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The sigma invariants for the golden mean Thompson group

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Abstract. We use a method of Bieri, Geoghegan and Kochloukova to calculate the BNSR-invariants for the irrational slope Thompson's group F_τ. To do so we establish conditions under which the Sigma invariants coincide with those of a subgroup of finite index, addressing a problem posed by Strebel.

Contents

1. Introduction

The study of what is now known as the first Sigma invariant or the Bieri–Neumann–Strebel invariant $\Sigma^1(G)$ for a finitely generated group G goes back to [5, 3] and was later extended by Bieri and Renz to a sequence of homotopical invariants

$$\cdots \subseteq \Sigma^n(G) \subseteq \Sigma^{n-1}(G) \subseteq \cdots \subseteq \Sigma^1(G) \subseteq S(G)$$

and homological invariants

$$\cdots \subseteq \Sigma^n(G,R) \subseteq \Sigma^{n-1}(G,R) \subseteq \cdots \subseteq \Sigma^1(G,R) \subseteq S(G),$$

where R is a commutative ring; cf. [22, 4].

In this note, we compute the Sigma invariants for the Golden Mean Thompson group F_τ defined by Cleary in [10], see also [9]. We prove:

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Theorem 1.1. *Let $\lambda, \rho : F_\tau \to \mathbb{R}$ be the characters given by:*

$$\lambda(f) = \log_\tau(f'(0)) \quad \textit{and} \quad \rho(f) = \log_\tau(f'(1)).$$

Then the Sigma invariants of F_τ are as follows:

(1) $\Sigma^1(F_\tau) = \Sigma^1(F_\tau, \mathbb{Z}) = S(F_\tau) \setminus \{[-\lambda], [-\rho]\}$, *and*
(2) $\Sigma^\infty(F_\tau) = \Sigma^\infty(F_\tau, \mathbb{Z}) = \Sigma^2(F_\tau) = \Sigma^1(F_\tau) \setminus \{[-a\lambda - b\rho] \mid a, b > 0\}$.

Note that $\Sigma^1(F_\tau)$ was already known, see Citation 1.3 below. The computation of Σ^1 and higher Sigma invariants is of interest for various topological reasons; see, for instance, [3, 4, 21, 2, 24, 15, 14, 19]. Particularly, we obtain the following information about coabelian subgroups of F_τ.

Corollary 1.2. *Let $N \trianglelefteq F_\tau$ be a normal subgroup of homological type $\mathtt{FP}_2$ for which the quotient F_τ/N is abelian. Then N is of homotopical type $\mathtt{F}_\infty$.*

Proof. Immediate from Theorem 1.1, [22, Satz C] and [4, Theorem B]. □

Theorem 1.1 confirms that, similarly to the case of R. Thompson's original group F [2], the Sigma invariants of F_τ are determined by an integral polytope (in the sense of [14]). The same behaviour is seen in other Thompson groups that 'resemble' F (e.g., [2, 25, 26, 19]), though not all of them; see [23].

While no unexpected phenomenon for the Sigma invariants of F_τ is observed, their computation slightly diverges from those in the above mentioned works. More precisely, as a first step we consider the behaviour of the Sigma invariants $\Sigma^n(G)$ under passage to subgroups of finite index — which, to our knowledge, was not needed so far for other Thompson groups. Using this, the computations for the Sigma invariants for F_τ then follow from methods similar to those of Bieri–Geoghegan–Kochloukova in [2].

Throughout the paper, we denote by G a finitely generated group and by $G_{ab} \cong H_1(G;\mathbb{Z})$ its abelianisation. We consider nontrivial characters $\chi \in \mathrm{Hom}(G, \mathbb{R}) \cong H^1(G;\mathbb{R})$. Define an equivalence relation by $\chi \sim \chi'$ if and only if there exists an $a \in \mathbb{R}_{>0}$ such that $\chi = a\chi'$. The set of equivalence classes is a sphere in $\mathbb{R}^n$, called the character sphere $S(G)$. Its dimension is determined by the torsion-free rank $r_0(G_{ab})$ of G_{ab} (equivalently, the first Betti number $b_1(G)$ of the group G) and given by $r_0(G_{ab}) - 1$; see [6, Lemma 1.1]. Now consider the following subset of the Cayley graph $\Gamma(G)$ with respect to some finite generating set: $\Gamma_\chi(G)$ is the subgraph of $\Gamma(G)$ consisting of those vertices with $\chi(g) > 0$, and edges that have both initial and terminal vertices in $\Gamma_\chi(G)$. The first homotopical Sigma invariant is now defined as

$$\Sigma^1(G) = \{[\chi] \in S(G) \mid \Gamma_\chi(G) \text{ is connected}\}.$$

Note that this is independent of the choice of finite generating set for the Cayley graph [3]. For certain groups of homeomorphisms of the real line, including Thompson's group F and the Golden Mean Thompson's group F_τ we have a complete description of $\Sigma^1(G)$:

Citation 1.3 ([6, Chapter IV, Corollary 3.4]). *Let G be an irreducible subgroup of the group of piecewise linear homeomorphisms of the interval $[0,1]$. Take the characters $\chi_1(g) = ln(g'(0))$ as the natural log of the right derivative of an element $g \in G$ at 0 and $\chi_2(g) = ln(g'(1))$ as the natural log of the left derivative of that element at 1. If $G = \ker\chi_1 \cdot \ker\chi_2$, then $\Sigma^1(G)^c = \{[\chi_1],[\chi_2]\}$.*

In the 1990s, Bieri and Strebel gave a formula to compute the complement $\Sigma^1(G)^c$ using $\Sigma^1(H)^c$ and a subsphere of $S(H)$ in case H is a subgroup of finite index in G; see [6, Chapter III, Proposition 2.9] and [24, Proposition B1.11]. In higher dimensions, a related formula was recently considered by Koban–Wong in [15]. In his notes [24, Section B1.2c], Strebel goes on to wonder about the applicability of this formula, and poses the following.

Citation 1.4 ([24, Problem B1.13]). *Find situations where one is interested in $\Sigma^1(G)$ with G admitting a subgroup of finite index which is easier to deal with and for which Σ^1 can be computed.*

We give a positive contribution towards Strebel's problem and find a sufficient condition for 'equality' of Sigma invariants with those of subgroups of finite index.

Theorem 1.5. *Let G be a group of type $\mathtt{F}_n$ with $H \leq G$ a subgroup of finite index and write $\iota : H \hookrightarrow G$ for the inclusion. If $r_0(G_{ab}) = r_0(H_{ab})$, then $\iota^* : S(G) \to S(H)$ is a well-defined homeomorphism and for all n it holds*

$$\iota^*(\Sigma^n(G)) = \Sigma^n(H).$$

Theorem 1.6. *Let A be a $\mathbb{Z}G$-module of type $\mathtt{FP}_n$. Suppose $H \leq G$ is a subgroup of finite index and write $\iota : H \hookrightarrow G$ for the inclusion. If $r_0(G_{ab}) = r_0(H_{ab})$, then $\iota^* : S(G) \to S(H)$ is a well-defined homeomorphism and for all n it holds*

$$\iota^*(\Sigma^n(G,A)) = \Sigma^n(H,A).$$

Recalling the definition of ι^*, any character $\chi \in \mathrm{Hom}(G,\mathbb{R})$ can be restricted to a character of H, and we set $\iota^*(\chi) = \chi_{|H} \in \mathrm{Hom}(H,\mathbb{R})$. In general, this map does not induce a function between character spheres. Thus, the above statements also mean that the assignment $\iota^*([\chi]) = [\chi_{|H}]$ can be made on the level of character spheres, and we abuse notation also denoting this map by $\iota^* : S(G) \to S(H)$. We refer the reader to Section 3 for the proofs of Theorems 1.5 and 1.6 — the main issue, as should be known to experts, is whether characters of the subgroup H can be extended to characters of the whole group G. Examples 1.7, 1.8, and 3.2 illustrate how the equalities $\iota^*(\Sigma^n(G)) = \Sigma^n(H)$ and $\iota^*(\Sigma^n(G,A)) = \Sigma^n(H,A)$ can fail.

We note that the problems of extending characters and of computing Sigma invariants from those of given subgroups appear in various guises in the literature; see, for example, [6, 21, 15, 17, 13, 18]. However, we were unable to find explicit references of statements along the lines of Theorems 1.5 and 1.6. We will make use of the homological result in [21] (included here as Citation 3.5 in Section 3) during our proof of Theorem 1.6.

Expanding on some related work, the authors in [15] study Sigma invariants for finite-index normal subgroups $N \trianglelefteq G$, obtaining the image of $\Sigma^n(G)$ as an intersection of $\Sigma^n(N)$ with a certain subset of $\mathrm{Hom}(N, \mathbb{R})$ invariant under a G/N-action. In [17], extensions of characters from coabelian normal subgroups play a central role. More recently in [13, 18], the authors give conditions under which one can extend a character from certain normal subgroups of infinite index. Our formulation of Theorems 1.5 and 1.6 gives a simple, easy-to-check condition on the Sigma invariants for (not necessarily normal) finite-index subgroups. We also remark that, over $\mathbb{Z}$ or a field, Theorem 1.6 can be alternatively proved using techniques from Novikov homology and a recent generalisation of Sikorav's theorem due to Fisher [12]; see Remark 3.6. Our proof, in turn, uses only elementary methods.

We stress that neither the equality $b_1(G) = r_0(G) = r_0(H) = b_1(H)$ nor finite index alone suffice as hypotheses, as the following examples show.

Example 1.7. Note that $r_0(G) = r_0(H)$ is insufficient to show an embedding of character spheres via ι^*. As a counterexample, consider Thompson's original group $F = \langle x_0, x_1, \dots \mid x_i^{-1}x_jx_i = x_{j+1} \text{ for } 0 \le i < j\rangle$ and the subgroup $F[1] = \langle x_1, x_2, \dots \mid x_i^{-1}x_jx_i = x_{j+1} \text{ for } 1 \le i < j\rangle$. Clearly $F \cong F[1]$, and so $r_0(F) = r_0(F_1) = 2$. But any character $\chi \in \mathrm{Hom}(F, \mathbb{R})$ with $\chi(x_1) = 0$ restricts to the trivial character on $F[1]$, and all other character classes in $S(F)$ restrict to $[\pm\chi_1]$, where $\chi_1(x_0) = 0, \chi_1(x_1) = 1$. Hence ι^* is only defined on a proper subset of $S(F)$, and the character classes in $\Sigma^n(F)$ on which ι^* is defined are mapped to a proper subset of $\Sigma^n(F[1])$.

Example 1.8. Similarly, $|G : H| < \infty$ alone does not guarantee the existence of a bijection between Sigma invariants of G and H. For instance, the infinite dihedral group $D_\infty \cong \mathbb{Z} \rtimes C_2$ contains $\mathbb{Z}$ as a subgroup of index two. While $S(\mathbb{Z}) = \Sigma^1(\mathbb{Z})$ is the 0-sphere (and thus consists of two points), one has that $S(D_\infty)$ — thus also $\Sigma^n(D_\infty)$ — is empty as the abelianisation of D_∞ is finite.

Also note that the implications in Theorems 1.5 and 1.6 cannot be reversed:

Example 1.9. Let $\mathbb{F}_n$ denote the free group on n letters. It is known, see [6, Proposition III.4.5], that $\Sigma^1(\mathbb{F}_n) = \varnothing$ for all $n \ge 2$. Furthermore, $\mathbb{F}_n$ embeds with finite index in $\mathbb{F}_2$ [20, Proposition I.3.9]. However, the torsion-free ranks of these groups are not equal as long as $n > 2$.

We begin by establishing facts about both the Sigma invariants and F_τ. In Section 3 we prove Theorems 1.5 and 1.6. And finally, in Section 4 we compute the Sigma invariants for F_τ.

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2. Background

2.1. Higher homotopical sigma invariants. We will begin with recalling some general definitions and facts that can be found, for example, in [11]. An Eilenberg–MacLane space, denoted $K(G,1)$, is an aspherical CW-complex Y with $\pi_1(Y) = G$. Its universal cover X is contractible and has G acting freely by deck transformations. Such a space is also called a model for EG and is unique up to G-homotopy. A group G is said to be of type $\mathtt{F}_n$ if there is a model for EG with finite n-skeleton modulo the G-action, in which case we also say that this model has G-finite n-skeleton. Finally, G is said to be of type $\mathtt{F}_\infty$ if it is of type $\mathtt{F}_n$ for all $n \in \mathbb{N}$.

From now on, let G be of type $\mathtt{F}_n$ and let X be a model for EG with G-finite n-skeleton. The following construction is due to Renz [22, Kapitel II, Abschnitt 2], see also [6, Appendix B, Section B1.1]: For a given character $\chi \in \operatorname{Hom}(G, \mathbb{R})$, one defines an action of G on $\mathbb{R}$ by $g \cdot r = r + \chi(g)$ for all $g \in G$ and $r \in \mathbb{R}$, which can be extended to a corresponding continuous G-equivariant map $h_\chi : X \to \mathbb{R}$, also called a height function. Any such height function gives rise to an $\mathbb{R}$-filtration of X given by the closed subspaces $h_\chi^{-1}([r, \infty))$. We shall consider $X_{h_\chi}^{[r,+\infty)}$, defined as the largest subcomplex of X such that

$$x \in X_{h_\chi}^{[r,+\infty)} \implies h_\chi(x) \in [r, +\infty).$$

When considering $X_{h_\chi}^{[0,+\infty)}$, we shall use the notation X_{h_χ}.

Definition 2.1 ([22, Kapitel II, Definition 3.4] or [6, Appendix B, Definition in p. 194])**.** Let G be of type $\mathtt{F}_n$. Then the n-th Sigma invariant $\Sigma^n(G) \subseteq S(G)$ is defined as follows: $[\chi] \in \Sigma^n(G)$ if there exists a model X for EG with G-finite n-skeleton and a corresponding height function h_χ on X such that X_{h_χ} is $(n-1)$-connected.

There are a priori different ways of extending the character χ to a G-equivariant height function h_χ, though Renz shows that this distinction is immaterial and $\Sigma^n(G)$ is well-defined; cf. [22, Kapitel II, Bemerkungen 3.5]. This allows us to write h instead of h_χ for an admissible height function extending a character χ, if no confusion arises.

While the connectivity condition in Definition 2.1 might not hold for every model of EG with G-finite n-skeleton, Renz [22] also showed that the model may be arbitrary if one considers essential connectivity instead.

Definition 2.2 ([22, Kapitel II, Definition 3.6] or [6, Appendix B, Section B1.2])**.** For $X_h^{[r,+\infty)}$ as defined above, we say that $X_h^{[r,+\infty)}$ is essentially k-connected for $k \in \mathbb{Z}_{\geq -1}$ if there is a real number $d \geq 0$ such that the map $\iota_j : \pi_j(X_h^{[r,+\infty)}) \to$

$\pi_j(X_h^{[r-d,+\infty)})$ induced by the inclusion $\iota : X_h^{[r,+\infty)} \hookrightarrow X_h^{[r-d,+\infty)}$ is the zero map for all $j \leq k$.

Citation 2.3 ([22, Kapitel IV, Satz 3.4] or [6, Appendix B, Theorem B1.1]). *Let G be a group of type $\mathtt{F}_n$ and let X be an arbitrary model for EG with G-finite n-skeleton. Let $\chi : G \to \mathbb{R}$ be a nontrivial character and $h : X \to \mathbb{R}$ a corresponding height function as above. Then*

$$[\chi] \in \Sigma^n(G) \iff X_h \textit{ is essentially } (n-1)\textit{-connected.}$$

2.2. The homological invariant $\Sigma^n(G, A)$. We will now give a brief overview of the definition and essential properties of the homological invariants $\Sigma^n(G, A)$, where A is a $\mathbb{Z}G$-module, see [4]. We follow the convention of Bieri, Renz, and Strebel [1, 22, 4, 6] of working with left modules. In particular, a group or monoid acting on a module acts on the left.

Definition 2.4 ([7, Chapter VIII.4]). Given a unital ring R, a (left) R-module A is said to be of type $\mathtt{FP}_n$ over R if it admits a resolution of the form

$$\mathbf{P} \quad : \quad ... \to P_i \to P_{i-1} \to ... \to P_1 \to P_0 \to A \to 0$$

where the P_i are free (left) R-modules which are finitely generated for $i \leq n$. In case G is a group or monoid, we say that G is of type $\mathtt{FP}_n$ if the trivial $\mathbb{Z}G$-module $A = \mathbb{Z}$ is of type $\mathtt{FP}_n$ over $R = \mathbb{Z}G$.

One can analogously define '(right) type $\mathtt{FP}_n$', i.e., using right actions and right modules, and a group being of type $\mathtt{FP}_n$ does not depend on whether one works from the left or right; cf. [1]. However, the same is not true in the case of monoids (see, for instance, [16]), whence the importance of fixing a convention for the actions when working with both groups and monoids.

Definition 2.5 ([4, Section 1.3]). Let G be a group and A a $\mathbb{Z}G$-module of type $\mathtt{FP}_n$. The n-th homological Sigma invariant $\Sigma^n(G, A) \subseteq S(G)$ is defined as follows:

$$[\chi] \in \Sigma^n(G, A) \iff A \text{ is of type } \mathtt{FP}_n \text{ over the subring } \mathbb{Z}G_\chi \subseteq \mathbb{Z}G,$$

where G_χ is the submonoid $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$.

Now let G be a group of type $\mathtt{F}_n$. Then (cf. [22, Satz B] and also [4]) it holds:

$$\begin{aligned} \Sigma^1(G) &= \Sigma^1(G; \mathbb{Z}) \\ \Sigma^n(G) &= \Sigma^2(G) \cap \Sigma^n(G; \mathbb{Z}) \text{ for } n \geq 2 \end{aligned} \tag{1}$$

Similarly to the homotopical case, it was shown that the definition of $\Sigma^n(G; A)$ does not depend on the partial finitely generated free resolution of the $\mathbb{Z}G$-module A, see [4, Theorem 3.2].

2.3. Background on the golden mean Thompson group F_τ. Let τ denote the small Golden Ratio, that is, the positive solution $\tau = \frac{\sqrt{5}-1}{2}$ to the equation $x^2 + x = 1$.

Definition 2.6 ([10]). The group F_τ is defined as the subgroup of piecewise linear, orientation-preserving homeomorphisms of the interval $[0,1]$ with slopes in the group $\langle\tau\rangle$ and breakpoints in $\mathbb{Z}[\tau]$.

Citation 2.7 ([9, Theorem 4.4]). *F_τ has the (infinite) presentation*

$$F_\tau \cong \langle x_i, y_i \mid a_j b_i = b_i a_{j+1}, y_i^2 = x_i x_{i+1}; a, b \in \{x, y\}, 0 \le i < j\rangle. \tag{2}$$

In the above, $i, j \in \mathbb{Z}_{\ge 0}$. We can write the generators of F_τ as functions on the interval $[0,1]$ in the following forms:

$$\begin{aligned} x_i(n) &= \begin{cases} n & \text{for } 0 \le n \le 1-\tau^i, \\ \tau^{-2}n - \tau^{-1}(1-\tau^i) & \text{for } 1-\tau^i \le n \le 1-\tau^i+\tau^{i+4}, \\ n + \tau^{i+3} & \text{for } 1-\tau^i+\tau^{i+4} \le n \le 1-\tau^{i+1}, \\ \tau n + \tau^2 & \text{for } 1-\tau^{i+1} \le n \le 1, \end{cases} \\ y_i(n) &= \begin{cases} n & \text{for } 0 \le n \le 1-\tau^i, \\ \tau^{-1}n - \tau^{-1}(1-\tau^i) & \text{for } 1-\tau^i \le n \le 1-\tau^{i+1}, \\ \tau n + \tau^2 & \text{for } 1-\tau^{i+1} \le n \le 1. \end{cases} \end{aligned} \tag{3}$$

These elements can also be understood as equivalence classes of ordered tree-pairs, as described in [9, Section 4]. As for the original Thompson group F, the elements of F_τ have a unique normal form [9, Theorem 7.3]. We shall use the following normal form:

Citation 2.8 ([9, Section 7]). *Any element $f \in F_\tau$ can be uniquely expressed in the form*

$$f = x_0^{i_0} y_0^{\epsilon_0} x_1^{i_1} y_1^{\epsilon_1} \cdots x_n^{i_n} y_n^{\epsilon_n} x_m^{-j_m} x_{m-1}^{-j_{m-1}} \cdots x_0^{-j_0},$$

where $i_k, j_k \in \mathbb{Z}_{\ge 0}$, $\epsilon_k \in \{0,1\}$, $0 \le k \le n$, and moreover the following hold for all k:

1. *If $i_k \ne 0 \ne j_k$, then at least one of i_{k+1}, j_{k+1}, ϵ_k, ϵ_{k+1} is nonzero;*
2. *In case f contains a subword of the form $x_k y_k x_{k+2} u x_{k+1}^{-1} x_k^{-1}$, then the middle subword u contains a generator indexed either by $k+1$ or $k+2$.*

Like F, the group F_τ also enjoys the strong homotopical and homological finiteness properties.

Citation 2.9 ([10]). *The Golden Mean Thompson group F_τ is of type $\mathtt{F}_\infty$.*

3. Sigma invariants and finite index

In this section, we prove Theorems 1.5 and 1.6. We begin by discussing, for $H \le G$, maps between $H^1(H;\mathbb{R}) \cong \mathrm{Hom}(H,\mathbb{R})$ and $H^1(G;\mathbb{R}) \cong \mathrm{Hom}(G,\mathbb{R})$.

Lemma 3.1. *Suppose G is a finitely generated group, let $H \leq G$, and write $\pi : G \twoheadrightarrow G_{\mathrm{ab}}$ for the canonical projection and $\iota : H \hookrightarrow G$ for the inclusion. Then the following hold.*

(1) *If $|G : H| < \infty$, then the map $\iota^* : \mathrm{Hom}(G, \mathbb{R}) \to \mathrm{Hom}(H, \mathbb{R})$ induced by the inclusion is injective.*
(2) *If the image $\pi(H)$ is infinite, then there exists a nontrivial morphism $e : \mathrm{Hom}(H, \mathbb{R}) \to \mathrm{Hom}(G, \mathbb{R})$. That is, any character ψ of $H \leq G$ gives rise to a character $e(\psi)$ of G and the image $e(\mathrm{Hom}(H, \mathbb{R})) \subseteq \mathrm{Hom}(G, \mathbb{R})$ is a nonzero subspace.*

Lemma 3.1(2) was observed by Kochloukova–Vidussi; cf. [18, Proof of Theorem 1.1]. Kochloukova and Vidussi work with characters in G that are already assumed to be extensions of characters of a subgroup $H \leq G$. However, in the form we state Lemma 3.1, the character $e(\psi) \in \mathrm{Hom}(G, \mathbb{R})$ need not be a valid extension of the original character $\psi \in \mathrm{Hom}(H, \mathbb{R})$. That is, it might be the case that $\iota^* \circ e(\psi) \neq \psi$; see Example 3.2 below.

From now on, when working in the abelianisation of a group, we will write the group operation additively.

Proof. Part (1): Take a nonzero character $\chi \in \mathrm{Hom}(G, \mathbb{R})$ and suppose that $\iota^*(\chi) = \chi_{|H} = 0$. As $\chi(G) \neq 0$, there exists $g \in G$ such that $\chi(g) \neq 0$, but as $\chi(H) = 0$ one has $g \notin H$. Furthermore, we can say $g^n \notin H$ for all $n \in \mathbb{N}$, as

$$\begin{aligned} g^n \in H &\implies \chi(g^n) = 0 \\ &\iff n\chi(g) = 0 \\ &\iff \chi(g) = 0, \end{aligned}$$

contradicting $\chi(g) \neq 0$. Thus $g^n H$ are all distinct cosets of H, which means that H is not finite index, contradicting our assumption. Hence, $\chi(H) \neq 0$.

Part (2): Consider the (finite dimensional) $\mathbb{Q}$-vector space $V = G_{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Since the image $\pi(H) \subseteq G_{\mathrm{ab}}$ is infinite, the set $\pi(H)$ contains some torsion-free elemenet and thus $\pi(H) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a partial basis for V, say $\mathcal{B}' = \{\overline{h}_1, \dots, \overline{h}_m\}$, where each $\overline{h}_i$ is the image in G_{ab} of some $h_i \in H$. Extend this to a basis $\mathcal{B} = \{\overline{h}_1, \dots, \overline{h}_m, \overline{g}_{m+1}, \dots, \overline{g}_r\}$ of V, again with $\overline{g}_j$ being the image of some $g_j \in G$. Since the image of characters of a group factors through their abelianisation, we may define

$$e(\psi)(g) := \sum_{i=1}^{m} a_i \psi(h_i),$$

where the a_x with $x \in \mathcal{B}$ are the coordinates of the image of g in $G_{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}$ in the basis $\mathcal{B}$. It is straightforward to check that e is a homomorphism from $\mathrm{Hom}(H, \mathbb{R})$ to $\mathrm{Hom}(G, \mathbb{R})$. Again because $\pi(H) \subseteq G_{\mathrm{ab}}$ is infinite and G is finitely generated, the induced map $H \to \pi(H) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^m$ gives a nontrivial character, call it $\psi \in \mathrm{Hom}(H, \mathbb{R})$, by projecting onto the line spanned by a nonzero vector of $\pi(H)\otimes_{\mathbb{Z}}\mathbb{R}$. By construction, the character $e(\psi) \in \mathrm{Hom}(G,\mathbb{R})$ is also nontrivial. □

Example 3.2. As mentioned above, the proof of Lemma 3.1(2) might yield an 'extension' of a character ψ of H such that $\iota^* \circ e(\psi) \neq \psi$. For example, let $H = \mathbb{Z}\times\mathbb{Z} \leq G = D_\infty \times \mathbb{Z}$ and take ψ to be a character of H which is nonzero on the first coordinate.

However, provided H is of finite index in G and their first Betti numbers agree, one can always construct a lift from $\mathrm{Hom}(H,\mathbb{R})$ to $\mathrm{Hom}(G,\mathbb{R})$ that circumvents these problems. We summarise these properties in the following.

Proposition 3.3. *Let G be a finitely generated group and $H \leq G$ a subgroup of finite index. Then the following are equivalent:*

(1) $r_0(G) = r_0(H)$.
(2) $\iota^* : \mathrm{Hom}(G,\mathbb{R}) \to \mathrm{Hom}(H,\mathbb{R})$ *is an isomorphism of* $\mathbb{R}$*-vector spaces.*
(3) *The assignment* $\iota^*([\chi]) := [\chi_{|H}]$ *is defined on all character classes* $[\chi] \in S(G)$*, and the corresponding map* $\iota^* : S(G) \to S(H)$ *is a homeomorphism.*
(4) *Every character* $\chi \in \mathrm{Hom}(H,\mathbb{R})$ *admits a lift* $\chi' \in \mathrm{Hom}(G,\mathbb{R})$ *such that* $\chi'|_H = \chi$ *and* $\chi \neq 0 \iff \chi' \neq 0$.

Proof. The equivalences of (1), (2), and (3) are immediate from Lemma 3.1(1) as $\dim_{\mathbb{R}}(\mathrm{Hom}(\Gamma,\mathbb{R})) = r_0(\Gamma)$ for any group Γ. Item (4) is equivalent to (2) as the function $e : \mathrm{Hom}(H,\mathbb{R}) \to \mathrm{Hom}(G,\mathbb{R})$ given by $e(\chi) = \chi'$ is a right inverse to ι^*. □

Example 3.4. It is not hard to explicitly construct the 'extension map' $e : \mathrm{Hom}(H,\mathbb{R}) \to \mathrm{Hom}(G,\mathbb{R})$ of Proposition 3.3. Let $\{x_i,\dots,x_n\}$ be a generating set for G and write $r_0(G) = r_0(H) = k \leq n$. Without loss of generality one can assume that $\{\overline{x_1},\dots,\overline{x_k}\}$ generates $(G_{ab})_0$. Since $|G : H| < \infty$, for each $i = 1,\dots,n$, there exists an $\alpha_i \in \mathbb{N}$ such that $x_i^{\alpha_i} \in H$. Hence, using functoriality of abelianisations, and the fact that $\bar{x}_i$ has infinite order in G_{ab}, we have that $0 \neq \alpha_i\overline{x_i} \in (H_{ab})_0$ for all $i = 1,\dots,k$. Let $\alpha = \mathrm{lcm}\{\alpha_1,\dots,\alpha_k\}$. Given a character $\chi : H \to \mathbb{R}$, we define its lift $e(\chi) = \chi' : G \to \mathbb{R}$ by

$$\chi'(x_i) = \frac{1}{\alpha}\chi(x_i^{\alpha}), \text{ for all } i = 1,\dots,n.$$

To finish off Theorem 1.6, we make use of the following:

Citation 3.5 ([21, Proposition 9.3]). *Suppose that $H \leq G$ is a subgroup of finite index and A a $\mathbb{Z}G$-module of type* FP_n, *and suppose that $\chi : G \to \mathbb{R}$ restricts to a nonzero homomorphism of H. Then*

$$[\chi_{|H}] \in \Sigma^n(H,A) \iff [\chi] \in \Sigma^n(G,A).$$

Proof of Theorem 1.6. Immediate from Proposition 3.3 and Citation 3.5. □

Remark 3.6. In case $A = \mathbb{Z}$ or a field, Theorem 1.6 can also be proved as follows: by change of rings [1] and noting that the Novikov ring $\widehat{A[G]}^{\chi}$ is isomorphic to the tensor product $\widehat{A[H]}^{\chi|_H} \otimes_{A[H]} A[G]$, an application of Proposition 3.3 combined with the equivalence (1) $\iff$ (5) from a result of Fisher [12, Theorem 5.3] proves the claim.

For completeness, we now give an elementary proof of the homotopical part, which needs the following.

Proposition 3.7. *Let G be a group of type $\mathtt{F}_n$ and H a subgroup of finite index such that $r_0(G) = r_0(H)$. With the notation of Proposition 3.3 we have*

$$[\chi] \in \Sigma^n(G) \implies [\chi_{|H}] \in \Sigma^n(H),$$

and

$$[\chi] \in \Sigma^n(H) \implies [\chi'] \in \Sigma^n(G).$$

Proof. To prove the first claim, consider a model X for EG with G-finite n-skeleton. Now suppose $[\chi] \in \Sigma^n(G)$, hence $X_{h_\chi}^{[0,+\infty)}$ is $(n-1)$-connected for the height function h_χ corresponding to χ. Since H is finite index in G, the space X is also a model for EH with H-finite n-skeleton, and $h_\chi = h_{\chi_{|H}}$. Hence, $[\chi_{|H}] \in \Sigma^n(H)$.
Let us now assume $[\chi] \in \Sigma^n(H)$. Again using $|G : H| < \infty$, choose a model for X for EH as above: X is a simplicial complex with G-finite n-skeleton and one G-orbit of zero-cells labeled by G.

We now fix a set $T = \{t_0, \dots, t_{m-1}\}$ of coset representatives of H in G, put $t_0 = e$, and construct an H-equivariant height function $h_\chi : X \to \mathbb{R}$ on the vertices of X as follows: For $\gamma \in H$ we put $h_\chi(\gamma) = \chi(\gamma)$ and set $h_\chi(t_i) = 0$. Hence, since every $g \in G$ has a unique expression as $g = t_i\gamma$, we get

$$h_\chi(g) = h_\chi(t_i) + h_\chi(\gamma) = \chi(\gamma).$$

Finally, we extend this function piecewise linearly to the entire n-skeleton on X. Hence $X_{h_\chi}^{[0,+\infty)}$ is essentially $(n-1)$-connected, see Citation 2.3.

It remains to show that this connectivity property remains true using a height function $h_{\chi'}$ corresponding to a lift χ' of χ. Note that $\chi'(t_i)$ is not necessarily equal to 0. Define $d = \mathbf{min}\{\chi(t_i)\}$.

We claim that, for every $g \in G$, $h_\chi(g) \geq 0$ if and only if $\chi'(g) \geq d$. To see this, write $g = t_i\gamma$ as above. Since $h_\chi(g) = \chi(\gamma)$ and $\chi'(g) = \chi'(t_i\gamma) = \chi'(t_i) + \chi(\gamma)$, we get

$$h_\chi(g) \geq 0 \iff \chi(\gamma) \geq 0 \iff \chi'(t_i) + \chi(\gamma) \geq d + 0 \iff \chi'(g) \geq d,$$

as required.

This now implies implies that the 0-skeleton of $X_{h_\chi}^{[0,+\infty)}$ is precisely the same as the 0-skeleton of $X_{h_{\chi'}}^{[d,+\infty)}$. As the space $X_{h_\chi}^{[r,+\infty)}$ is defined as the maximal subcomplex of X contained in $h_\chi^{-1}([r, +\infty))$, where an m-cell is included if all of its boundary cells are included [6, Appendix B, p. 194], we have shown that $X_{h_\chi}^{[0,+\infty)} = X_{h_{\chi'}}^{[d,+\infty)}$. Hence $[\chi'] \in \Sigma^n(G)$ again by Citation 2.3. □

Proof of Theorem 1.5. This follows from Propositions 3.3 and 3.7. □

4. The sigma invariants for F_τ

We begin by collecting some properties of F_τ and its characters as well as exhibiting a finite index subgroup which satisfies the assumptions of Theorems 1.5 and 1.6.

It was shown in [9, Chapter 5] that

$$(F_\tau)_{ab} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}.$$

Hence,

$$S(F_\tau) = S^1.$$

Similarly to the original Thompson's group case, we have the two linearly independent characters λ and ρ given by some logarithm of the slopes at 0 and 1 respectively, such that $[\lambda]$ and $[\rho]$ span $S(F_\tau)$. In particular, these, for every $f \in F_\tau$, are given by

$$\lambda(f) = \log_\tau(f'(0)) \quad \text{and} \quad \rho(f) = \log_\tau(f'(1)).$$

By taking appropriate subdivisions of $[0,1]$, one can construct elements $f \in F_\tau$ with support in $[0,b] \cap \mathbb{Z}[\tau]$ for some $b < 1$ and such that $f'(0) = \tau$. Analogously, one can find $g \in F_\tau$ with support in $[a,1]$ for some $a > 0$ and with $g'(1) = \tau$. Hence $\lambda(f) = 1 = \rho(g)$, $\lambda(g) = 0 = \rho(f)$ and thus λ and ρ are linearly independent.

As an example, we can use the following elements:

Example 4.1.

$$f(x) = \begin{cases} \tau x & \text{for } 0 \le x \le \tau^2 \\ \tau^{-1}x - \tau^2 & \text{for } \tau^2 \le x \le \tau \\ x & \text{for } \tau \le x \le 1 \end{cases}$$

$$g(x) = \begin{cases} x & \text{for } 0 \le x \le \tau^2 \\ \tau^{-1}x - \tau^3 & \text{for } \tau^2 \le x \le \tau \\ \tau x + \tau^2 & \text{for } \tau \le x \le 1. \end{cases}$$

Proposition 4.2. *Let K denote the subgroup of F_τ generated by $\{x_0, x_1, y_1, x_2, y_2, \ldots\}$. Then $|F_\tau : K| = 2$ and $K_{ab} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$.*

Proof. We claim $F_\tau = K \sqcup y_0 K$. To do so, consider $g \in F_\tau$ in normal form, see Citation 2.8:

$$g = x_0^{i_0} y_0^{\epsilon_0} x_1^{i_1} y_1^{\epsilon_1} \cdots x_n^{i_n} y_n^{\epsilon_n} x_m^{-j_m} x_{m-1}^{j_{m-1}} \cdots x_0^{j_0}$$

where $i_0, \ldots, i_n, j_0, \ldots, j_m \in \mathbb{Z}_{\ge 0}$ and $\epsilon_0, \ldots, \epsilon_m \in \{0,1\}$. Hence,

$$gK = x_0^{i_0} y_0^{\epsilon_0} K.$$

When $\epsilon_0 = 0$ we have $g \in K$, and when $\epsilon_0 = 1$ a repeated application of the following computation gives $g \in y_0 K$:

$$\begin{aligned} x_0^{i_0} y_0 &= x_0^{i_0-1} x_0 y_0 = x_0^{i_0-1} x_0 x_1 x_1^{-1} y_0 = x_0^{i_0-1} y_0^2 x_1^{-1} y_0 = \\ &= x_0^{i_0-1} y_0^2 y_0 x_2^{-1} = x_0^{i_0-1} y_0 y_0^2 x_2^{-1} = x_0^{i_0-1} y_0 x_0 x_1 x_2^{-1}. \end{aligned} \tag{4}$$

Consider any word in the generators x_i and y_j ($i \geq 0, j \geq 1$) in F_τ. The relations of F_τ, see Eq. (2), imply that in any other expression on this element, the occurrence of y_0^k will have k an even integer. Hence, in the normal form of Citation 2.8 such an element will have no occurrence of y_0. This implies that K is a proper subgroup of F_τ, and moreover $|F_\tau : K| = 2$.

To determine the abelianisation, we do a similar calculation to that in [9, Section 5]: We denote the images of an element $f \in F_\tau$ in the abelianisation by $\bar{f}$ and write the group operation additively. From the relations, it follows immediately that $\bar{x}_i = \bar{x}_{i+1}$ and that $2\bar{y}_i = 2\bar{x}_1$ for all $i \geq 1$. Substituting $\bar{z} = \bar{y}_1 - \bar{x}_1$, we have the two infinite order generators $\bar{x}_0$ and $\bar{x}_1$ as well an order 2 generator $\bar{z}$ as required. □

Let H be a group and $\sigma : H \to H$ a monomorphism. An ascending HNN extension (with base H) is a group given by the presentation

$$H *_{t,\sigma} = \langle H, t \mid tht^{-1} = \sigma(h); h \in H \rangle.$$

We now consider the subgroup $F_\tau[1] \leq F_\tau$ generated by $\{x_1, y_1, x_2, y_2, ...\}$. In analogy to Thompson's F, there is a well-known monomorphism $\sigma : F_\tau \to F_\tau$ given by $\sigma(x_n) = x_{n+1}$ and $\sigma(y_n) = y_{n+1}$, whose image is clearly $F_\tau[1] \subsetneq F_\tau$. Restricting to $F_\tau[1]$ gives a monomorphism $\sigma : F_\tau[1] \to F_\tau[1]$ whose image is the proper subgroup $F_\tau[2] \subsetneq F_\tau[1]$ generated by $\{x_2, y_2, x_3, y_3, ...\}$, and so on. Hence, any $F_\tau[m]$ is isomorphic to F_τ and thus of type $\mathtt{F}_\infty$. Much like F is an HNN extension over a copy of itself (cf. [8, Proposition 1.7]), the group K — which contains $F_\tau[1]$ by definition — differs from its subgroup $F_\tau[1] \cong F_\tau$ by the stable letter x_0.

Lemma 4.3. *The subgroup $K \leq F_\tau$ is isomorphic to the HNN extension*

$$K \cong F_\tau[1] *_{t,\sigma} = \langle F_\tau[1], t \mid tgt^{-1} = \sigma(g); g \in F_\tau[1] \rangle$$

by mapping t to x_0^{-1} and $F_\tau[1]$ to itself.

Proof. For this proof, we implicitly use standard facts about presentations and HNN extensions; cf. [20, Chapter IV, Section 2].

Let $\langle X \mid R \rangle$ denote the obvious presentation of $F_\tau[1]$, that is, the same as that of F_τ from Eq. (2) but with decorated generating set $X = \{\widetilde{x}_i, \widetilde{y}_i \mid i \geq 1\}$ and indices starting from 1. The HNN extension $F_\tau[1] *_{t,\sigma}$ is thus given by the (abstract) group presentation

$$F_\tau[1] *_{t,\sigma} \cong L := \langle X, t \mid R,\ t\widetilde{x}_i t^{-1} = \widetilde{x}_{i+1},\ t\widetilde{y}_i t^{-1} = \widetilde{y}_{i+1} \text{ for all } i \geq 1 \rangle.$$

The obvious map

$$\phi : L \to K \text{ induced by } t \mapsto x_0^{-1},\ \widetilde{x}_i \mapsto x_i,\ \widetilde{y}_i \mapsto y_i$$

is a well-defined group homomorphism since all defining relations in L hold in K. It is surjective by construction, and we want to check that it is also injective. Note that, since L is an HNN extension, the group $F_\tau[1]$ effectively embeds in L as its obvious subgroup $\langle X\rangle$. The restriction of ϕ to $\langle X\rangle$ is thus an isomorphism onto its image $F_\tau[1] \leq K$. In particular, if $g \in \langle X\rangle$, the isomorphisms $\langle X\rangle \cong F_\tau[1] \cong F_\tau$ and Citation 2.8 yield a (unique) normal form for g matching the (unique) normal form of $\phi(g) \in K \subseteq F_\tau$ (by dropping the tildes), and such a normal form of $\phi(g)$ in K does not involve the generator x_0.

Now let $w \in \ker(\phi) \trianglelefteq L$. As L is an HNN extension, we may write w in normal form

$$w = g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \cdots g_{n-1} t^{\varepsilon_{n-1}} g_n$$

with each $\varepsilon_i \in \{\pm 1\}$ and $g_i \in \langle X\rangle$. If $\varepsilon_i = -1$, repeated applications of the defining relations in L yield

$$g_i t^{-1} = t^{-1} g_i' \text{ for some } g_i' \in \langle\{\widetilde{x}_j, \widetilde{y}_j \mid j \geq 2\}\rangle \leq \langle X\rangle.$$

Similarly, if $\varepsilon_i = 1$, then

$$t g_i = g_i' t \text{ for some } g_i' \in \langle\{\widetilde{x}_j, \widetilde{y}_j \mid j \geq 2\}\rangle \leq \langle X\rangle.$$

Thus, writing $a = \#\{i \mid \varepsilon_i < 0\} \geq 0$ and $b = \#\{i \mid \varepsilon_i > 0\} \geq 0$, the word w can be rewritten as

$$w = t^{-a} g' t^b \text{ where } g' \in \langle X\rangle.$$

As $g' \in \langle X\rangle$, we may replace it by its (unique) normal form in $\langle X\rangle \cong F_\tau[1]$, if necessary. Mapping over to K, we obtain

$$\phi(w) = \phi(t)^{-a}\phi(g')\phi(t)^b = x_0^a \phi(g') x_0^{-b},$$

where the subword $\phi(g')$ lies in $F_\tau[1]$ and is written in its (unique) normal form, not involving the letter x_0. In particular, the word $x_0^a\phi(g')x_0^{-b} \in K \subseteq F_\tau$ can be written in a normal form as in Citation 2.8.

Suppose first that $x_0^a\phi(g')x_0^{-b}$ is already in normal form, see Citation 2.8. Since $1 = \phi(w) = x_0^a\phi(g')x_0^{-b}$ by assumption, the above considerations imply that $g' = 1$ and $a = b$, whence w is trivial in L.

If $x_0^a\phi(g')x_0^{-b}$ is not in normal form, then $\phi(g')$ has no occurrences of the letters x_1 or y_1. We can assume that $a \geq b$. Hence $x_0^a\phi(g')x_0^{-b} = x_0^{a-b}\phi(g'[b])$, where $g'[b]$ denotes the word g' with the indices of the x_i and y_i increased by b. This is now in normal form as in Citation 2.8, and as above it means that $g'[b] = 1$, hence $g' = 1$, and that $a - b = 0$. Again, w is trivial in L. This finishes the proof. □

We can finally adapt the calculations for Thompson's group F as in [2] to compute the Sigma invariants for F_τ.

Citation 4.4 ([2, Theorem 2.1]). *Let G decompose as an ascending HNN extension $H *_{t,\sigma}$. Let χ be a character such that $\chi(H) = 0$, $\chi(t) = 1$.*

- *Suppose H is of type* F_n, *then* $[\chi] \in \Sigma^n(G)$.
- *Suppose H is of type* FP_n, *then* $[\chi] \in \Sigma^n(G; \mathbb{Z})$.

- *If H is finitely generated and σ is not surjective, then $[-\chi] \notin \Sigma^1(G)$.*

Lemma 4.5. *Let λ and ρ be the characters defined at the beginning of this section. Then*

$$[\lambda], [\rho] \in \Sigma^\infty(K) \cap \Sigma^\infty(F_\tau) \quad \textit{and} \quad [-\lambda], [-\rho] \notin \Sigma^1(K) \cup \Sigma^1(F_\tau),$$
$$[\lambda], [\rho] \in \Sigma^\infty(K; \mathbb{Z}) \cap \Sigma^\infty(F_\tau; \mathbb{Z}) \textit{ and } [-\lambda], [-\rho] \notin \Sigma^1(K; \mathbb{Z}) \cup \Sigma^1(F_\tau; \mathbb{Z}).$$

Proof. We begin by determining the result for $[\lambda]$ and $[-\lambda]$. The support of $F_\tau[1]$ lies in $[\tau^2, 1]$ and hence $\lambda(F_\tau[1]) = 0$. The slope of x_0 at 0 is τ^{-2}. Hence, taking the character $\chi := \frac{1}{2}\lambda \in [\lambda]$, we obtain $\chi(t) = 1$. We can thus apply Citation 4.4 to conclude that $[\lambda] \in \Sigma^\infty(K)$ and $[-\lambda] \notin \Sigma^1(K)$. By Theorem 1.5, it follows that $[\lambda] \in \Sigma^\infty(F_\tau)$ and $[-\lambda] \notin \Sigma^1(F_\tau)$.

As in [2, Section 1.4], we now consider a specific automorphism ν of F_τ to clear the case of ρ. Viewing the group F_τ as a group of PL homeomorphisms of the unit interval, ν is given by conjugation by $t \mapsto 1 - t$. This induces a homeomorphism of the character sphere that in particular swaps $[\lambda]$ with $[\rho]$, and also $[-\lambda]$ with $[-\rho]$, thus proving the lemma for F_τ. A further application of Theorem 1.5 also yields the result for K.

The homological variant of the lemma follows similarly; see also Eq. (1). □

We shall now consider the arcs between $[-\lambda]$ and $[-\rho]$ on the character sphere $S(F_\tau)$. Since $[-\lambda]$ and $[-\rho]$ are not antipodal points, there is a unique (closed) geodesic segment in $S(F_\tau)$ connecting them, which we denote by $\mathrm{conv}([-\lambda], [-\rho])$. In the other direction, there is a unique local geodesic from $[-\lambda]$ and $[-\rho]$, which we call the *long arc*, whose union with $\mathrm{conv}([-\lambda], [-\rho])$ yields the great circle in $S(F_\tau)$ containing $[-\lambda]$ and $[-\rho]$, in particular in this one-dimensional case, this is just $S(F_\tau)$ itself. We will need the following:

Citation 4.6 ([2, Theorem 2.3])**.** *Let G decompose as an ascending HNN extension $G = H *_{t,\sigma}$. Let χ be a character of G such that $\chi_{|H} \neq 0$. If H is of type $\mathtt{F}_\infty$ and $\chi|_H \in \Sigma^\infty(H)$, then $\chi \in \Sigma^\infty(G)$.*

Proposition 4.7. *All of $S(F_\tau)$, except possibly the closed geodesic*

$$\mathrm{conv}([-\lambda], [-\rho]),$$

lies in $\Sigma^\infty(F_\tau)$ and in $\Sigma^\infty(F_\tau, \mathbb{Z})$.

Proof. Again, we use our previous expression of the subgroup K as an HNN extension of $H = F_\tau[1]$. By Lemma 4.5, we know that $[\rho] \in \Sigma^\infty(K) \cap \Sigma^\infty(F_\tau)$. Now let $\chi \in \mathrm{Hom}(F_\tau, \mathbb{R})$ be arbitrary. We claim that

$$\chi(x_1) > 0 \iff \chi_{|H} \in [\rho|_H]. \tag{5}$$

In effect, $\chi = r\lambda + s\rho$ for some (unique) $r, s \in \mathbb{R}$ as λ and ρ are linearly independent and $\dim_{\mathbb{R}}(\mathrm{Hom}(F_\tau, \mathbb{R})) = 2$. Since $\lambda(x_1) = \lambda(y_1) = 0$ and $a_j = a_0 a_{j+1} a_0^{-1}$ for any $j \geq 1$ and $a \in \{x, y\}$, it follows that $\chi(w) = s\rho(w)$ for any $w \in H = F_\tau[1]$. This means that $\chi|_H \in \{[\rho|_H], [-\rho|_H]\}$. Finally, $\rho(x_1) = 1$ implies that $\chi(x_1) = s$, whence $\chi(x_1) > 0$ if and only if $\chi_{|H} \in [\rho|_H]$.

From here, we highlight that $H = F_\tau[1]$ is isomorphic to F_τ, via the isomorphism γ such that $\gamma(x_i) = x_{i-1}$ and $\gamma(y_i) = y_{i-1}$ for $i \geq 1$. The homeomorphism $S(F_\tau[1]) \cong S(F_\tau)$ induced by γ maps $[\rho|_H]$ to $[\rho]$. As $[\rho] \in \Sigma^\infty(F_\tau)$, this means $[\rho|_H] \in \Sigma^\infty(F_\tau[1])$. In particular, if $\chi \in \mathrm{Hom}(F_\tau, \mathbb{R})$ is positive on x_1, Claim (5) yields $[\chi_{|H}] = [\rho|_H] \in \Sigma^\infty(F_\tau[1])$. From here, we can apply Citation 4.6 to conclude that $[\chi_{|K}] \in \Sigma^\infty(K)$. Thus, $\chi(x_1) > 0 \implies [\chi_{|K}] \in \Sigma^\infty(K)$, whence $[\chi] \in \Sigma^\infty(F_\tau)$ by Theorem 1.5.

A straightforward computation shows that any character χ on the straight line from λ to ρ in $\mathrm{Hom}(F_\tau, \mathbb{R})$ satisfies $\chi(x_1) > 0$. The same holds for any character on the straight line from ρ to $-\lambda$. Hence, we have that the open arc in $S(F_\tau)$ from $[\lambda]$ to $[-\lambda]$ that contains $[\rho]$ actually lies in $\Sigma^\infty(F_\tau)$. Arguing again with the symmetry in $S(F_\tau)$ given by the automorphism ν induced by conjugation with $t \mapsto 1 - t$, we conclude that the open arc from $[\rho]$ to $[-\rho]$ containing $[\lambda]$ is also in $\Sigma^\infty(F_\tau)$. Altogether, the long (open) arc from $[-\lambda]$ to $[-\rho]$ is in $\Sigma^\infty(F_\tau)$, as claimed. The homological version follows directly from Eq. (1). $\square$

It now remains to consider the remaining short arc $\mathrm{conv}([-\lambda], [-\rho])$. To do this we will follow the approach of [2, Section 2.3]. We need the following two results:

Citation 4.8 ([2, Corollary 1.2]). *The kernel of a nonzero discrete character χ has type FP_n over the ring R if and only if both $[\chi]$ and $[-\chi]$ lie in $\Sigma^n(G, R)$.*

Citation 4.9 ([2, Theorem 2.7]). *Assume G contains no nonabelian free subgroups and is of type FP_2 over a ring R. Let $\widetilde{\chi} : G \to \mathbb{R}$ be a nonzero discrete character. Then G decomposes as an ascending HNN extension $H *_{t,\sigma}$, where H is a finitely generated subgroup of* $\ker(\widetilde{\chi})$ *and $\widetilde{\chi}(t)$ generates the image of $\widetilde{\chi}$.*

Proposition 4.10. *Let R be a ring. Then*

$$\mathrm{conv}([-\lambda], [-\rho]) \cap \Sigma^2(F_\tau, R) = \varnothing.$$

Proof. It suffices to show that no discrete character $\chi \in \mathrm{conv}([-\lambda], [-\rho])$ lies in $\Sigma^2(F_\tau, R)$ because such characters are dense in $\mathrm{conv}([-\lambda], [-\rho])$ and $\Sigma^2(F_\tau, R)$ is open; see, e.g., [2, Proposition 2.9]. Observe further that $[-\lambda]$, $[-\rho] \notin \Sigma^2(F_\tau, R)$ by Lemma 4.5.

So let χ be a discrete character of the form $\chi = a\lambda + b\rho$, with $a, b \in \mathbb{Q} \setminus \{0\}$. Using the elements $f, g \in F_\tau$ of Example 4.1, we can construct elements $t \in F_\tau$ with the following properties:

$$\lambda(t) = mb \quad \text{and} \quad \rho(t) = -ma \text{ for some } m \in \mathbb{Q} \setminus \{0\}. \tag{6}$$

In particular, $\chi(t) = 0$. Since λ has discrete image in $\mathbb{R}$ and $a \neq 0$, there exists t_0 satisfying condition (6) such that $|\lambda(t_0)|$ is minimal among all elements t fulfilling the properties listed in (6). Moreover, $\lambda(t_0) \neq 0$ for otherwise t_0 would not fulfill (6).

Let $G = \ker(\chi)$. Then, since the abelianisation of F_τ is $\mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z}$, we have that $G = \langle \sqrt{F'_\tau}, t_0 \rangle = \sqrt{F'_\tau} \rtimes \langle t_0 \rangle$, where $\sqrt{F'_\tau} := \{f \in F_\tau \mid f^n \in F'_\tau \text{ for some } n\}$.

Note that $\lambda|_G$ is a discrete nonzero character vanishing on the subgroup $\sqrt{F'_\tau} \leq G$ and such that $\mathrm{im}(\lambda|_G)$ is generated by $\lambda(t_0)$.

Now suppose G has type FP_2 over a ring R. By Citation 4.9, we can decompose G as the HNN extension $H *_{t,\sigma}$, where H is a finitely generated subgroup of $\sqrt{F'_\tau}$. As H is generated by a finite set of elements of F_τ, and each generator has support away from 0 and 1, there exists a value $\varepsilon'' > 0$ such that all elements of H are supported in the interval $[\varepsilon'', 1-\varepsilon'']$. Similarly, as t_0 has finitely many breakpoints, there is a value $\varepsilon' > 0$ such that t_0 is linear on the intervals $[0, \varepsilon']$ and $[1-\varepsilon', 1]$. Let $\varepsilon = \mathbf{min}\{\varepsilon', \varepsilon''\}$, giving us a value with both of these properties.

Since $\sqrt{F'_\tau} \rtimes \langle t_0 \rangle = G \cong H *_{t,\sigma}$, we can say that $\sqrt{F'_\tau} = \bigcup_{n\geq 1} t^n H t^{-n}$. Hence for each $f \in \sqrt{F'_\tau}$, there is a value n such that $t^{-n} f t^n \in H$, hence $t^{-n} f t^n$ is supported in $[\varepsilon, 1-\varepsilon]$. From here, we can see that any $f \in \sqrt{F'_\tau}$ must be supported in $[t_0^n(\varepsilon), t_0^n(1-\varepsilon)]$ for some n. As $\sqrt{F'_\tau}$ has support $(0,1)$, we can see that $(t_0^n(\varepsilon))_{n\in\mathbb{N}}$ must have a subsequence that converges to 0 and $(t_0^n(1-\varepsilon))_{n\in\mathbb{N}}$ must have a subsequence that converges to 1. As t_0 is linear on the intervals $[0,\varepsilon]$ and $[1-\varepsilon, 1]$, it holds $t_0(\varepsilon) < \varepsilon$ and $t_0(1-\varepsilon) > 1-\varepsilon$. Hence t_0 must have slope smaller than 1 near 0 and slope bigger than 1 near 1. Therefore, $ab < 0$. Given that we started with the assumption that $G = \ker(\chi)$ was of type FP_2, we obtain the implication

$$\chi = a\lambda + b\rho \quad \text{and} \quad \ker(\chi) \text{ of type } \mathrm{FP}_2 \implies ab < 0$$

whenever $a, b \in \mathbb{Q}\setminus\{0\}$. The contrapositive of this is that $ab > 0$ implies $\ker(\chi)$ is not of type FP_2. Combining this with Citation 4.8, we see that we cannot have both $[\chi]$ and $[-\chi]$ in $\Sigma^2(F_\tau, R)$. In particular, if the antipodal point $[-\chi]$ lies in $\Sigma^2(F_\tau, R)$, then by Proposition 4.7 we have that $[\chi] \notin \Sigma^2(F_\tau, R)$.

Transfering this result to the homotopical invariant with the use of Eq. (1), we conclude that if $[\chi] \notin \Sigma^2(F_\tau, R)$, then $[\chi] \notin \Sigma^2(F_\tau)$. □

This finishes off the proof of Theorem 1.1. □

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