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# Linear independence between odd and even periods of modular forms

**Hui Xue**

ABSTRACT. We investigate the linear dependence between an odd period and an even period of modular forms. We show that two periods of different parity are linearly independent provided that the even period has index at least 6 or the odd period has index at least 7.

CONTENTS



## 1. Introduction and statements of results

For each even integer $k \geq 4$, let $M_k$ be the space of modular forms of weight $k$ for $\mathrm{SL}_2(\mathbb{Z})$, and let $S_k$ be its subspace of cuspforms. For each $0 \leq t \leq k-2$, the $t^{\text{th}}$ period of $f \in S_k$ is defined as [6]

$$r_t(f) := \int_0^{i\infty} f(z) z^t dz = \frac{t!}{(-2\pi i)^{t+1}} L(f, t+1). \tag{1}$$

Here, the $L$-series of a cuspform $f(z) = \sum_{n=1}^{\infty} a_n(f) e^{2\pi i n z} \in S_k$ is $L(f, s) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s}$. Each $r_t$ defines a linear map from $S_k$ to $\mathbb{C}$, that is $r_t \in S_k^*$ (the dual space of $S_k$).

The set of odd periods $\{r_{2i+1}\}_{i=0}^{k/2-2}$ and the set of even periods $\{r_{2i}\}_{i=0}^{k/2-1}$ behave differently, and they are subject to many linear dependence relations, called the Eichler-Shimura relations; see Manin [6] for more details. However, not much is known about the linear independence of a subset of the periods: the first work in this direction seems to be [1], in which Fukuhara found an explicit subset of odd periods that forms a basis for $S_k^*$. As a corollary, Fukuhara [2] found a special basis for $M_k$ consisting of products of two Eisenstein series.

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Most recently, Lei et al. [5, 4] have provided some evidence for the linear independence of odd periods and even periods, respectively. The main theme of [5, 4] is that odd or odd periods of modular forms are linearly independent unless forced by dimension considerations. On the other hand, very little seems to be known about the relationship between even and odd periods: for instance, the Eichler-Shimura relations only address them separately. In the present paper, we will extend the ideas of [5, 4] to provide some evidence for the linear independence between odd and even periods. More precisely, we will show the following.

**Theorem 1.1.** *Let $\ell$ and $\ell'$ be positive even integers such that $\ell < \frac{k}{2} - 1$ and $\ell' \leq \frac{k}{2}$, and suppose that $\ell \geq 6$ or $\ell' \geq 8$. If $\dim S_k \geq 2$, then the even period $r_\ell$ and the odd period $r_{\ell'-1}$ are linearly independent.*

The restriction to periods $r_\ell$ and $r_{\ell'-1}$ for even integers $\ell < \frac{k}{2}-1$ and $\ell' \leq \frac{k}{2}$ is due to the Eichler-Shimura relations $r_\ell + r_{k-2-\ell} = 0$ and $r_{\ell'-1} - r_{k-1-\ell'} = 0$. In fact, numerical computation done by Daozhou Zhu ([10]) shows that Theorem 1.1 holds true for all positive even integers $\ell$ and $\ell'$ for $k \leq 100$ and $\dim S_k \geq 2$. So, we propose the following natural speculations.

**Conjecture 1.2.** *(1) Let $\ell$ and $\ell'$ be positive even integers such that $\ell < \frac{k}{2} - 1$ and $\ell' \leq \frac{k}{2}$. If $\dim S_k \geq 2$, then the even period $r_\ell$ and the odd period $r_{\ell'-1}$ are linearly independent.*
*(2) More generally, suppose $2 \leq \ell_1 < \ell_2 < \cdots < \ell_a < \frac{k}{2} - 1$ and $2 \leq \ell'_1 < \cdots < \ell'_b \leq \frac{k}{2}$ are even integers. If $a + b \leq \dim S_k$, then the set of periods $\{r_{\ell_1}, \cdots, r_{\ell_a}, r_{\ell'_1-1}, \cdots, r_{\ell'_b-1}\}$ is linearly independent.*

We now give an account of the main idea of the proof. For an even integer $k \geq 2$, let $E_k(z)$ denote the normalized Eisenstein series of weight $k$ given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $B_k$ is the $k$-th Bernoulli number, $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and $q = e^{2\pi i z}$. It should be noted that $E_2(z)$, although holomorphic, is not a modular form; it is a quasi-modular form instead [9].

Let $f \in M_k$ and $g \in M_\ell$. For some integer $d \geq 0$, the $d$-th Rankin-Cohen bracket of $f$ and $g$ is defined as [8, (1)]:

$$[f, g]_d = \sum_{0 \leq r \leq d} (-1)^r \binom{d+k-1}{d-r} \binom{d+\ell-1}{r} f^{(r)} g^{(d-r)},$$

where $f^{(r)} := \frac{1}{(2\pi i)^r} \frac{d^r f}{dz^r}$ is the normalized $r^{\text{th}}$ derivative of $f$ with respect to $z$. In this paper, we are only interested in the cases when both $f$ and $g$ are Eisenstein series and $d = 0, 1$. In order to include $E_2$ in the Rankin-Cohen brackets, we also define ([3, p. 214 (ii)] and [4, (1.2)])

$$[E_k, E_2]_d := \sum_{0 \le r \le d} (-1)^r \binom{d+k-1}{d-r}\binom{d+\ell-1}{r} E_k^{(r)} E_2^{(d-r)} - (-1)^d \frac{12}{k+d} E_k^{(d+1)} \tag{2}$$

for $k > 2$, where $E_2^{(i)}$ on the right hand side is the normalized $i^{\text{th}}$ derivative of $E_2$ with respect to the variable $z$, for $0 \le i \le d$. Then, $[E_k, E_2]_d$ is a modular form in $M_{k+2+2d}$.

Next, we recall the Rankin's identity for the two cases $d = 0, 1$. Let $k > \ell \ge 2$ and $k' \ge \ell' \ge 2$ be even integers such that $K := k + \ell + 2 = k' + \ell'$. Then, by ([7, (77)], [3, pp. 213-215]) we have the following formulas for the Petersson inner products

$$\langle g, E_{k'}E_{\ell'} - E_K\rangle = (-1)^{\frac{k'}{2}} \frac{\Gamma(K-1)\Gamma(k')}{(4\pi)^{K-1}(2\pi)^{k'}} \frac{2k'}{B_{k'}} \frac{2\ell'}{B_{\ell'}} L(g, K-1)L(g, k') \tag{3}$$

and

$$\langle g, [E_k, E_\ell]_1\rangle = (-1)^{\frac{k}{2}+1} \frac{\Gamma(K-1)\Gamma(k+1)}{(4\pi)^{K-1}(2\pi)^{k}} \frac{2k}{B_{k}} \frac{2\ell}{B_{\ell}} L(g, K-2)L(g, k+1) \tag{4}$$

For later application we need to normalize $[E_k, E_\ell]_1$ and $E_kE_\ell - E_K$, so that their $q$-coefficients become 1. It follows from [4, Section 2] that

$$a_1(E_{k'}E_{\ell'} - E_K) = \begin{cases} -\frac{2k'}{B_{k'}} - \frac{2\ell'}{B_{\ell'}} + \frac{2K}{B_K} & \text{if } \ell' \ge 4 \\ -24\left(1 + \frac{K-2}{12B_{K-2}} - \frac{1}{B_{K-2}} - \frac{K}{12B_K}\right) & \text{if } \ell' = 2 \end{cases}, \tag{5}$$

and

$$a_1([E_k, E_\ell]_1) = \begin{cases} \frac{2k\ell}{B_k} - \frac{2k\ell}{B_\ell} & \text{if } \ell \ge 4 \\ \frac{4k}{B_k} - \frac{24k}{(k+1)B_k} - \frac{4k}{B_2} & \text{if } \ell = 2 \end{cases}. \tag{6}$$

We normalize $[E_k, E_\ell]_1$, denoted $\Delta^1_{k,\ell}(z)$, so that its $q$-coefficient $a_1(\Delta^1_{k,\ell})$ becomes 1. Similarly, we normalize $E_kE_\ell - E_K$, denoted $\Delta_{k',\ell'}(z)$, so that $a_1(\Delta_{k',\ell'})$ $= 1$. The following result follows immediately from (3)-(6).

**Proposition 1.3.** *Let $k > \ell \ge 2$ and $k' \ge \ell' \ge 2$ be even integers such that $K := k + \ell + 2 = k' + \ell'$. Let $\mathcal{H}_K$ denote the set of normalized Hecke eigenforms in $S_K$. Then*

$$\Delta^1_{k,\ell} = A^1_{k,\ell} \cdot \sum_{g \in \mathcal{H}_K} \frac{L(g, K-2)L(g, k+1)}{\langle g, g\rangle} g,$$

*where*

$$A^1_{k,\ell} := (-1)^{\frac{k}{2}+1} \frac{\Gamma(K-1)\Gamma(k+1)}{(4\pi)^{K-1}(2\pi)^{k}} \frac{2k}{B_{k}} \frac{2\ell}{B_{\ell}} \cdot \frac{1}{a_1([E_k, E_\ell]_1)}$$

*with* $a_1([E_k, E_\ell]_1)$ *given by* (6). *Also,*

$$\Delta_{k',\ell'} = A_{k',\ell'} \cdot \sum_{g \in \mathcal{H}_K} \frac{L(g, K-1)L(g, k')}{\langle g, g\rangle} g,$$

*where*

$$A_{k',\ell'} := (-1)^{\frac{k'}{2}} \frac{\Gamma(K-1)\Gamma(k')}{(4\pi)^{K-1}(2\pi)^{k'}} \frac{2k'}{B_{k'}} \frac{2\ell'}{B_{\ell'}} \cdot \frac{1}{a_1(E_{k'}E_{\ell'} - E_K)}$$

*with* $a_1(E_{k'}E_{\ell'} - E_K)$ *given in* (5).

*Remark* 1.4. We want to point out that the actual values of $A^1_{k,\ell}$ and $A_{k',\ell'}$ are not important, as long as they are nonzero and are independent of $g \in \mathcal{H}_K$; see Section 2. It is also important to note that for each $g \in \mathcal{H}_K$ the value $L(g, k+1)$ is positive because $k+1 > \frac{K+1}{2}$ is within the region of absolute convergence for the Euler product of $L(g, s)$.

Now, assume that $\ell$ and $\ell'$ satisfy the conditions of Theorem 1.1, and that $r_\ell$ and $r_{\ell'}$ are linearly dependent. Our strategy is to compare the $a_2$ Fourier coefficients of $\Delta^1_{k,\ell}$ and $\Delta_{k',\ell'}$, to reach a contradiction. On one hand, using results from [4, 5], we show (Lemma2.2) that for $\ell \geq 6$ or $\ell' \geq 8$ and $K \geq 100$

$$|a_2(\Delta^1_{k,\ell}) - a_2(\Delta_{k',\ell'})| > 18. \tag{7}$$

On the other hand, by some detailed analysis on the $L$-values $L(g, K-1)$ and $L(g, K-2)$ for $g \in \mathcal{H}_K$, we obtain (Proposition 2.6) for $K \geq 100$

$$|a_2(\Delta^1_{k,\ell}) - a_2(\Delta_{k',\ell'})| < 16.007. \tag{8}$$

These arguments enable us to finish the proof for $K \geq 100$. The case $K < 100$ has been verified numerically by Daozhou Zhu; see [10]. Altogether, the proof of Theorem 1.1 is complete.

## 2. The proofs

We shall first establish a lower bound for $|a_2(\Delta^1_{k,\ell}) - a_2(\Delta_{k',\ell'})|$. In order to do this we first recall the following results on the $a_2$-coefficients of $\Delta^1_{k,\ell}$ and $\Delta_{k',\ell'}$ for large $K$.

**Proposition 2.1.** *We have*

$$\lim_{k\to\infty} \frac{a_2(\Delta^1_{k,\ell})}{2(1+2^{\ell-1})} = 1, \quad \textit{and} \quad \lim_{k'\to\infty} \frac{a_2(\Delta_{k',\ell'})}{1+2^{\ell'-1}} = 1.$$

*Moreover, when* $K \geq 100$, *we have*

$$a_2(\Delta^1_{k,\ell}) = (2+2^\ell)(1+\delta^1), \quad \textit{and} \quad a_2(\Delta_{k',\ell'}) = (1+2^{\ell'-1})(1+\delta),$$

*where* $|\delta^1| < 0.11294$, *and* $|\delta| < 0.21703$.

**Proof.** The bound for $|\delta|$ is obtained by plugging $K = 100$ in the calculations in [5, Lemma 3.8]. The bound for $|\delta^1|$ is obtained by plugging $K = 100$ into the calculations in [4, Proposition 3.4]. $\square$

**Lemma 2.2.** *Let $K \geq 100$. If $\ell \geq 6$ or $\ell' \geq 8$ are even, then*

$$|a_2(\Delta^1_{k,\ell}) - a_2(\Delta_{k',\ell'})| > 18.$$

**Proof.** Let us first assume that $\ell' \geq 8$. Then there are two cases to consider. Case 1: $\ell \geq \ell'$. By Proposition 2.1

$$\begin{aligned} |a_2(\Delta^1_{k,\ell}) - a_2(\Delta_{k',\ell'})| &= |(2+2^\ell)(1+\delta^1) - (1+2^{\ell'-1})(1+\delta)| \\ &= (2+2^\ell)\left|(1+\delta^1) - \frac{(1+2^{\ell'-1})}{2(1+2^{\ell-1})}(1+\delta)\right| \\ &> (2+2^\ell)(1-0.11294-0.5(1+0.21763)) \\ &> (2+2^8)\cdot 0.27825 \\ &> 71. \end{aligned}$$

Case 2: $\ell < \ell'$. Note that $(2+2^\ell)/(1+2^{\ell'-1})$ maximizes at $\ell = 6$ and $\ell' = 8$. Thus

$$\begin{aligned} |a_2(\Delta^1_{k,\ell}) - a_2(\Delta_{k',\ell'})| &= |(2+2^\ell)(1+\delta^1) - (1+2^{\ell'-1})(1+\delta)| \\ &= (1+2^{\ell'-1})\left|\frac{2+2^\ell}{1+2^{\ell'-1}}(1+\delta^1) - (1+\delta)\right| \\ &> (1+2^{\ell'-1})\left(1 - |\delta| - \frac{2+2^6}{1+2^7}(1+|\delta^1|)\right) \\ &> (1+2^{\ell'-1})\cdot\left(1-0.21763 - \frac{66}{129}(1+0.11294)\right) \\ &> 27. \end{aligned}$$

Now, assume that $\ell \geq 6$. Similarly, when $\ell \geq \ell'$, we have

$$\begin{aligned} |a_2(\Delta^1_{k,\ell}) - a_2(\Delta_{k',\ell'})| &> (2+2^\ell)(1-0.11294-0.5(1+0.21763)) \\ &> (2+2^6)\cdot 0.27825 \\ &> 18. \end{aligned}$$

When $\ell < \ell'$, then $\ell' \geq 8$ and we have

$$\begin{aligned} |a_2(\Delta^1_{k,\ell}) - a_2(\Delta_{k',\ell'})| &> (1+2^{\ell'-1})\cdot\left(1-0.21763 - \frac{66}{129}(1+0.11294)\right) \\ &> 27. \end{aligned}$$

The proof is now complete. $\square$

*Remark* 2.3. If neither condition of Lemma 2.2 is met, for instance if $\ell' = 6$ and $\ell = 4$, then

$$|2+2^4-1-2^5| = 15 < 16,$$

which does not contradict the upper bound obtained in Proposition 2.6. Therefore, it seems that Lemma 2.2 is optimal.

Next, we shall establish some estimates on the values $L(g,K-1)$ and $L(g,K-2)$ for each Hecke eigenform $g\in\mathcal{H}_K$ and for $K\geq 100$.

By Deligne's bound $|a_n(g)|\leq d(n)n^{(K-1)/2}$ with $d(n)$ being the number of divisor function and the fact that $\sum_{n=1}^{\infty}\frac{d(n)}{n^s}=\zeta(s)^2$, we get

$$\left|L(g,K-1)-1-\frac{a_2(g)}{2^{K-1}}\right|\leq\sum_{n=3}\frac{d(n)}{n^{(K-1)/2}}=\zeta\left(\frac{K-1}{2}\right)^2-1-2^{1-(K-1)/2}.$$

Noting that

$$\begin{aligned}\zeta\left(\frac{K-1}{2}\right)&=1+2^{-(K-1)/2}+3^{-(K-1)/2}+\sum_{n=4}^{\infty}n^{-(K-1)/2}\\&\leq 1+2^{-(K-1)/2}+3^{-(K-1)/2}+\int_3^{\infty}t^{-(K-1)/2}\,dt\\&=1+2^{-(K-1)/2}+3^{-(K-1)/2}+\frac{6}{K-3}\cdot 3^{-(K-1)/2}\\&=1+2^{-(K-1)/2}+\frac{K+3}{K-3}3^{-(K-1)/2}.\end{aligned}$$

This means that

$$\begin{aligned}&\zeta\left(\frac{K-1}{2}\right)^2-1-2^{1-(K-1)/2}\\=&2\cdot 3^{-(K-1)/2}\frac{K+3}{K-3}+2^{-(K-1)}+2^{1-(K-1)/2}3^{-(K-1)/2}\frac{K+3}{K-3}\\&+\left(\frac{K+3}{K-3}\right)^2 3^{-(K-1)}.\end{aligned}$$

Thus, for $K\geq 100$, we may write

$$L(g,K-1)=1+\frac{a_2(g)}{2^{K-1}}+\delta_{K-1}\cdot 3^{-(K-1)/2},\tag{9}$$

where

$$\begin{aligned}&|\delta_{K-1}|\\\leq&2\frac{K+3}{K-3}+\left(\frac{3}{4}\right)^{(K-1)/2}+2^{1-(K-1)/2}\frac{K+3}{K-3}+\left(\frac{K+3}{K-3}\right)^2 3^{-(K-1)/2}\\<&2.12372.\end{aligned}\tag{10}$$

Similarly, for $L(g,K-2)$ we have

$$\left|L(g,K-2)-1-\frac{a_2(g)}{2^{K-2}}\right|\leq\sum_{n=3}\frac{d(n)}{n^{(K-3)/2}}=\zeta\left(\frac{K-3}{2}\right)^2-1-2^{1-(K-3)/2},$$

and for $K \geq 100$ we obtain

$$L(g, K-2) = 1 + \frac{a_2(g)}{2^{K-2}} + \delta_{K-2} \cdot 3^{-(K-3)/2}, \tag{11}$$

where

$$\begin{aligned} &|\delta_{K-2}| \\ \leq &2\frac{K+1}{K-5} + \left(\frac{3}{4}\right)^{(K-3)/2} + 2^{1-(K-3)/2}\frac{K+1}{K-5} + \left(\frac{K+1}{K-5}\right)^2 3^{-(K-3)/2} \\ < &2.12632. \end{aligned} \tag{12}$$

From now on, let us assume on the contrary that the odd period $r_{\ell'-1}$ and the even period $r_\ell$ are linearly dependent. By (1) and the Eichler-Shimura relations $r_\ell + r_{K-2-\ell} = 0$ and $r_{\ell'-1} - r_{K-1-\ell'} = 0$, this means that there is some constant $c$ such that for all $g \in \mathcal{H}_K$, where $K = k + \ell + 2 = k' + \ell'$, such that

$$L(g, k') = c \cdot L(g, k+1). \tag{13}$$

Our strategy is to derive a contradiction from (13) on the value

$$|a_2(\Delta^1_{k,\ell}) - a_2(\Delta_{k',\ell'})|.$$

A lower bound of it has been established in Lemma 2.2. Our next task is to find an upper bound. We shall first derive some information from the $a_1$-coefficients. By Proposition 1.3 and (11), we obtain

$$\begin{aligned} 1 &= A^1_{k,\ell} \sum_{g \in \mathcal{H}_K} \frac{L(g, K-2)L(g, k+1)}{\langle g, g\rangle} \\ &= A^1_{k,\ell} \sum_{g \in \mathcal{H}_K} \frac{L(g, k+1)(1 + a_2(g)2^{2-K} + \delta_{K-2} \cdot 3^{-(K-3)/2})}{\langle g, g\rangle}. \end{aligned}$$

As $|a_2(g)2^{2-K}| \leq 2 \cdot 2^{(3-K)/2}$, by (12), for $K \geq 100$

$$|a_2(g)2^{2-K}| + |\delta_{K-2} \cdot 3^{-(K-3)/2}| < 2.0001 \cdot 2^{(3-K)/2}.$$

Since $L(g, k+1)$ is positive for all $g$ (Remark 1.4), for $K \geq 100$

$$\begin{aligned} 1 &= \left| A^1_{k,\ell} \sum_{g \in \mathcal{H}_K} \frac{L(g, k+1)(1 + a_2(g)2^{2-K} + \delta_{K-2} \cdot 3^{-(K-3)/2})}{\langle g, g\rangle} \right| \\ &\geq |A^1_{k,\ell}| \sum_{g \in \mathcal{H}_K} \frac{L(g, k+1)(1 - |a_2(g)2^{2-K}| - |\delta_{K-2} \cdot 3^{-(K-3)/2}|)}{\langle g, g\rangle} \\ &\geq \left| A^1_{k,\ell} \sum_{g \in \mathcal{H}_K} \frac{L(g, k+1)}{\langle g, g\rangle} \right| \cdot (1 - 2.0001 \cdot 2^{(3-K)/2}) \end{aligned}$$

and

$$\begin{aligned}1 = &\left|A^1_{k,\ell}\sum_{g\in\mathcal{H}_K}\frac{L(g,k+1)(1+a_2(g)2^{2-K}+\delta_{K-2}\cdot 3^{-(K-3)/2})}{\langle g,g\rangle}\right|\\ \leq &|A^1_{k,\ell}|\sum_{g\in\mathcal{H}_K}\frac{L(g,k+1)(1+|a_2(g)2^{2-K}|+|\delta_{K-2}\cdot 3^{-(K-3)/2}|)}{\langle g,g\rangle}\\ \leq &\left|A^1_{k,\ell}\sum_{g\in\mathcal{H}_K}\frac{L(g,k+1)}{\langle g,g\rangle}\right|\cdot(1+2.0001\cdot 2^{(3-K)/2}).\end{aligned}$$

Thus, for $K\geq 100$

$$A^1_{k,\ell}\sum_{g\in\mathcal{H}_K}\frac{L(g,k+1)}{\langle g,g\rangle} = 1+\epsilon^1_K(1), \tag{14}$$

for some $|\epsilon^1_K(1)| < 2.001\cdot 2^{(3-K)/2}$.

Similarly, by Proposition 1.3, (11) and taking (13) into account, we get

$$\begin{aligned}1 &= A_{k',\ell'}\sum_{g\in\mathcal{H}_K}\frac{L(g,K-1)L(g,k')}{\langle g,g\rangle}\\ &= cA_{k',\ell'}\sum_{g\in\mathcal{H}_K}\frac{L(g,k+1)(1+a_2(g)2^{1-K}+\delta_{K-1}\cdot 3^{-(K-1)/2})}{\langle g,g\rangle}.\end{aligned}$$

For $K\geq 100$, as

$$|a_2(g)2^{1-K}|\leq 2\cdot 2^{(1-K)/2}$$

and

$$|a_2(g)2^{1-K}+\delta_{K-1}\cdot 3^{-(K-1)/2}| < 2.0001\cdot 2^{(1-K)/2},$$

we obtain analogously

$$cA_{k',\ell'}\sum_{g\in\mathcal{H}_K}\frac{L(g,k+1)}{\langle g,g\rangle} = 1+\epsilon_K(1) \tag{15}$$

for some $|\epsilon_K(1)|\leq 2.001\cdot 2^{(1-K)/2}$.

We next investigate and compare the $a_2$-coefficients of $\Delta^1_{k,\ell}$ and $\Delta_{k,\ell}$. Again, by Proposition 1.3 and (11)

$$\begin{aligned}a_2(\Delta^1_{k,\ell}) &= A^1_{k,\ell}\sum_{g\in\mathcal{H}_K}\frac{L(g,K-2)L(g,k+1)}{\langle g,g\rangle}a_2(g)\\ &= A^1_{k,\ell}\sum_{g\in\mathcal{H}_K}\frac{L(g,k+1)}{\langle g,g\rangle}a_2(g)+A^1_{k,\ell}\sum_{g\in\mathcal{H}_K}\frac{L(g,k+1)}{\langle g,g\rangle}a_2^2(g)2^{2-K}\\ &+A^1_{k,\ell}\sum_{g\in\mathcal{H}_K}\frac{L(g,k+1)}{\langle g,g\rangle}\cdot\delta_{K-2}\cdot 3^{-(K-3)/2}a_2(g).\end{aligned} \tag{16}$$

We denote the last term of (16) by $\epsilon_K^1(2)$. Then, for $K \geq 100$, by (12)

$$\begin{aligned}|\epsilon_K^1(2)| =& \left| A_{k,\ell}^1 \sum_{g \in \mathcal{H}_K} \frac{L(g,k+1)}{\langle g,g \rangle} \cdot \delta_{K-2} \cdot 3^{-(K-3)/2} a_2(g) \right| \\ \leq& \left| A_{k,\ell}^1 \sum_{g \in \mathcal{H}_K} \frac{L(g,k+1)}{\langle g,g \rangle} \right| \cdot |\delta_{K-2}| \cdot 2 \cdot \left(\frac{2}{3}\right)^{(K-3)/2} \\ <& 0.0001.\end{aligned}$$

Here we have again used the fact that $L(g, k+1)$ is positive, see Remark 1.4. Analogously, for $K \geq 100$ we have

$$\begin{aligned}& a_2(\Delta_{k',\ell'}) \\ =& cA_{k',\ell'} \sum_{g \in \mathcal{H}_K} \frac{L(g,k+1)}{\langle g,g \rangle} a_2(g) + cA_{k',\ell'} \sum_{g \in \mathcal{H}_K} \frac{L(g,k+1)}{\langle g,g \rangle} a_2^2(g) 2^{1-K} \\ &+ \epsilon_K(2),\end{aligned} \tag{17}$$

where

$$|\epsilon_K(2)| < 0.0001. \tag{18}$$

**Lemma 2.4.** *Let $k > \ell \geq 2$ and $k' \geq \ell' \geq 2$ be even integers such that $K := k + \ell + 2 = k' + \ell'$. Assume that (13): $L(g, k')) = cL(g, k+1)$ holds for all $g \in \mathcal{H}_K$. Then for $K \geq 100$*

$$\left| A_{k,\ell}^1 \sum_{g \in \mathcal{H}_K} \frac{L(g,k+1)}{\langle g,g \rangle} a_2^2(g) 2^{1-K} \right| \leq 4(1 + |\epsilon_K^1(1)|).$$

**Proof.** This is due to the Deligne's bound $|a_2(g)| \leq 2 \cdot 2^{(K-1)/2}$, (14) and positivity of $L(g, k+1)$. $\square$

**Lemma 2.5.** *Retain the assumptions of Lemma 2.4. Then for $K \geq 100$*

$$\left| (A_{k,\ell}^1 - cA_{k',\ell'}) \left( \sum_{g \in \mathcal{H}_K} \frac{L(g,k+1)}{\langle g,g \rangle} a_2(g) \right) \right| < 12.006,$$

*and*

$$\left| A_{k,\ell}^1 - cA_{k',\ell'} \right| \sum_{g \in \mathcal{H}_K} \frac{L(g,k+1)}{\langle g,g \rangle} a_2^2(g) 2^{1-K} \leq 4(|\epsilon_K^1(1)| + |\epsilon_K(1)|).$$

**Proof.** By (14) and (15), we get

$$\left| (A_{k,\ell}^1 - cA_{k',\ell'}) \sum_{g \in \mathcal{K}} \frac{L(g,k+1)}{\langle g,g \rangle} \right| \leq |\epsilon_K^1(1)| + |\epsilon_K(1)|.$$

Remembering that $|a_2(g)| \leq 2 \cdot 2^{(K-1)/2}$, $|\epsilon_K^1(1)| \leq 4.002 \cdot 2^{(1-K)/2}$, $|\epsilon_K(1)| \leq 2.001 \cdot 2^{(1-K)/2}$, and that $L(g,k+1) > 0$, thus

$$\begin{aligned}
&\left|(A^1_{k,\ell} - cA_{k',\ell'})\left(\sum_{g\in\mathcal{H}_K} \frac{L(g,k+1)}{\langle g,g\rangle} a_2(g)\right)\right| \\
\leq& 2\cdot 2^{(K-1)/2}\left|(A^1_{k,\ell} - cA_{k',\ell'})\sum_{g\in\mathcal{K}} \frac{L(g,k+1)}{\langle g,g\rangle}\right| \\
\leq& 2\cdot 2^{(K-1)/2}\cdot(|\epsilon_K^1(1)| + |\epsilon_K(1)|) \\
\leq& 12.006,
\end{aligned}$$

giving the first inequality. The second inequality follows exactly the same way by noticing that $|a_2(g)^2 2^{1-K}| \leq 4$. $\square$

We are now ready to establish an upper bound for $|\Delta^1_{k,\ell} - \Delta_{k',\ell'}|$. Note that this result does not require that $\ell$ or $\ell'$ satisfy the conditions in Lemma 2.2.

**Proposition 2.6.** *Let $k > \ell \geq 2$ and $k' \geq \ell' \geq 2$ be even integers such that $K := k + \ell + 2 = k' + \ell'$. Assume that (13): $L(g,k')) = cL(g,k+1)$ holds for all $g \in \mathcal{H}_K$. Then, for $K \geq 100$*

$$|a_2(\Delta^1_{k,\ell}) - a_2(\Delta_{k',\ell'})| < 16.007.$$

**Proof.** By (16) and (17)

$$\begin{aligned}
&|a_2(\Delta^1_{k,\ell}) - a_2(\Delta_{k',\ell'})| \\
\leq& \left|A^1_{k,\ell} - cA_{k,\ell}\right|\left|\sum_{g\in\mathcal{H}_K} \frac{L(g,k+1)}{\langle g,g\rangle} a_2(g)\right| + \left|A^1_{k,\ell} - cA_{k',\ell'}\right| \sum_{g\in\mathcal{H}_K} \frac{L(g,k+1)}{2^{K-1}\langle g,g\rangle} a_2^2(g) \\
&+ \left|A^1_{k,\ell}\sum_{g\in\mathcal{H}_K} \frac{L(g,k+1)}{\langle g,g\rangle} a_2^2(g)2^{1-K}\right| + |\epsilon_K^1(2)| + |\epsilon_K(2)|.
\end{aligned}$$

Now, by Lemma 2.4

$$\left|A^1_{k,\ell}\sum_{g\in\mathcal{H}_K} \frac{L(g,k+1)}{\langle g,g\rangle} a_2^2(g)2^{1-K}\right| \leq 4(1 + |\epsilon_K^1(1)|).$$

By Lemma 2.5

$$\left|(A^1_{k,\ell} - cA_{k',\ell'})\left(\sum_{g\in\mathcal{H}_K} \frac{L(g,k+1)}{\langle g,g\rangle} a_2(g)\right)\right| < 12.006,$$

and hence

$$\left|A^1_{k,\ell} - cA_{k',\ell'}\right| \sum_{g\in\mathcal{H}_K} \frac{L(g,k+1)}{\langle g,g\rangle} a_2^2(g)2^{1-K} \leq 4(|\epsilon_K^1(1)| + |\epsilon_K(1)|).$$

The desired upper bound then follows by taking into consideration of the bounds $|\epsilon_K^1(1)| < 2.001 \cdot 2^{(3-K)/2}$, $|\epsilon_K(1)| < 2.001 \cdot 2^{(1-K)/2}$, $|\epsilon_K^1(2)|, |\epsilon_K(2)| < 0.0001$. $\square$

As having been explained in the introduction, the lower bound Lemma 2.2 and the upper bound Proposition 2.6 imply that Theorem 1.1 holds true for $K \geq 100$. The remaining finitely many cases have been numerically verified by Daozhou Zhu ([10]).

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(Hui Xue) School of Mathematical and Statistical Sciences, Clemson University, Clemson, SC 29634-0975, USA
huixue@clemson.edu

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