking the adjoint, we get an equivalence of motivic spaces

$$T \wedge KO_f^{[n]}(-) \xrightarrow{\sim} KO_f^{[n+1]}(-) \tag{4.1}$$

for each n . The sequence $(KO_f^{[n]})_{n \geq 0}$ along with structure maps 4.1 defines a T -spectrum $\mathbf{KO} \in \mathbf{Spt}(S)_T$. As we have to make choices for fibrant replacements $KO_f^{[n]}$ and the structure maps, the T -spectrum \mathbf{KO} is not unique. However, by Theorem A.5 we get a unique object in $SH(S)$ up to (not necessarily unique) isomorphism.

Definition 4.4. We define \mathbf{KO} to be the naive T -spectrum given by the sequence $\mathbf{KO} = (KO^{[n]})_{n \geq 0}$ and the structure maps $T \wedge_L KO^{[n]} \rightarrow KO^{[n+1]}$ in $H_*(S)$ induced by the weak equivalences in 4.1. By Theorem A.5, \mathbf{KO} defines a unique object in $SH(S)$ up to isomorphism. By abuse of notation, we will refer to this object also as \mathbf{KO} .

From Theorems 2.8 and 4.3, we get the following corollary.

Corollary 4.5. *Let S be regular noetherian scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. For all $X \in Sm_S$, there is an isomorphism of groups*

$$[\Sigma^\infty X_+, S^{p,q} \wedge \mathbf{KO}] \cong KO_{2q-p}^{[q]}(X)$$

where $2q \geq p \geq q \geq 0$ and $[-, -]$ denotes $\text{Hom}_{SH(S)}(-, -)$.

Proof. As $S^{p,q} = S^{p-q} \wedge \mathbb{G}_m^q$ and $T \cong S^{2,1}$ in $H_*(S)$ we have a sequence of bijections

$$\begin{aligned} [\Sigma^\infty X_+, S^{p,q} \wedge \mathbf{KO}] &\cong [S^{2q-p} \wedge \Sigma^\infty X_+, T^q \wedge \mathbf{KO}] \\ &\cong [S^{2q-p} \wedge \Sigma^\infty X_+, \mathbf{KO}[-q]] \cong [S^{2q-p} \wedge X_+, \mathbf{KO}^{[q]}]_{\mathbb{A}^1} \cong \mathbf{KO}_{2q-p}^{[q]}(X) \end{aligned}$$

for all $2q \geq p \geq q \geq 0$. \square

The above result suggests a definition of $\mathbf{KO}_i^{[n]}(X)$ for $i < 0$. Indeed Schlichting in [Sch17] constructs an Ω -spectrum $\mathbb{G}\mathbb{W}^n(X)$ for any scheme X with an ample family of line bundles which satisfies

$$\pi_i \mathbb{G}\mathbb{W}^n(X) = \begin{cases} \mathbf{KO}_i^{[n]}(X) & i \geq 0 \\ W^{n-i}(X) & i < 0 \end{cases} \quad (4.2)$$

for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}$. Here $W^n(X)$ are Balmer's triangular Witt groups [Bal00]. Setting $\mathbf{KO}_i^{[n]}(X) = \pi_i \mathbb{G}\mathbb{W}^n(X)$ for all i and using [PW18, Lem. 5.2] we get the following generalisation.

Theorem 4.6. *Let S be regular noetherian scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. For all $X \in \text{Sm}_S$, there is an isomorphism of groups*

$$[\Sigma^\infty X_+, S^{p,q} \wedge \mathbf{KO}] \cong \mathbf{KO}_{2q-p}^{[q]}(X)$$

where $p \geq q \geq 0$.

5. The geometric \mathbf{HP}^1 -spectrum \mathbf{KO}^{geo}

As before, the stable homotopy category $SH(S)$ is the stabilization of $H_*(S)$ with respect to the functor $(X, x_0) \mapsto T \wedge (X, x_0)$ where $T = \mathbb{A}^1/(\mathbb{A}^1 - 0)$. There is a Quillen equivalence between the categories $\mathbf{Spt}(S)_T$ and $\mathbf{Spt}(S)_{T^{\wedge 2}}$, of T - and $T^{\wedge 2}$ -spectra, given by the adjoint pair $(X_n) \mapsto (X_{2n})$ and $(X_n) \mapsto (X_0, T \wedge X_0, X_1, T \wedge X_1, \dots)$, inducing an isomorphism of stable homotopy categories $SH(S)_T \cong SH(S)_{T^{\wedge 2}}$ [Jar00, Prop. 2.13]. To construct our desired spectrum, we need a different model of the stable homotopy category which utilizes the quaternionic projective space $HP^1 = HGr(2, 4)$. From this point onwards, we will denote the base scheme S by pt to simplify notation.

Theorem 5.1. *Let $x_0 : pt \rightarrow HP^1$ be the distinguished point corresponding to the subbundle $[H_- \oplus 0]$. There is an isomorphism $\eta : (HP^1, x_0) \cong T^{\wedge 2}$ in $H_*(S)$.*

Proof. This is [PW18, Thm. 9.8]. To introduce notation and to illustrate the geometry present when discussing higher Grassmannians, we elaborate some of the arguments below. Consider the open subscheme $\mathbb{A}^4 \cong N \hookrightarrow Gr(2, 4)$ classifying rank 2 subbundles $U \rightarrow \mathcal{O}^4$ whose projection onto $0 \oplus 0 \oplus \mathcal{O}^2$ is an isomorphism. We have $N = N^+ \oplus N^-$, where $N^+ = HP^1 \cap Gr(2, 0 \oplus \mathcal{O} \oplus \mathcal{O}^2)$ and $N^- = HP^1 \cap Gr(2, \mathcal{O} \oplus 0 \oplus \mathcal{O}^2)$ are closed subschemes of HP^1 which are

isomorphic to \mathbb{A}^2 . By [PW21, Thm. 3.4], $HP^1 - N^+$ is the quotient of $\mathbb{A}^5 = \mathbb{A}^2 \times \mathbb{A}^2 \times \mathbb{A}^1$ by the free \mathbb{G}_a -action,

$$t \cdot (a, b, r) = (a, b + ta, r + t(1 - a \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} b)).$$

The inclusion $\mathbb{A}^1 \hookrightarrow \mathbb{A}^5$ given by $t \mapsto (0, 0, 0, 0, t)$ is then \mathbb{G}_a -equivariant and an \mathbb{A}^1 -equivalence. Therefore, we have an induced \mathbb{A}^1 -equivalence of the quotients $x_0 : pt \rightarrow HP^1 - N^+$ given by the subspace $0 \oplus 0 \oplus H_-$. The commutative square,

$$\begin{array}{ccc} pt & \longrightarrow & HP^1 \\ \downarrow \sim & & \downarrow = \\ HP^1 - N^+ & \hookrightarrow & HP^1 \end{array}$$

gives us an equivalence of pointed spaces $(HP^1, x_0) \xrightarrow{\sim} HP^1/(HP^1 - N^+)$. We have a similar square induced by the \mathbb{A}^1 -equivalence $N^- \xrightarrow{\sim} N$,

$$\begin{array}{ccc} N^- - 0 & \longrightarrow & N^- \\ \downarrow \sim & & \downarrow \sim \\ N - N^+ & \hookrightarrow & N \end{array}$$

where the left hand side is an equivalence as $N - N^+ = \mathbb{A}^4 - \mathbb{A}^2$ is a rank 2 vector bundle over $N^- - 0$. Hence we have $N^-/(N^- - 0) \cong N/(N - N^+)$ in $H_*(S)$ and there is a zigzag of \mathbb{A}^1 -equivalences

$$\begin{array}{ccccc} T^2 \cong \mathbb{A}^2/(\mathbb{A}^2 - 0) \cong N^-/N^- - 0 & \longrightarrow & HP^1/(HP^1 - N^+) & \longleftarrow \sim & (HP^1, x_0) \\ & \searrow & \uparrow \text{excision} & & \\ & & N \cap HP^1/((N \cap HP^1) - N^+) & & \\ & \swarrow \text{excision} & & & \\ N/N - N^+ & & & & \end{array}$$

Using the 2 out of 3 property twice, this gives us an \mathbb{A}^1 -equivalence $T^{\wedge 2} \xrightarrow{\sim} HP^1/(HP^1 - N^+)$ and hence $T^{\wedge 2} \cong (HP^1, x_0)$ in $H_*(S)$. \square

Theorem 5.1 provides equivalences of stable homotopy categories $SH(S) \cong SH(S)_{T^{\wedge 2}} \cong SH(S)_{HP^1}$. We now have the desired model of $SH(S)$. We will construct \mathbf{KO}^{geo} as a naive HP^1 -spectrum in the sense of appendix A. For any $n \in \mathbb{Z}$, we have a canonical decomposition $V_n \perp V_n^\perp \cong H_+^n$ over $RGr(n, 2n)$ which corresponds to the identity map $RGr(n, 2n) \rightarrow RGr(n, 2n)$. Similarly, we have the canonical decomposition $U_{2n} \perp U_{2n}^\perp \cong H_-^{2n}$ over $HGr(2n, 4n)$. Let $\mathbb{Z} \times HGr := \text{colim}_n [-n, n] \times HGr(2n, 4n)$, where $[-n, n] \times X$ denotes the disjoint union of $2n + 1$ copies of X . The morphism $HGr(2n, 4n) \rightarrow HGr(2n + 2, 4n + 4)$ is given by the subbundle

$$U_{2n} \perp H_- \rightarrow (H_-^n \perp H_-) \perp (H_-^n \perp 0) \rightarrow (H_-^n \perp H_-) \perp (H_-^n \perp H_-)$$

where $HGr(2n, 4n)$ is identified with $HGr(2n, H_-^n \perp H_-^n)$. Defining $[-n, n]'$ by

$$[-n, n]' = \{i \in \mathbb{Z} \mid -n \leq i \leq n \text{ and } i \equiv n \pmod{2}\}$$

the infinite real Grassmannian,

$$\mathbb{Z} \times RGr := \operatorname{colim}_n [-2n, 2n]' \times RGr(2n, 4n) \cup [-2n+1, 2n-1]' \times RGr(2n-1, 4n-2)$$

is defined along similar lines. The distinguished points $h_0 : pt \rightarrow \mathbb{Z} \times HGr$ and $r_0 : pt \rightarrow \mathbb{Z} \times RGr$ correspond to the subbundles $H_-^n \perp 0 \rightarrow H_-^n \perp H_-^n$ and $H_+^n \perp 0 \rightarrow H_+^n \perp H_+^n$ over $\{0\} \times HGr(2n, 4n) \subset [-n, n] \times HGr(2n, 4n)$ and $\{0\} \times RGr(2n, 4n) \subset [-2n, 2n]' \times RGr(2n, 4n)$ respectively for each n .

Lemma 5.2. *For all $n \geq 0$, there exist morphisms of pointed smooth schemes*

$$f_{2n} : ([-n, n] \times HGr(2n, 4n)) \times HP^1 \rightarrow RGr(16n, 32n)$$

such that the following hold.

- (1) *The restriction $f_{2n}|_{(0, H_-^n \perp 0) \times HP^1}$ is given by the subbundle $H_+^{8n} \perp 0$.*
- (2) *The restriction $f_{2n}|_{[-n, n] \times HGr(2n, 4n) \times (H_- \perp 0)}$ is given by an embedding $H_+^{8n} \rightarrow H_+^{16n}$ which is \mathbb{A}^1 -homotopic to the embedding given by $H_+^{8n} \perp 0$.*
- (3) *These morphisms and \mathbb{A}^1 -homotopies are compatible with inclusions of schemes $HGr(2n, 4n) \rightarrow HGr(2(n+1), 4(n+1))$ and $RGr(16n, 32n) \rightarrow RGr(16(n+1), 32(n+1))$.*

Proof. For simplicity, given two vector bundles U and V over two distinct schemes X and Y , we denote by $U \boxtimes V$ the vector bundle $p_1^*U \otimes p_2^*V$ over $X \times Y$ where p_i are the projections. We have decompositions $U_{2n} \perp U_{2n}^\perp \cong H_-^{2n}$ for all n . Remember that the tensor product of two symplectic spaces is a symmetric space and in particular $H_-^n \boxtimes H_-^m \cong H_+^{2mn}$. For each $n \in \mathbb{N}$ and $i \in [-n, n]$, we have inclusions of symmetric spaces over $HGr(2n, 4n) \times HP^1$,

$$\begin{aligned} U_{2n} \boxtimes U_2 &\rightarrow H_-^{2n} \boxtimes U_2, \\ H_-^{n-i} \boxtimes U_2^\perp &\rightarrow H_-^{2n} \boxtimes U_2^\perp, \\ U_{2n}^\perp \boxtimes H_- &\rightarrow H_-^{2n} \boxtimes H_-, \\ H_-^{n+i} \boxtimes H_- &\rightarrow H_-^{2n} \boxtimes H_-. \end{aligned} \tag{5.1}$$

Putting these together, we get

$$(U_{2n} \boxtimes U_2) \perp (H_-^{n-i} \boxtimes U_2^\perp) \perp (U_{2n}^\perp \boxtimes H_-) \perp (H_-^{n+i} \boxtimes H_-) \tag{5.2}$$

which is a rank $16n$ symmetric subspace of the rank $32n$ symmetric space

$$(H_-^{2n} \boxtimes U_2) \perp (H_-^{2n} \boxtimes U_2^\perp) \perp (H_-^{2n} \boxtimes H_-) \perp (H_-^{2n} \boxtimes H_-). \tag{5.3}$$

This space is isometric to H_+^{16n} and a choice of isometry gives us $2n+1$ subspaces of H_+^{16n} (one for each i), each of which is rank $16n$. Hence, we have a morphism

$$f_{2n} : ([-n, n] \times HGr(2n, 4n)) \times HP^1 \rightarrow RGr(16n, 32n). \tag{5.4}$$

The composition

$$HGr(2n, 4n) \times HP^1 \rightarrow HGr(2n+2, 4n+4) \times HP^1 \rightarrow RGr(16n+16, 32n+32),$$

for a fixed i then is given by the subspace

$$((U_{2n} \perp H_-) \boxtimes U_2) \perp ((U_{2n}^\perp \perp H_-) \boxtimes H_-) \perp ((H_-^{n-i} \perp H_-) \boxtimes U_2^\perp) \perp H_+^{2n+2+2i}.$$

To make this equal to the composition

$$HGr(2n, 4n) \times HP^1 \rightarrow RGr(16n, 32n) \rightarrow RGr(16n + 16, 32n + 32),$$

we need to choose the isometry used to define f_{2n} carefully. First, we index H_-^{2n} as $\perp_{i=1}^{2n} H_-^{(i)}$. We then get isometries

$$(H_-^{(i)} \boxtimes U_2) \perp (H_-^{(i)} \boxtimes U_2^\perp) \perp (H_-^{(i)} \boxtimes H_-) \perp (H_-^{(i)} \boxtimes H_-) \cong H_+^8 \quad (5.5)$$

for each $i \in \{1, \dots, 2n\}$. Putting these together we get,

$$(H_-^{2n} \boxtimes U_2) \perp (H_-^{2n} \boxtimes U_2^\perp) \perp (H_-^{2n} \boxtimes H_-) \perp (H_-^{2n} \boxtimes H_-) \cong H_+^{16n} \quad (5.6)$$

which is the desired isometry. Pulling back this isometry along the inclusion

$$(0, H_-^n \perp 0) \times HP^1 \hookrightarrow [-n, n] \times HGr(2n, 4n) \times HP^1$$

turns the symplectic subspace 5.3 into $H_+^{8n} \perp 0$ and pulling back the isometry along $[-n, n] \times HGr(2n, 4n) \times (H_- \perp 0)$ turns it into

$$(U_{2n} \boxtimes (H_- \perp 0)) \perp (H_-^{n-i} \boxtimes (0 \perp H_-)) \perp (U_{2n}^\perp \boxtimes (H_- \perp 0)) \perp (H_-^{n+i} \boxtimes (0 \perp H_-))$$

which is a subbundle isometric to H_+^{8n} . Permuting the summands of H_+^{16n} , we can send the above subbundle to $H_+^{8n} \perp 0 \rightarrow H_+^{8n} \perp H_+^{8n}$. As given in the proof of Theorem 3.1, these permutations are \mathbb{A}^1 -homotopic to identity. \square

The next lemma can be proved analogously,

Lemma 5.3. *For all $n \geq 0$, there exist morphisms of pointed smooth schemes*

$$g_n : ([-n, n]' \times RGr(n, 2n)) \times HP^1 \rightarrow HGr(8n, 16n)$$

such that the following holds:

- (1) The restriction $g_n|_{(0, H_+^{2n} \perp 0) \times HP^1}$ is given by the subbundle $H_-^{8n} \perp 0$ when n is even.
- (2) The restriction $g_n|_{[-n, n] \times RGr(2n, 4n) \times (H_+ \perp 0)}$ is given by an embedding $H_-^{8n} \rightarrow H_-^{16n}$ which is \mathbb{A}^1 -homotopic to the embedding given by $H_-^{8n} \perp 0$.
- (3) These morphisms and \mathbb{A}^1 -homotopies are compatible with inclusions of schemes

$$RGr(n, 2n) \rightarrow RGr(n+1, 2(n+1))$$

and

$$HGr(8n, 16n) \rightarrow HGr(8(n+1), 16(n+1)).$$

Proof. The proof is the same as for Lemma 5.2, except for a change in indices. The morphism g_n is defined on the i^{th} component, where $i \in [-n, n]'$, by the subspace

$$(V_n \boxtimes V_1) \perp (H_+^{\frac{n-i}{2}} \boxtimes V_1^\perp) \perp (V_n^\perp \boxtimes H_+) \perp (H^{n+1}) \quad (5.7)$$

\square

Using the morphisms of schemes f_n and g_n , we can construct a naive HP^1 -spectrum.

Theorem 5.4. *For any scheme S , there exists a naive HP^1 -spectrum \mathbf{KO}_S^{geo} such that for all $i \geq 0$, $\mathbf{KO}_{2i+1}^{geo} = \mathbb{Z} \times HGr$ and $\mathbf{KO}_{2i}^{geo} = \mathbb{Z} \times RGr$ as motivic spaces and $\Omega_{HP^1}^2 \mathbf{KO}^{geo} \cong \mathbf{KO}^{geo}$ in $SH(S)_{HP^1}$.*

Here we are abusing notation and using \mathbf{KO}_S^{geo} to mean both the naive HP^1 -spectrum and the associated object in $SH(S)$.

Proof. From Lemma 5.2 and Lemma 5.3 we get morphisms of ind-schemes,

$$f : HP^1 \times \mathbb{Z} \times HGr \rightarrow RGr \quad g : HP^1 \times \mathbb{Z} \times RGr \rightarrow HGr \quad (5.8)$$

such that $f|_{HP^1 \vee HGr}$ and $g|_{HP^1 \vee RGr}$ are \mathbb{A}^1 -homotopic to the constant zero morphism. These induce structure maps $f : HP^1 \wedge_{\mathbb{L}} (\mathbb{Z} \times HGr) \rightarrow \mathbb{Z} \times RGr$ and $g : HP^1 \wedge_{\mathbb{L}} (\mathbb{Z} \times RGr) \rightarrow \mathbb{Z} \times HGr$ in $H_*(S)$, where $\wedge_{\mathbb{L}}$ is the left derived functor of the smash product. Periodicity follows by construction. \square

The periodicity above implies (8, 4)-periodicity in the standard bigrading of motivic spectra.

Remark 5.5. Our construction of \mathbf{KO}^{geo} is along the same lines as the construction of \mathbf{BO} in [PW18]. However, we have shown that the condition of 2 being invertible is not needed. We have also used the theory of naive spectra elaborated in Appendix A which simplifies that construction of the structure maps of \mathbf{KO}^{geo} and shows that only the class of the structure maps in $H_*(S)$ matter.

6. Properties of \mathbf{KO}^{geo}

Having constructed \mathbf{KO}^{geo} , we will look at some of its properties. The first notable property is that it is absolute in the following sense.

Theorem 6.1. *For any morphism $f : S \rightarrow T$ of schemes, there is a canonical isomorphism $Lf^* \mathbf{KO}_T^{geo} \xrightarrow{\sim} \mathbf{KO}_S^{geo}$ in $SH(S)$. In particular, for any scheme S , \mathbf{KO}_S^{geo} is isomorphic to the pullback of $\mathbf{KO}_{\mathbb{Z}}^{geo} = \mathbf{KO}_{\text{Spec}(\mathbb{Z})}^{geo}$ by the structure map $S \rightarrow \text{Spec}(\mathbb{Z})$.*

Proof. Let $f : S \rightarrow T$ be a morphism of schemes. Pullback induces canonical identifications $f^*(HGr(2r, 2n)_T) \cong HGr(2r, 2n)_S$ and $f^*(\mathbb{Z} \times HGr_T) \cong \mathbb{Z} \times HGr_S$. This pullback isomorphism is also compatible with the structure maps of \mathbf{KO}^{geo} , as the tensor product of forms is preserved under pullback. Using the closed model structure given in [PPR09] f^* is a left Quillen functor completing the proof. \square

The next interesting property is that \mathbf{KO}^{geo} over $\text{Spec}(\mathbb{Z}[\frac{1}{2}])$ gives us back the motivic spectrum \mathbf{BO} constructed in [PW18]. This implies a corresponding representability result for hermitian K-theory. To state this result, we will need a model of \mathbf{BO} as a naive HP^1 -spectrum. For any rank 2 symplectic bundle

(E, ϕ) over a scheme X , the structure map $\phi : E \otimes E \rightarrow \mathcal{O}_X$ induces an isomorphism $\Lambda^2 E \xrightarrow{\sim} \mathcal{O}_X$. The canonical morphism $U_2 \rightarrow H_-^2$ over HP^1 restricts to a set of four maps $U_2 \rightarrow \mathcal{O}_{HP^1}$. The pair which factors through $U_2 \rightarrow H_-$ differ up to isomorphism only by a sign. Therefore, denote these pairs by $(x_0, -x_0)$ and $(x_\infty, -x_\infty)$ respectively (this notation is consistent with the fact these are isomorphisms when pulled back along points x_0 and x_∞). Consider the symmetric form

$$\begin{array}{ccccc}
 \mathcal{O}_{HP^1} & \longrightarrow & U_2 & \xrightarrow{x_0} & \mathcal{O}_{HP^1} \\
 \downarrow -1 & & \downarrow \phi_{2,4} & & \downarrow 1 \\
 \mathcal{O}_{HP^1} & \xrightarrow{(-x_\infty)^\vee} & U_2^\vee & \longrightarrow & \mathcal{O}_{HP^1}
 \end{array}$$

in $Ch^b(\text{Vect}(HP^1 \times X), -\eta, q)$, indexed from degrees 0 to 2. By construction, this form is equal to $[U_2] - [H_-]$ in $KSp(HP^1 \times X)$ under

$$KO^{[2]}(HP^1 \times X) \xrightarrow{\sim} KSp^{[0]}(HP^1 \times X) \xrightarrow{\sim} KSp(HP^1 \times X)$$

and is the pullback of $\kappa(U_2, \phi)$ along the zero section $z : HP^1 \rightarrow U_2$. We will call this element of $KO_0^{[2]}(HP^1 \times X)$ the *Borel class* $-b_1(U_2)$. The Borel class will give us the desired structure map. Let us denote by \mathbf{BO}_{HP^1} the image of \mathbf{BO} in $SH(S)_{HP^1}$.

Theorem 6.2. *Let S be a regular Noetherian scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(\mathcal{O}_S, S)$. The structure morphisms of \mathbf{BO}_{HP^1} are represented by maps*

$$KO^{[n]}(-) \rightarrow KO^{[n+2]}(- \times HP^1), \quad C. \mapsto C. \boxtimes (-b_1(U_2))$$

for all n .

Proof. It is enough to show that the image of the Borel class under the zigzag of weak equivalences between HP^1 and $T^{\wedge 2}$ is $th(\mathcal{O}, id) \boxtimes th(\mathcal{O}, id)$. Firstly the sequence $\mathcal{O}_{HP^1} \rightarrow U_2 \xrightarrow{x_0} \mathcal{O}_{HP^1}$ is exact when restricted to $HP^1 - N^+$. Hence $-b_1(U_2)$ is an element of $KO_0^{[n+2]}(HP^1, HP^1 - N^+)$. The pullback of the morphism of vector bundles $U_2 \xrightarrow{x_0} \mathcal{O}_{HP^1}$ along $\mathbb{A}^2 \cong N^- \rightarrow HP^1$ is $\mathcal{O}_{\mathbb{A}^2}^{\oplus 2} \xrightarrow{(t_0, t_1)} \mathcal{O}_{\mathbb{A}^2}$, where t_0 and t_1 are the variables in $\mathbb{A}_{\text{Spec}(R)}^2 \cong \text{Spec}(R[t_0, t_1])$. Therefore $-b_1(U_2)$ becomes

$$\begin{array}{ccccc}
 \mathcal{O}_{\mathbb{A}^2} & \longrightarrow & \mathcal{O}_{\mathbb{A}^2}^{\oplus 2} & \xrightarrow{(t_0, t_1)} & \mathcal{O}_{\mathbb{A}^2} \\
 \downarrow -1 & & \downarrow & & \downarrow 1 \\
 \mathcal{O}_{\mathbb{A}^2} & \xrightarrow{(-t_1, -t_0)} & \mathcal{O}_{\mathbb{A}^2}^{\oplus 2} & \longrightarrow & \mathcal{O}_{\mathbb{A}^2}
 \end{array}$$

which is $th(\mathcal{O}, id) \boxtimes th(\mathcal{O}, id)$ as required. □

We can now state the desired result.

Theorem 6.3. *There exists an isomorphism $\mathbf{KO}^{\text{geo}} \cong \mathbf{BO}_{\mathbb{HP}^1}$ as objects in $SH(\text{Spec}(\mathbb{Z}[\frac{1}{2}]))$. Hence \mathbf{KO}^{geo} represents hermitian K-theory over regular Noetherian schemes of finite Krull dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$.*

Proof. The isomorphisms $\tau : \mathbb{Z} \times HGr \xrightarrow{\sim} KSp$ in $H_*(S)$ follows from Theorem 3.1. To show that the diagram

$$\begin{array}{ccc} HP^1 \wedge_{\mathbb{L}} HP^1 \wedge (\mathbb{Z} \times HGr) & \longrightarrow & \mathbb{Z} \times HGr \\ \downarrow 1 \wedge 1 \wedge \tau_{4n-2} & & \downarrow \tau_{4n+2} \\ HP^1 \wedge_{\mathbb{L}} HP^1 \wedge_{\mathbb{L}} KO^{[4n-2]} & \longrightarrow & KO^{[4n+2]} \end{array}$$

commutes in $H_*(S)$, we use the fact that $\mathbb{Z} \times HGr$ is an ind-scheme and hence the restriction $\{i\} \times HGr(2r, 2n) \hookrightarrow \mathbb{Z} \times HGr \xrightarrow{\sim} KSp$ is classified by an element in $KSp_0(HGr(2r, 2n))$. The map $BSp_{2n} \rightarrow Bi\mathbf{Sym}$ in $H_*(S)$ is given at the level of schemes by sending principal Sp_{2n} -bundles to the associated symplectic bundle. Consequently, the map $HGr(2r, 2n) \rightarrow BSp_{2n} \rightarrow Bi\mathbf{Sym}$ corresponds to the tautological symplectic bundle $U_{2r, 2n} \rightarrow HGr(2r, 2n)$. From this and the definition of the map $\mathbb{Z} \times M_{\infty} \rightarrow R\Omega^1 BM$, it follows that the isomorphism $\tau : \mathbb{Z} \times HGr \rightarrow KSp$ satisfies

$$\begin{aligned} \tau_{\{i\} \times HGr(2r, 2n)} &= [U_{2r, 2n}] + (i - r)[H_-] \\ &\in KSp_0(HGr(2r, 2n)) \rightarrow Hom_{H_*(S)}(HGr(2r, 2n), KSp). \end{aligned}$$

The last map is an isomorphism over regular Noetherian schemes S with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. We denote by τ_{4k+2} the composition

$$\mathbb{Z} \times HGr \xrightarrow{\tau} KSp \xrightarrow{\sim} KO^{[4k+2]}$$

where $KSp \xrightarrow{\sim} KO^{[4k+2]}$ is the isomorphism 2.2 for $n = 0$. We then have

$$\tau_{4k+2, \{i\} \times HGr(2n, 4n)} = ([U_{2n, 4n}] + (i - 2n)[H_-])[2k + 1]$$

and similarly the map $HP^1 \wedge HP^1 \wedge (\mathbb{Z} \times HGr) \rightarrow \mathbb{Z} \times HGr \rightarrow KSp$ restricted to $HP^1 \times HP^1 \times \{i\} \times HGr(2n, 4n)$ is given by

$$([U_2] - [H_-]) \boxtimes ([U_2] - [H_-]) \boxtimes ([U_{2n, 4n}] + (i - 2n)[H_-]).$$

To see this, note that $[U_{2n, 4n}^{\perp}] = 2n[H] - [U_{2n, 4n}]$. But, tensoring twice with $([U_2] - [H_-])$ is exactly the structure map of $\mathbf{BO}_{\mathbb{HP}^1}$ (6.2) and hence the diagram commutes when restricted to the finite Grassmannians. As $\mathbb{Z} \times HGr$ is the colimit of $\{i\} \times HGr(2n, 4n)$ we have a map

$$Hom_{H_*(S)}(\text{colim}_n HGr(2n, 4n), X) \rightarrow \lim_n Hom_{H_*(S)}(HGr(2n, 4n), X)$$

for any $X \in \mathbf{Spc}_*(S)$. This is an isomorphism if

$$Hom_{H_*(S)}(S_S^1 \wedge HGr(2n + 2, 4n + 4), X) \rightarrow Hom_{H_*(S)}(S_S^1 \wedge HGr(2n, 4n), X)$$

is a surjection. To see this, take a fibrant replacement of X to get the set of simplicial homotopy classes. Surjectivity then implies that we can lift a collection of homotopy classes, uniquely up to homotopy, to the colimit. This holds for $X = KO^{[k]}$ as then we have

$$\text{Hom}_{H_*(S)}(S^1_s \wedge HGr(2n, 4n), KO^{[k]}) \cong KO^{[k]}_1(HGr(2n, 4n))$$

and the maps $KO^{[k]}_i(HGr(2n+2, 4n+4)) \rightarrow KO^{[k]}_i(HGr(2n, 4n))$ are surjections by [PW21, Thm. 11.4] applied to $KO^{[*]}_*$ which is a cohomology theory with a -1 -commutative ring structure [PW18, Thm. 1.4]. \square

Remark 6.4. Note that we needed $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$ to get the isomorphism $KSp \xrightarrow{\sim} KO^{[2k+1]}$ as only then do we have the identification between skew-symmetric and alternating forms.

We can also extend the cellularity result in [RSØ18] to arbitrary schemes. We will use the definition of cellular spectra from [DI05].

Definition 6.5. For any scheme S , let $\mathbf{Spt}_{cell}(S)$ be the smallest full subcategory of $\mathbf{Spt}(S)$ satisfying the following:

- (1) One has $S^{p,q} \in \mathbf{Spt}_{cell}(S)$ for all $p, q \in \mathbb{Z}$.
- (2) If \mathbf{F} is stably equivalent to \mathbf{E} for some $\mathbf{E} \in \mathbf{Spt}_{cell}(S)$ then $\mathbf{F} \in \mathbf{Spt}_{cell}(S)$.
- (3) For any diagram $D \rightarrow \mathbf{Spt}_{cell}(S)$, $\text{hocolim } D$ is in $\mathbf{Spt}_{cell}(S)$.

Since $\mathbf{Spt}_{cell}(S)$ is closed under stable equivalences, it defines a subcategory $SH(S)_{cell}$ of $SH(S)$. We call elements of $\mathbf{Spt}_{cell}(S)$ *cellular spectra*. Given a morphism $f : S_1 \rightarrow S_2$, we have the following.

Lemma 6.6. For any morphism of schemes $f : S_1 \rightarrow S_2$, $Lf^* : SH(S_2) \rightarrow SH(S_1)$ restricts to a morphism of cellular objects $Lf^* : SH(S_2)_{cell} \rightarrow SH(S_1)_{cell}$.

Proof. This follows from the fact that Lf^* preserves all motivic spheres and homotopy colimits. \square

We wish to prove the following.

Theorem 6.7. Let S be any scheme. The motivic spectrum \mathbf{KO}_S^{geo} is cellular.

To prove this, we will first show that the suspension spectra $\Sigma^\infty HG(2r, 2n)_+$ are cellular. They constitute the case $m = 0$ of the following statement.

Lemma 6.8. Let $m \geq 0$. The suspension spectrum of the Thom space of $U_{2r,2n}^{\oplus m}$ on $HGr(2r, 2n)$ is a finite cellular spectrum.

Proof. By Lemma 6.6 it is enough to prove this over $\text{Spec}(\mathbb{Z})$. The proof is by induction on r and n . As $HGr(0, 2n) \cong pt$, the statement holds for $r = 0$. Extending the definitions in Theorem 5.1 we define $N^+ = HGr(2r, 2n) \cap Gr(2r, 0 \oplus \mathcal{O}^{2r} \oplus \mathcal{O}^{2n})$ and $N^- = HGr(2r, 2n) \cap Gr(2r, \mathcal{O}^{2r} \oplus 0 \oplus \mathcal{O}^{2n})$ as closed subschemes of $HGr(2r, 2n)$. The direct sum of these bundles $N = N^+ \oplus N^-$ is the normal bundle of the embedding $HGr(2r, 2n - 2) \rightarrow HGr(2r, 2n)$ [PW21,

Thm 4.1]. Let Y be the open subscheme $HGr(2r, 2n) \setminus N^+$, then by [Spi10, Lem.3.5] the cofiber of the map

$$Th(U_{2r,2n|Y}^{\oplus m}) \rightarrow Th(U_{2r,2n}^{\oplus m}),$$

is isomorphic to $Th(U_{2r,2n|N^+}^{\oplus m} \oplus \mathcal{N})$ where \mathcal{N} is the normal bundle of the closed embedding $N^+ \rightarrow HGr(2r, 2n)$. We have $U_{2r,2n|N^+} \cong \pi_+^* U_{2r,2n-2}$ by [PW21] and in fact the proof shows us that $\pi_+^* U_{2r,2n-2} \cong \mathcal{N}$. We therefore have a cofiber sequence

$$Th(U_{2r,2n|Y}^{\oplus m}) \rightarrow Th(U_{2r,2n}^{\oplus m}) \rightarrow Th(\pi_+^* U_{2r,2n-2}^{\oplus m+1}).$$

where π_+ is the structure map of a vector bundle. Therefore, the induced morphism $Th(\pi_+^* U_{2r,2n-2}^{\oplus m+1}) \rightarrow Th(U_{2r,2n-2}^{\oplus m+1})$ is an unstable weak equivalence. By induction on n , we have reduced to showing that $\Sigma^\infty Th(U_{2r,2n|Y}^{\oplus m})$ is cellular. By [PW21, Thm. 5.1], we have a zig-zag

$$Y \leftarrow Y_1 \leftarrow Y_2 \rightarrow HG(2r-2, 2n-2)$$

where every map is an affine bundle, such that moreover there is an isomorphism of symplectic bundles

$$U_{2r,2n|Y_2} \cong \mathcal{O}_{Y_2}^{\oplus 2} \oplus U_{2r-2,2n|Y_2}$$

and the map $Y_2 \rightarrow HG(2r-2, 2n-2)$ has a section by the proof of [PW21, Thm. 5.2], whence every scheme in the sequence has a point. By Theorem B.1 we then have equivalences

$$Th(U_{2r,2n|Y}) \simeq Th(U_{2r,2n|Y_2}) \cong Th(\mathcal{O}_{Y_2}^{\oplus 2m} \oplus U_{2r-2,2n|Y_2}^{\oplus m}) \simeq S^{4m,2m} \wedge Th(U_{2r-2,2n}^{\oplus m}).$$

Induction completes the proof. \square

This is enough to prove the required result.

Proof of Theorem 6.7. The proof is essentially given in [RSØ18]. As before we have $\mathbf{KO}^{geo} = \text{hocolim}_n \Sigma^{-4n, -2n} \Sigma^\infty \mathbb{Z} \times HGr$. Therefore by [DI05, Lemma 3.4] and Definition 6.5 (3) it is enough to show that $\Sigma^\infty HGr(2n, 4n)_+$ is cellular for each n . \square

A. Naive spectra

Throughout this section, \mathcal{C} is a pointed cofibrantly generated model category with fibrant and cofibrant replacement functors R and Q respectively. We denote by $H(\mathcal{C})$ the corresponding homotopy category. Given a Quillen adjunction

$$(T, U, \eta) : \mathcal{C} \rightleftarrows \mathcal{C},$$

the category of T -spectra $Sp^{\mathbb{N}}(\mathcal{C}, T)$ has as objects sequences $\{E_i\}_{i \geq 0}$ of objects in \mathcal{C} along with assembly maps

$$e_i : T(E_i) \rightarrow E_{i+1}.$$

The category $Sp^{\mathbb{N}}(\mathcal{C}, T)$ inherits several model structures from \mathcal{C} . Here we will consider the levelwise and stable projective model structures given in [Hov01]. We denote the associated homotopy categories by $H^l(Sp^{\mathbb{N}}(\mathcal{C}, T))$ and $SH(\mathcal{C}, T)$ respectively.

Definition A.1 (Naive spectra). A naive T -spectrum $(E., e.)$ is a sequence $\{E_n\}_{n \in \mathbb{N}}$ of objects in $H(\mathcal{C})$ equipped with assembly morphisms

$$e_n \in Hom_{H(\mathcal{C})}(\mathbf{LT}(E_n), E_{n+1}).$$

A morphism of naive T -spectra $\phi : (E., e.) \rightarrow (F., f.)$ is a collection of morphisms $\phi_n : E_n \rightarrow F_n$ in $H(\mathcal{C})$ such that the relevant diagrams commute in $H(\mathcal{C})$. We denote this category by $Sp^{\text{naive}}(\mathcal{C}, T)$.

Any T -spectrum \mathbf{E} defines canonically a naive T -spectrum $\text{Naive}(\mathbf{E})$ with underlying sequence \mathbf{E}_n and assembly morphisms the image of

$$\mathbf{LT}(\mathbf{E}_n) = T(Q\mathbf{E}_n) \rightarrow T(\mathbf{E}_n) \xrightarrow{e_n} \mathbf{E}_{n+1}$$

in $H(\mathcal{C})$ for every n . This gives us a functor $\text{Naive} : Sp(\mathcal{C}, T) \rightarrow Sp^{\text{naive}}(\mathcal{C}, T)$.

Remark A.2. The notion of naive spectra presented here is a generalization of the one given in [Riou07].

Lemma A.3. The functor $\text{Naive} : Sp^{\mathbb{N}}(\mathcal{C}, T) \rightarrow Sp^{\text{naive}}(\mathcal{C}, T)$ is essentially surjective.

Proof. Given any naive T -spectrum $(E_n, e_n)_{n \in \mathbb{N}}$, choose bifibrant models $E'_n \in \mathcal{C}$ of E_n . We then have isomorphisms

$$Hom_{H(\mathcal{C})}(\mathbf{LT}(E_n), E_{n+1}) \cong Hom_{\mathcal{C}}(T(E'_n), E'_{n+1}) / \simeq$$

allowing us to choose a lift $e'_n : T(E'_n) \rightarrow E'_{n+1}$ of e_n for every n . We then have a T -spectrum \mathbf{E}' whose image under Naive is isomorphic to $(E_n, e_n)_{n \in \mathbb{N}}$. \square

Lemma A.4. Let $\mathbf{E} = (E_0, E_1, \dots), \mathbf{F} = (F_0, F_1, \dots) \in Sp^{\mathbb{N}}(\mathcal{C}, T)$ with assembly maps $e_n : T(E_n) \rightarrow E_{n+1}$ and $f_n : T(F_n) \rightarrow F_{n+1}$ respectively. If there is an isomorphism $\phi : \text{Naive}(\mathbf{E}) \xrightarrow{\sim} \text{Naive}(\mathbf{F})$ in $Sp^{\text{naive}}(\mathcal{C}, T)$, then \mathbf{E} and \mathbf{F} are isomorphic in $H^l(Sp^{\mathbb{N}}(\mathcal{C}, T))$.

Proof. We have isomorphisms $\phi_n : E_n \xrightarrow{\sim} F_n$ in $H(\mathcal{C})$ such that the diagrams

$$\begin{array}{ccc} \mathbf{LT}(E_n) & \longrightarrow & E_{n+1} \\ \downarrow \mathbf{LT}(\phi_n) & & \downarrow \phi_{n+1} \\ \mathbf{LT}(F_n) & \longrightarrow & F_{n+1} \end{array}$$

commute for all n . As \mathcal{C} is cofibrantly generated, so is $H^l(Sp^{\mathbb{N}}(\mathcal{C}, T))$ and hence it has a fibrant and cofibrant replacement functor [Hov01]. Using the cofibrant-fibrant replacement of the spectra \mathbf{E} and \mathbf{F} we can reduce to case when the spectra are levelwise cofibrant-fibrant. In this case each ϕ_n lifts to a weak equivalence in \mathcal{C} and commutativity of the diagrams implies that the two maps $\phi_{n+1}e_n$

and $f_n T(\phi_n)$ are homotopic [Hov99, Prop. 1.2.5]. For each $n \in \mathbb{N}$, let $E_n \times I$ be the functorial cylinder object for E_n . As $T(E_n)$ is cofibrant and F_{n+1} is fibrant, $T(E_n \times I)$ is a cylinder object for $T(E_n)$ and there is a left homotopy

$$\begin{array}{ccc} T(E_n) & \xrightarrow{e_n} & E_{n+1} \\ \downarrow i_0 & & \downarrow \phi_{n+1} \\ T(E_n \times I) & \xrightarrow{H_n} & F_{n+1} \\ \uparrow i_1 & & \uparrow f_n \\ T(E_n) & \xrightarrow{T(\phi_n)} & T(F_n), \end{array}$$

between $\phi_{n+1} e_n$ and $f_n T(\phi_n)$. The mapping cylinder M_{ϕ_n} is the pushout of the diagram

$$\begin{array}{ccc} E_n & \xrightarrow{\phi_n} & F_n \\ \downarrow i_1 & & \downarrow \\ E_n \times I & \longrightarrow & M_{\phi_n} \end{array}$$

and hence the morphism $F_n \rightarrow M_{\phi_n}$ is an acyclic cofibration with a left inverse induced by $id_{F_n} : F_n \rightarrow F_n$ and $E_n \times I \rightarrow E_n \xrightarrow{\phi_n} F_n$. As T is a left adjoint it preserves pushouts and hence $(T(\phi_n), H_n)$ induces a map

$$T(M_{\phi_n}) \rightarrow F_{n+1} \rightarrow M_{\phi_{n+1}}.$$

This gives us a spectrum $Cyl(\phi) = (M_{\phi_0}, M_{\phi_1}, \dots)$ with levelwise weak equivalences $\mathbf{E} \rightarrow Cyl(\phi)$ and $\mathbf{F} \rightarrow Cyl(\phi)$ given by

$$E_n \xrightarrow{i_0} E_n \times I \rightarrow M_{\phi_n} \quad \text{and} \quad F_n \rightarrow M_{\phi_n},$$

where $E_n \times I \rightarrow M_{\phi_n}$ is a weak equivalence by the 2-out-of-3 property. Hence \mathbf{E} and \mathbf{F} are isomorphic in $H(Sp^{\mathbb{N}}(\mathcal{C}, T))$. \square

These two lemmas show that a naive T -spectrum defines a unique object in $H^1(Sp^{\mathbb{N}}(\mathcal{C}, T))$ up to isomorphism.

Theorem A.5. *For any naive T -spectrum $(E_n, e_n)_{n \in \mathbb{N}}$, there exists a T -spectrum $\mathbf{E} \in Sp^{\mathbb{N}}(\mathcal{C}, T)$, with assembly maps $f'_n : T(\mathbf{E}_n) \rightarrow \mathbf{E}_{n+1}$, such that*

- (1) $\text{Naive}(\mathbf{E}) \xrightarrow{\sim} (E_n, e_n)_{n \in \mathbb{N}}$ in $SH(\mathcal{C}, T)$;
- (2) any other $\mathbf{E}' \in Sp^{\mathbb{N}}(\mathcal{C}, T)$ satisfying the above condition is isomorphic to \mathbf{E} in $H^1(Sp^{\mathbb{N}}(\mathcal{C}, T))$.

In particular every naive T -spectrum defines a unique object in $SH(\mathcal{C}, T)$ up to isomorphism.

Proof. Statement (2) follows from statement (1) and Lemma A.4. Statement (1) is just a reformulation of Lemma A.3. The last sentence follows because every levelwise weak equivalence is a stable equivalence. \square

Remark A.6. Using the projective model structure is not necessary; all we need is a model structure on $S\mathcal{P}^{\mathbb{N}}(\mathcal{C}, T)$ where the weak equivalences are precisely the levelwise weak equivalences and every bifibrant object is also levelwise bifibrant. Therefore the injective model structure works as well.

B. Thom spaces

The Thom space construction preserves \mathbb{A}^1 -weak equivalences for sufficiently well behaved schemes. Let S be a scheme which is ind-smooth over a Dedekind ring k with perfect residue fields.

Theorem B.1. *Let $X \in Sm_S$ be a smooth S -scheme with a rational point $x : S \rightarrow X$. For any \mathbb{A}^1 -equivalence of pointed smooth schemes $f : (Y, y) \rightarrow (X, x)$ and any vector bundle $E \rightarrow X$ of constant rank n , the induced map of Thom spaces $Th(f^*E) \rightarrow Th(E)$ is an \mathbb{A}^1 -equivalence.*

To prove this we use the following lemma.

Lemma B.2. *Let $E \rightarrow B$ be a principal GL_n -bundle over S . Given a point $b : S \rightarrow B$, the diagram*

$$GL_n \rightarrow E \rightarrow B$$

coming from the pullback

$$\begin{array}{ccc} GL_n & \longrightarrow & E \\ \downarrow & & \downarrow \\ S & \xrightarrow{b} & B \end{array}$$

is an \mathbb{A}^1 -local fiber sequence. Furthermore, for any scheme F with a GL_n -action $\sigma : GL_n \times F \rightarrow F$ the induced diagram

$$F \rightarrow E \times_{\sigma} F \rightarrow B$$

is an \mathbb{A}^1 -local fiber sequence.

Proof. Note that for any locally trivial bundle $P \rightarrow X$ with fiber F and any point x in X the pullback diagram

$$F \rightarrow P \rightarrow X$$

is a simplicial fiber sequence (taking stalks gives a fiber sequence of simplicial sets). For any smooth S -scheme B , there is a sequence of bijections

$$Vect_n(B) \cong P_{Nis}(B, GL_n) \cong P_{Nis}(B, GL_n) \cong Hom_{H^s(S)}(B, BGL_n)$$

where $P_{\tau}(B, GL_n)$ is the set of τ -locally trivial GL_n bundles by [AHW18, Ex.2.3.4] and [MV99, Prop.4.1.15]. This implies that the map $E \rightarrow B$ is a pullback of the GL_n -bundle $E_{Nis}GL_n \rightarrow B_{Nis}GL_n$ where $B_{Nis}GL_n$ is a Nisnevich fibrant replacement. As every vector bundle (and hence every GL_n -torsor) is Zariski locally trivial, BGL_n satisfies Nisnevich descent and hence we have a bijection

$$\pi_0(B_{Nis}GL_n(X)) \cong \pi_0(BGL_n(X))$$

for any $X \in Sm_S$. By [AHW18, Thm. 5.2.3], the set of rank n vector bundles $Vect_n(-)$ is \mathbb{A}^1 -invariant for affine schemes over S and hence we have

$$Vect_n(X) \cong \pi_0(B_{Nis}GL_n(X)) \cong \pi_0(BGL_n(X))$$

for X affine. By [AHW18, Thm. 2.2.5]

$$G \rightarrow E_{Nis}G \rightarrow B_{Nis}G$$

is an \mathbb{A}^1 -local fiber sequence hence by [Wen11, Prop. 2.3]

$$GL_n \rightarrow E \rightarrow B$$

is an \mathbb{A}^1 -local fiber sequence. For any scheme F with a GL_n -action, we can show that

$$F \rightarrow E_{Nis}(GL_n) \times_{\sigma} F \rightarrow B_{Nis}(GL_n)$$

is an \mathbb{A}^1 -local fiber sequence along the lines of [Wen11, Prop. 5.1]. The simplicial fiber sequence

$$F \rightarrow E \times_{\sigma} F \rightarrow B$$

is a pullback of the universal sequence and hence is also \mathbb{A}^1 -local. \square

Proof of Theorem B.1. Given any vector bundle $E \rightarrow X$ of rank n , the 2 out of 3 property implies that an \mathbb{A}^1 -weak equivalence $f : Y \rightarrow X$ induces an \mathbb{A}^1 -weak equivalence $f^*E \rightarrow E$. The complement of the zero section $E - X \rightarrow X$ is a locally trivial bundle with fiber $\mathbb{A}^n - 0$. The fiber sequence

$$\mathbb{A}^n - 0 \rightarrow E - X \rightarrow X$$

is obtained by twisting the GL_n -torsor associated to the vector bundle $E \rightarrow X$ by the standard GL_n -action on $\mathbb{A}^n - 0$ and is hence an \mathbb{A}^1 -local fiber sequence by Lemma B.2. Thus the pullback of $E - X$ along an \mathbb{A}^1 -equivalence $f : Y \rightarrow X$ induces an \mathbb{A}^1 -equivalence $f^*E - Y \rightarrow E - X$. We therefore have an equivalence of cofibration sequences

$$\begin{array}{ccccc} f^*E - Y & \longrightarrow & f^*E & \longrightarrow & Th(f^*E) \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ E - X & \longrightarrow & E & \longrightarrow & Th(E) \end{array}$$

giving the desired \mathbb{A}^1 -equivalence of Thom spaces. \square

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