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Canonical components of character varieties of double twist links $J(2m+1, 2m+1)$

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Abstract. We show that a certain smooth projective model of the canonical component of the $SL_2(\mathbb{C})$-character variety of the double twist link $J(2m+1, 2m+1)$, where m is a positive integer, is the conic bundle over the projective line $\mathbb{P}^1$ which is isomorphic to the surface obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by repeating a one-point blow-up $6m+3$ times.

Contents

1. Introduction

For a complete finite-volume hyperbolic 3-manifold with cusps, the $SL_2(\mathbb{C})$-character variety of M, denoted by $X(M)$, is a complex algebraic set associated to representations of $\pi_1(M)$ into $SL_2(\mathbb{C})$. Thurston [8] showed that any irreducible component of such a variety containing the character of a discrete faithful representation has complex dimension equal to the number of cusps of M. Such components are called canonical components and are denoted by $X_0(M)$. Character varieties have been important tools in studying the topology of M, and canonical components encode a lot of topological information about M. They contain subvarieties corresponding to Dehn fillings of M and their ideal points can be used to determine essential surfaces in M (see [1]).

Let $J(k, l)$ denote the double twist knot/link indicated in Figure 1, where the integers k and l determine the number of half twists in the boxes; positive numbers correspond to right-handed twists and negative numbers correspond

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to left-handed twists. This is the rational knot/link $C(k, -l)$ in the Conway's notation, which corresponds to the continued fraction $[k, -l] = k - 1/l$. It is a knot when kl is even and a two-component link when kl is odd. These are hyperbolic exactly when $|k|$ and $|l|$ are greater than one; the $J(\pm 1, l) = J(l, \pm 1)$ knot/links are torus knots/links.

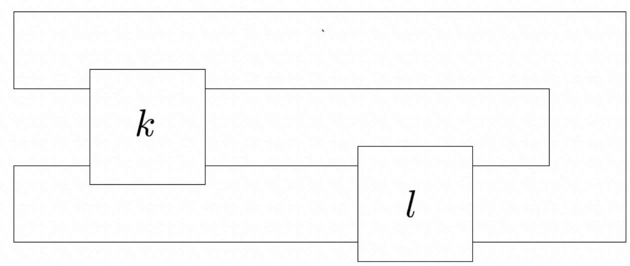

FIGURE 1. The double twist knot/link $J(k, l)$.

Character varieties of the $J(k, l)$ knots and links were computed and analyzed in [6] and [7] respectively. For the Whitehead link 5^2_1, which is $J(3, 3)$, Landes [5] showed that a certain smooth projective model of the canonical component in $\mathbb{P}^2 \times \mathbb{P}^1$ is the conic bundle over the projective line $\mathbb{P}^1$ which is isomorphic to the surface obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by repeating a one-point blow-up nine times. Equivalently, it is isomorphic to the surface obtained from $\mathbb{P}^2$ by repeating a one-point blow-up ten times. Harada [2] proved similar results for the links 6^2_2 and 6^2_3 in the Rolfsen's table. Note that a blow-up of $\mathbb{P}^2$ at two points is isomorphic to a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point, although a blow-up of $\mathbb{P}^2$ at one point is not isomorphic $\mathbb{P}^1 \times \mathbb{P}^1$ (see e.g. [3, Example 7.22]).

In [7], Petersen and the first author generalized Landes' result to the double twist links $J(3, 2m + 1)$ which contain the Whitehead link $J(3, 3)$, and proved that a certain smooth projective model of the canonical component of $J(3, 2m+1)$ in $\mathbb{P}^2 \times \mathbb{P}^1$ is the conic bundle over $\mathbb{P}^1$ which is isomorphic to the surface obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by repeating a one-point blow-up $9m$ times if $m \geq 1$, and $-(9m + 6)$ times if $m \leq -2$. An important step in proving this result is to show that each singular point of a certain singular projective model of the canonical component of $J(3, 2m + 1)$ in $\mathbb{P}^2 \times \mathbb{P}^1$ requires only one blow-up to resolve. However, this step was assumed without proof in [7]. Note that Harada [2] proved that for the link 6^2_3, which is not a double twist link, a certain singular projective model of the canonical component in $\mathbb{P}^2 \times \mathbb{P}^1$ has singular points which require more than one blow-up to resolve.

In this paper, we consider the hyperbolic double twist links $J(2m + 1, 2m + 1)$ which also contain the Whitehead link $J(3, 3)$, and identify their canonical components topologically. Since $J(-(2m + 1), -(2m + 1))$ is the mirror image of $J(2m+1, 2m+1)$, we only need to consider the case $m \geq 1$. We will show the following.

Theorem 1. *The smooth projective model of the canonical component of the* $\mathrm{SL}_2(\mathbb{C})$*-character variety of the double twist link* $J(2m+1, 2m+1)$*,* $m \geq 1$*, is the conic bundle over the projective line* $\mathbb{P}^1$ *which is isomorphic to the surface obtained from* $\mathbb{P}^1 \times \mathbb{P}^1$ *by repeating a one-point blow-up* $6m+3$ *times. Equivalently, it is isomorphic to the surface obtained from* $\mathbb{P}^2$ *by repeating a one-point blow-up* $6m+4$ *times.*

Let us explain the meaning of the smooth projective model in Theorem 1 and sketch the proof. An affine model of the canonical component of the $\mathrm{SL}_2(\mathbb{C})$-character variety of the double twist link $J(2m+1, 2m+1)$ is given by the zero set of a single polynomial in three complex variables, and it is known to be an affine surface birational to $\mathbb{C} \times \mathbb{C}$. (This fact actually holds true for all double twist links $J(2m+1, 2n+1)$, by [7].) For affine complex surfaces, choosing the right projective completion is not obvious since different projective completions might result in non-isomorphic smooth projective models. In the case of the canonical component of the double twist link $J(2m+1, 2m+1)$, choosing the projective completion in $\mathbb{P}^3$ seems natural. However, this projective model has infinitely many singular points. Following [5], we will choose the projective completion in $\mathbb{P}^2 \times \mathbb{P}^1$ which turns out to have finitely many singular points.

By compactifying the above affine model of the canonical component of $J(2m+1, 2m+1)$ in $\mathbb{P}^2 \times \mathbb{P}^1$, we obtain a projective model, denoted by S, birational to $\mathbb{P}^1 \times \mathbb{P}^1$. This projective model is not smooth; it has singular points. By resolving singular points of the surface S (using one-point blow-ups), we obtain a smooth projective model, denoted by $\tilde{S}$. *In this paper we refer to* $\tilde{S}$ *as the smooth projective model of the canonical component of the* $\mathrm{SL}_2(\mathbb{C})$*-character variety of* $J(2m+1, 2m+1)$.

The smooth projective model $\tilde{S}$ is also birational to $\mathbb{P}^1 \times \mathbb{P}^1$. It is known that for two birational varieties the birational equivalence between them can be written as a sequence of blow-ups and blow-downs, see e.g. [4, Chapter 5]. Since $\mathbb{P}^1 \times \mathbb{P}^1$ is a minimal smooth projective surface (in the sense that it is not a blow-up of any smooth projective surface), we conclude that $\tilde{S}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at N points. Moreover, this isomorphism (i.e. this number N) can be determined from the Euler characteristic of $\tilde{S}$ which, in turn, depends on the Euler characteristic and singular points of S.

An important part of the proof of Theorem 1 is to prove that each singular point of the singular projective model S requires only one blow-up to resolve, namely, the blow-up of S at each singular point is smooth everywhere except at the preimages of other singular points of S. A similar proof also works for $J(3, 2m+1)$ and therefore fixes the gap in [7]. The remaining of the proof is in the same line as those of [5, 7].

The paper is organized as follows. In Section 2 we review Chebyshev polynomials, character varieties of double twist links, and blowing up surfaces. In Section 3, we give a proof of Theorem 1 with the assumption that each singular point of the projective model S of the canonical component of $J(2m+1, 2m+1)$ requires only one blow-up to resolve (Proposition 3.4). Finally, we prove Proposition 3.4 in Section 4 and therefore complete the proof of Theorem 1.

2. Preliminaries

In this section, we first recall the definition of $\mathrm{SL}_2(\mathbb{C})$-character varieties of 3-manifolds. Then, we define Chebychev polynomials of the second kind and prove some of their properties. Next, we review character varieties of two-component double twist links from [7]. Finally, we recall the definition of blowing up varieties at a point.

2.1. Character varieties. Let M be a complete finite-volume hyperbolic 3-manifold with cusps. The $\mathrm{SL}_2(\mathbb{C})$-character variety of M is the set of all characters of representations $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$. The character associated to ρ is $\chi_\rho : \pi_1(M) \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \operatorname{tr} \rho(\gamma)$.

Let $X(M)$ denote the $\mathrm{SL}_2(\mathbb{C})$-character variety, that is

$$X(M) = \{\chi_\rho \mid \rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})\}.$$

The characters of reducible representations themselves form an algebraic set, which is a subset of $X(M)$. The closure of the set of characters of irreducible representations will be denoted by $X_{\mathrm{irr}}(M)$. Any irreducible component of $X(M)$ which contains the character of a discrete faithful representation is contained in $X_{\mathrm{irr}}(M)$ and is called a canonical component and denoted by $X_0(M)$.

Character varieties have been important tools in studying the topology of M, and canonical components encode a lot of topological information about M. They contain subvarieties corresponding to Dehn fillings of M and their ideal points can be used to determine essential surfaces in M (see [1]).

2.2. Chebychev polynomials. Let $S_k(z)$ be the Chebyshev polynomials of the second kind defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_{k+1}(z) = zS_k(z) - S_{k-1}(z)$ for all integers k.

It is elementary to verify the following lemma by induction.

Lemma 2.1. (1) *With $z = a + a^{-1}$ we have*

$$S_k(z) = \frac{a^{k+1} - a^{-k-1}}{a - a^{-1}}.$$

(2) *For $k \geq 1$, the polynomial $S_k(z)$ has degree k and leading term z^k.*

The following two lemmas can be verified by using Lemma 2.1.

Lemma 2.2. (1) *For $k \geq 1$, the polynomial $S_k(z) - S_{k-1}(z)$ has exactly k distinct roots given by $z = 2\cos\frac{(2j-1)\pi}{2k+1}$ where $1 \leq j \leq k$.*

(2) *For $k \geq 1$, the polynomial $S_k(z) + S_{k-1}(z)$ has exactly k distinct roots given by $z = 2\cos\frac{2j\pi}{2k+1}$ where $1 \leq j \leq k$.*

Lemma 2.3. *For any integer k we have*

$$S_k^2(z) + S_{k-1}^2(z) - zS_k(z)S_{k-1}(z) = 1.$$

We now prove the following two lemmas.

Lemma 2.4. *For $k \geq 1$, the polynomial $2z + (z^2-4)S_{k-1}(z)S_k(z)$ has exactly $2k+1$ distinct roots given by $z = 2\cos\frac{(2j-1)\pi}{2k}$ ($1 \leq j \leq k$) and $z = 2\cos\frac{(2j-1)\pi}{2k+2}$ ($1 \leq j \leq k+1$). In particular, it is a separable polynomial in $\mathbb{C}[z]$.*

Proof. Let $P(z) = 2z + (z^2-4)S_{k-1}(z)S_k(z)$. Consider $z = a + a^{-1}$ where $a \neq \pm 1$. Since $S_j(z) = \frac{a^{j+1}-a^{-j-1}}{a-a^{-1}}$ we have

$$\begin{aligned} P &= 2(a+a^{-1}) + (a^2+a^{-2}-2)\frac{a^k-a^{-k}}{a-a^{-1}}\frac{a^{k+1}-a^{-k-1}}{a-a^{-1}} \\ &= a + a^{-1} + a^{2k+1} + a^{-2k-1} \\ &= (a^k + a^{-k})(a^{k+1} + a^{-k-1}). \end{aligned}$$

Note that $P = 0$ if $a^{2k} = -1$ or $a^{2k+2} = -1$. Moreover, these two equations do not have any common roots. This implies that $z = 2\cos\frac{(2j-1)\pi}{2k}$, $1 \leq j \leq k$, and $z = 2\cos\frac{(2j-1)\pi}{2k+2}$, $1 \leq j \leq k+1$, are distinct roots of P. Since the degree of P is exactly $2k+1$, these are all the roots of P. Therefore, P is separable in $\mathbb{C}[z]$. □

Lemma 2.5. *For any integer k we have*

$$\frac{dS_k(z)}{dz} = \frac{kS_{k+1}(z) - (k+2)S_{k-1}(z)}{z^2-4}.$$

Proof. Write $z = a + a^{-1}$. Then $S_k(z) = \frac{a^{k+1}-a^{-k-1}}{a-a^{-1}}$ and so

$$\begin{aligned} \frac{dS_k(z)}{dz} &= \frac{dS_k(z)}{da}\Big/\frac{dz}{da} \\ &= \frac{(k+1)(a^k + a^{-k-2})(a-a^{-1}) - (a^{k+1}-a^{-k-1})(1+a^{-2})}{(a-a^{-1})^2(1-a^{-2})} \\ &= \frac{k\frac{a^{k+1}-a^{-k-3}}{1-a^{-2}} - (k+2)\frac{a^{k-1}-a^{-k-1}}{1-a^{-2}}}{z^2-4}. \end{aligned}$$

The lemma follows, since $\frac{a^j - a^{-j-2}}{1-a^{-2}} = \frac{a^{j+1}-a^{-j-1}}{a-a^{-1}} = S_j(z)$. □

2.3. Double twist links. Recall that $J(k,l)$ is the double twist knot/link indicated in Figure 1. It is a knot when kl is even and a two-component link when kl is odd. The knot/link $J(k,l)$ is hyperbolic exactly when $|k|$ and $|l|$ are greater than one; the $J(\pm 1, l) = J(l, \pm 1)$ knot/links are torus knots/links. Let $X(k,l)$ denote the $\mathrm{SL}_2(\mathbb{C})$-character variety of $S^3 \setminus J(k,l)$ and $X_0(k,l)$ its canonical component.

Character varieties of the $J(k,l)$ knots and links were computed in [6] and [7] respectively. We now review the computation for the $J(k,l)$ links with two components, so both k and l are odd. Suppose $k = 2m+1$ and $l = 2n+1$. By [6], the link group of $J(k,l)$ is $\pi_1(k,l) = \pi_1(S^3 \setminus J(k,l))$ and has presentation

$$\pi_1(k,l) = \langle a, b \mid a w_k^n b = w_k^{n+1} \rangle$$

where $w_k = (ab^{-1})^m ab (a^{-1}b)^m$. This is the Wirtinger presentation of a link diagram.

For a word u in two letters a and b, let $\overleftarrow{u}$ denote the word obtained from u by writing the letters in u in reversed order. By [7], the above presentation of the link group of $J(k,l)$ can be rewritten as

$$\pi_1(k,l) = \langle a, b \mid r = \overleftarrow{r} \rangle$$

where $r = w_k^n (ab^{-1})^m$.

For a representation $\rho\colon \pi_1(k,l) \to \mathrm{SL}_2(\mathbb{C})$, we let $x = \operatorname{tr}\rho(a)$, $y = \operatorname{tr}\rho(b)$ and $z = \operatorname{tr}\rho(ab^{-1})$. Then, by [9, Thm. 1] the algebraic set $X(k,l)$ is exactly the zero set of $\phi(x,y,z) = \operatorname{tr}\rho(rab) - \operatorname{tr}\rho(\overleftarrow{r}ab) \in \mathbb{C}[x,y,z]$. Moreover, by [7], this polynomial can be written in terms of Chebyshev polynomials as

$$\phi(x,y,z) = (xyz + 4 - x^2 - y^2 - z^2)\,(S_n(t)S_{m-1}(z) - S_{n-1}(t)S_m(z))\,,$$

where

$$t = \operatorname{tr}\rho(w_k) = xy - z + (xyz + 4 - x^2 - y^2 - z^2)S_m(z)S_{m-1}(z).$$

The character variety $X(k,l)$ is clearly reducible. The vanishing set of $xyz + 4 - x^2 - y^2 - z^2 \in \mathbb{C}[x,y,z]$ is the set of characters of reducible representations of $\pi_1(k,l)$ into $\mathrm{SL}_2(\mathbb{C})$. An affine model for the algebraic set $X_{\mathrm{irr}}(k,l)$ is the vanishing set of $S_n(t)S_{m-1}(z) - S_{n-1}(t)S_m(z) \in \mathbb{C}[x,y,z]$. Then we have the following.

Theorem 2.6. [7] *Let $k = 2m+1$ and $l = 2n+1$. The algebraic set $X_{\mathrm{irr}}(k,l)$ is birational to $C(k,l) \times \mathbb{C}$ where the curve $C(k,l)$ is given by*

$$C(k,l) = \{(t,z) \in \mathbb{C}^2 \mid S_n(t)S_{m-1}(z) - S_{n-1}(t)S_m(z) = 0\}.$$

If $k \neq l$ then $C(k,l)$ is irreducible and $X_0(k,l) = X_{\mathrm{irr}}(k,l)$ is birational to $C(k,l) \times \mathbb{C}$.

The curve $C(3,3) = C(-3,-3)$ is given by $t = z$. If $k = l$ and $|l| > 3$ then $C(l,l)$ is the union of exactly two irreducible components: $C_0(l,l)$, given by $t = z$, and $C_1(l,l)$, the scheme-theoretic complement of $C_0(l,l)$ in $C(l,l)$. The algebraic set $X_{\mathrm{irr}}(l,l)$ is given by the union $X_0(l,l) \cup X_1(l,l)$, where $X_0(l,l)$ is birational to $C_0(l,l) \times \mathbb{C}$ and $X_1(l,l)$ is birational to $C_1(l,l) \times \mathbb{C}$.

2.4. One-point blow-ups. Blowing up varieties is a standard tool for resolving singular points of surfaces. Since blowing up is a local process, it can be done in affine neighborhoods. For our purpose, understanding blowing up subvarieties of $\mathbb{A}^n$ at a point should be sufficient. For more details about blow-ups, see [3] and [4].

Blowing up $\mathbb{A}^n$ at a point $p \in \mathbb{A}^n$ can be described as replacing p by a copy of $\mathbb{P}^{n-1}$. To be precise, by taking $x_1, \cdots, x_n$ as affine coordinates for $\mathbb{A}^n$ and $y_1, \cdots, y_n$ as projective coordinates for $\mathbb{P}^{n-1}$, the blow-up of $\mathbb{A}^n$ at a point $p = (p_1, \cdots, p_n)$ is the closed subvariety

$$Y = \{((x_1, \cdots, x_n), [y_1 : \cdots : y_n]) \mid (x_i - p_i)y_j = (x_j - p_j)y_i \text{ for all } 1 \le i, j \le n\}$$

of $\mathbb{A}^n \times \mathbb{P}^{n-1}$. This blow-up comes with a natural map $\gamma : Y \to \mathbb{A}^n$ which is simply the projection onto the first factor. The preimage of any point $(x_1, \cdots, x_n) \neq (p_1, \cdots, p_n) \in \mathbb{A}^n$ is precisely one point in Y. However, the preimage of $(x_1, \cdots, x_n) = (p_1, \cdots, p_n)$ is the subset set $\{(p_1, \cdots, p_n)\} \times \mathbb{P}^{n-1}$ of Y. Since $\gamma|_{Y \setminus \gamma^{-1}(p)} : Y \setminus \gamma^{-1}(p) \to \mathbb{A}^n \setminus \{p\}$ is an isomorphism, γ is a birational map and $\mathbb{A}^n$ is birational to Y.

To blow up a subvariety $X \subset \mathbb{A}^n$ at a point p, we first take the blow-up Y of $\mathbb{A}^n$ at p. Then the blow-up of X at p is the Zariski closure of $\gamma^{-1}(X \setminus p)$ in Y.

In this paper, we obtain smooth projective models of singular projective surfaces by blowing them up at their singular points.

3. Proof of Theorem 1

Let m be a positive integer and $l = 2m+1$. By Theorem 2.6, an affine model of the canonical component $X_0(l, l)$ of the $\mathrm{SL}_2(\mathbb{C})$-character variety of the double twist link $J(l, l)$ is the zero set of the polynomial $t - z \in \mathbb{C}[x, y, z]$, where

$$t = xy - z + (xyz + 4 - x^2 - y^2 - z^2)S_m(z)S_{m-1}(z).$$

Moreover, it is birational to $C_0(l, l) \times \mathbb{C}$ where $C_0(l, l) = \{(t, z) \in \mathbb{C}^2 \mid t = z\}$. In particular, $X_0(l, l)$ is birational to $\mathbb{C} \times \mathbb{C}$.

3.1. Projective model. We begin by homogenizing the defining polynomial for $X_0(l, l)$.

Let $T_k = T_k(z, w) = w^k S_k(\frac{z}{w})$ for $k \ge 0$.

Lemma 3.1. *For $k \ge 1$ we have*
(1) $T_k(z, 0) = z^k$,
(2) $T_k^2 + w^2 T_{k-1}^2 - z\, T_k T_{k-1} = w^{2k}$,
(3) $w^{2k} + (z \pm 2w) T_k T_{k-1} = (T_k \pm w\, T_{k-1})^2$.

Proof. (1) follows from Lemma 2.1(2).

(2) follows from Lemma 2.3.

(3) From (2), we have $w^{2k} + z\, T_k T_{k-1} = T_k^2 + w^2 T_{k-1}^2$. Hence, $w^{2k} + (z \pm 2w) T_k T_{k-1} = T_k^2 + w^2 T_{k-1}^2 \pm 2w T_k T_{k-1} = (T_k \pm w\, T_{k-1})^2$. □

The homogenization of the defining polynomial $t - z = xy - 2z + (xyz + 4 - x^2 - y^2 - z^2)S_m(z)S_{m-1}(z)$ in $\mathbb{P}^2 \times \mathbb{P}^1 = \{([x : y : u], [z : w])\}$ is

$$F = (xyw - 2u^2 z)w^{2m} + (xyzw + 4u^2w^2 - x^2w^2 - y^2w^2 - u^2z^2)T_m T_{m-1}.$$

3.2. Singular points. We now determine the singular points of the projective model of $X_0(l,l)$. To do this, we consider solutions $([x : y : u],[z : w]) \in \mathbb{P}^2 \times \mathbb{P}^1$ of $F = F_x = F_y = F_u = F_z = F_w = 0$.

First, we compute these partial derivatives by direct calculations.

Lemma 3.2. *The first order partial derivatives of F are given by*

$$\begin{aligned}
F_x &= \left(yw^{2m} + (yz - 2xw)T_m T_{m-1}\right) w,\\
F_y &= \left(xw^{2m} + (xz - 2yw)T_m T_{m-1}\right) w,\\
F_u &= -2u\left(2zw^{2m} + (z^2 - 4w^2)T_m T_{m-1}\right),\\
F_z &= -2u^2w^{2m} + (xyw - 2u^2z)T_m T_{m-1}\\
&\quad + (xyzw + 4u^2w^2 - x^2w^2 - y^2w^2 - u^2z^2)(T_m T_{m-1})_z,\\
F_w &= (2m+1)xyw^{2m} - 4mu^2zw^{2m-1} + (xyz + 8u^2w - 2x^2w - 2y^2w)T_m T_{m-1}\\
&\quad + (xyzw + 4u^2w^2 - x^2w^2 - y^2w^2 - u^2z^2)(T_m T_{m-1})_w.
\end{aligned}$$

We can now determine the singular points.

Proposition 3.3. *The singular points $([x : y : u],[z : w]) \in \mathbb{P}^2 \times \mathbb{P}^1$ of F are*

- $s_1 = ([0 : 1 : 0],[1 : 0])$,
- $s_2 = ([1 : 0 : 0],[1 : 0])$,
- $s_3^{(k)} = \left([1 : 1 : 0],[z_3^{(k)} : 1]\right)$, *where* $z_3^{(k)} = 2\cos\frac{(2k-1)\pi}{2m+1}$, $1 \le k \le m$,
- $s_4^{(k)} = \left([1 : -1 : 0],[z_4^{(k)} : 1]\right)$, *where* $z_4^{(k)} = 2\cos\frac{2k\pi}{2m+1}$, $1 \le k \le m$.

The number of singular points is $2m+2$.

Proof. Consider the equations $F = F_x = F_y = F_u = F_z = F_w = 0$. We break the analysis down into two cases: $w = 0$ and $w \neq 0$.

Case 1: $w = 0$. We can assume $z = 1$. Note that $T_k(1,0) = 1$ for all $k \ge 1$. By Lemma 3.2, we have $F_x = F_y = 0$, $F = -u^2$ and $F_u = -2u$. Then $F = F_u = 0$ are equivalent to $u = 0$. Now we have $F_z = 0$ and $F_w = xy$. Thus $F_w = 0$ becomes $xy = 0$. In this case, there are two singular points $([0 : 1 : 0],[1 : 0])$ and $([1 : 0 : 0],[1 : 0])$.

Case 2: $w \neq 0$. In this case, we first solve $F_x = F_y = 0$ and then $F = F_u = 0$. Finally, we show that the equations $F_z = F_w = 0$ follow from $F = F_x = F_y = F_u = 0$.

Since $w \neq 0$, we can assume $w = 1$. We first claim that $(x,y) \neq (0,0)$. Indeed, assuming $(x,y) = (0,0)$ we have

$$F = -2z + (4 - z^2)S_{m-1}(z)S_m(z).$$

By Lemma 2.4, this polynomial is separable in $\mathbb{C}[z]$, so the equations $F = F_z = 0$ cannot occur. Hence, $(x,y) \neq (0,0)$.

Consider the equations $F_x = F_y = 0$. By Lemma 2.3, we have $S_m^2(z) + S_{m-1}^2(z) - zS_m(z)S_{m-1}(z) = 1$. This implies that

$$\begin{aligned} F_x &= y + (yz-2x)S_m(z)S_{m-1}(z) = y(S_m^2(z)+S_{m-1}^2(z)) - 2xS_m(z)S_{m-1}(z), \\ F_y &= x + (xz-2y)S_m(z)S_{m-1}(z) = x(S_m^2(z)+S_{m-1}^2(z)) - 2yS_m(z)S_{m-1}(z). \end{aligned}$$

Hence,

$$\begin{aligned} 2S_m(z)S_{m-1}(z)F_x + (S_m^2(z)+S_{m-1}^2(z))F_y &= x\left(S_m^2(z)-S_{m-1}^2(z)\right)^2, \\ 2S_m(z)S_{m-1}(z)F_y + (S_m^2(z)+S_{m-1}^2(z))F_x &= y\left(S_m^2(z)-S_{m-1}^2(z)\right)^2. \end{aligned}$$

Since x and y are not simultaneously equal to 0, the equations $F_x = F_y = 0$ imply that $S_m^2(z) - S_{m-1}^2(z) = 0$. We now consider the subcases $S_m(z) - S_{m-1}(z) = 0$ and $S_m(z) + S_{m-1}(z) = 0$ separately.

Subcase 2a: $S_m(z) - S_{m-1}(z) = 0$. By Lemma 2.2, $z = 2\cos\frac{(2k-1)\pi}{2m+1}$ for some $1 \le k \le m$. From $S_m^2(z)+S_{m-1}^2(z)-zS_m(z)S_{m-1}(z) = 1$ and $S_m(z)-S_{m-1}(z) = 0$, we have $S_m^2(z) = \frac{1}{2-z}$. This implies that $F_x = \frac{2(y-x)}{2-z}$ and $F_y = \frac{2(x-y)}{2-z}$. Hence, $F_x = F_y = 0$ are equivalent to $x = y$. Since $S_m^2(z) = \frac{1}{2-z}$, we have $F = u^2(2-z)$ and $F_u = 2u(2-z)$. Hence, $F = F_u = 0$ are equivalent to $u = 0$. Then, by Lemma 3.2 we have

$$\begin{aligned} F_z &= \left[S_m(z)S_{m-1}(z) + (z-2)(S_m(z)S_{m-1}(z))'\right]x^2 \\ F_w &= \left[(2m+1) + (z-4)S_m(z)S_{m-1}(z) + (z-2)(T_mT_{m-1})_w\right]x^2. \end{aligned}$$

We claim that $F_z = F_w = 0$. Indeed, by taking derivative of the identity $S_m^2(z) + S_{m-1}^2(z) - zS_m(z)S_{m-1}(z) = 1$ and using $S_m(z) = S_{m-1}(z)$, we get $(2-z)(S_m'(z) + S_{m-1}'(z)) = S_m(z)$. It follows that $F_z = 0$.

Similarly, by taking partial derivative w.r.t. w of the identity $T_m^2 + w^2T_{m-1}^2 - zT_mT_{m-1} = w^{2m}$ (by Lemma 3.1(2)) and using $S_m(z) = S_{m-1}(z)$, we get

$$(2-z)\left((T_m)_w + (T_{m-1})_w\right)S_m(z) + 2S_m^2(z) = 2m.$$

It follows that

$$(2m+1) + (z-4)S_m(z)S_{m-1}(z) + (z-2)(T_mT_{m-1})_w = 1 + (z-2)S_m^2(z) = 0.$$

Hence, $F_w = 0$.

We have proved that the singular points in this subcase are $([1:1:0],[z:1])$ where $z = 2\cos\frac{(2k-1)\pi}{2m+1}$ for some $1 \le k \le m$.

Subcase 2b: $S_m(z)+S_{m-1}(z) = 0$. Similar to the above, singular points in this subcase are $([1:-1:0],[z:1])$ where $z = 2\cos\frac{2k\pi}{2m+1}$ for some $1 \le k \le m$. □

Let $S = \mathcal{Z}(F) \subset \mathbb{P}^2 \times \mathbb{P}^1$ be the vanishing set of F.

Proposition 3.4. *Each singular point p of S requires only one blow-up to resolve. Namely, the blow-up of S at p is smooth everywhere except at the preimages of other singular points $q \neq p$ of S.*

We will prove Proposition 3.4 in the last section.

3.3. Euler characteristic. As in [5], to compute the Euler characteristic $\chi(S)$ we observe that $F = G + u^2H$, where G, H are polynomials independent of u. Explicitly,

$$\begin{aligned} G &= xyw^{2m+1} + (xyzw - x^2w^2 - y^2w^2)T_mT_{m-1}, \\ H &= -2zw^{2m} + (4w^2 - z^2)T_mT_{m-1}. \end{aligned}$$

Recall that $T_k = T_k(z,w) = w^kS_k(\frac{z}{w}) \in \mathbb{C}[z,w]$. By Lemma 3.1(2), we have $T_m^2 + w^2T_{m-1}^2 - z\,T_mT_{m-1} = w^{2m}$. Hence, we can write

$$G = (x\,T_m - yw\,T_{m-1})(y\,T_m - xw\,T_{m-1})w.$$

Due to the special form of F as above, we introduce the rational map

$$\varphi : S = \mathcal{Z}(F) \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

defined by $([x : y : u], [z : w]) \mapsto ([x : y], [z : w])$. This will play an important role in the computation of $\chi(S)$.

We first determine the domain of φ.

Lemma 3.5. *The domain of φ is the set $U = S \setminus A$, where A is the set of points $([0 : 0 : 1], [z : 1])$ in $\mathbb{P}^2 \times \mathbb{P}^1$ satisfying $-2z + (4 - z^2)S_m(z)S_{m-1}(z) = 0$.*

Proof. The map φ is not defined at points of the set

$$A = \{([0 : 0 : 1], [z : w]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid F = 0\} \subset S.$$

When $(x, y, u) = (0, 0, 1)$ we have $G = 0$ and so $F = H$. If $(z, w) = (1, 0)$ then $H = -T_m(1, 0)T_{m-1}(1, 0) = -1 \neq 0$. If $w = 1$ then $H = -2z + (4 - z^2)S_m(z)S_{m-1}(z)$. Hence, A is equal to the set of points $([0 : 0 : 1], [z : 1])$ in $\mathbb{P}^2 \times \mathbb{P}^1$ satisfying $-2z + (4 - z^2)S_m(z)S_{m-1}(z) = 0$. □

Note that the set A has cardinality $2m + 1$. We next determine the image $\varphi(U)$.

Lemma 3.6. *We have*

$$\varphi(U) = \mathbb{P}^1 \times \mathbb{P}^1 - B,$$

where B is the set of points $([x : y], [z : 1]) \in \mathbb{P}^1 \times \mathbb{P}^1$ satisfying $-2z + (4 - z^2)S_m(z)S_{m-1}(z) = 0$ and $(xS_m(z) - yS_{m-1}(z))(yS_m(z) - xS_{m-1}(z)) \neq 0$.

Proof. Note that a point $([x : y], [z : w]) \in \mathbb{P}^1 \times \mathbb{P}^1$ is not in the image $\varphi(U)$ if and only if $F([x : y : u], [z : w]) \in \mathbb{C}[u]$ is a nonzero constant. This is equivalent to $H = 0$ and $G \neq 0$. Recall that $G = (x\,T_m - yw\,T_{m-1})(y\,T_m - xw\,T_{m-1})w$.

Since $G \neq 0$, we have $w \neq 0$. We can assume $w = 1$, so $H = -2z + (4 - z^2)S_m(z)S_{m-1}(z)$ and $G = (xS_m(z) - yS_{m-1}(z))(yS_m(z) - xS_{m-1}(z))$. The lemma then follows. □

Lemma 3.7. *We have*

$$\chi(B) = 0.$$

Proof. Let $P(z) = -2z+(4-z^2)S_m(z)S_{m-1}(z)$. By Lemma 2.4, $P(z)$ is separable in $\mathbb{C}[z]$. Moreover, by Lemma 2.2, $P(z)$ and $S_m(z) \pm S_{m-1}(z)$ do not share any common roots. Hence, if $P(z) = 0$ then $S_m(z) \neq \pm S_{m-1}(z)$. We have

$$B = \bigsqcup_{z \in \mathcal{Z}(P)} (\mathbb{P}^1 \setminus \{[S_m(z) : S_{m-1}(z)], [S_{m-1}(z) : S_m(z)]\}) \times \{[z : 1]\}$$

Since $\mathbb{P}^1$ with two points removed has Euler characteristic zero, we obtain $\chi(B) = 0$. □

Let $C = \mathcal{Z}(G)$ be the zero set of G in $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 3.8. *We have*

$$\chi(C) = 4 - 2m.$$

Proof. To compute the Euler characteristic of C, we write $C = C_1 \cup C_2 \cup C_3$ where C_i's are subsets of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$\begin{aligned} C_1 &= \mathcal{Z}(w) = \mathbb{P}^1 \times \{(1 : 0)\}, \\ C_2 &= \mathcal{Z}(x T_m - yw T_{m-1}), \\ C_3 &= \mathcal{Z}(y T_m - xw T_{m-1}). \end{aligned}$$

Note that $C_1 \cap C_2 = \{([1 : 0], [1 : 0])\}$ and $C_1 \cap C_3 = \{([0 : 1], [1 : 0])\}$. Moreover, $([x : y], [z : w]) \in C_2 \cap C_3$ if and only if $x = y$ and $T_m = w T_{m-1}$, or $x = -y$ and $T_m = -w T_{m-1}$. If $(z, w) = (1, 0)$ then $T_k = 1$ and so $T_m \neq \pm w T_{m-1}$. If $w = 1$ then the equation $T_m = \pm w T_{m-1}$ is equivalent to $S_m(z) = \pm S_{m-1}(z)$. Hence,

$$\begin{aligned} C_2 \cap C_3 = \;& \{([1 : 1], [z : 1]) \mid S_m(z) - S_{m-1}(z) = 0\} \\ & \bigcup \{([1 : -1], [z : 1]) \mid S_m(z) + S_{m-1}(z) = 0\}, \end{aligned}$$

which has cardinality $2m$. Hence,

$$\begin{aligned} \chi(C) &= \chi(C_1) + \chi(C_2) + \chi(C_3) - \chi(C_1 \cap C_2) - \chi(C_1 \cap C_3) - \chi(C_2 \cap C_3) \\ &\quad + \chi(C_1 \cap C_2 \cap C_3) \\ &= 2 + 2 + 2 - 1 - 1 - 2m + 0 = 4 - 2m. \end{aligned}$$

Note that $C_1 \cap C_2 \cap C_3 = \emptyset$. □

We are now ready to compute the Euler characteristic of the surface $S = \mathcal{Z}(F)$.

Proposition 3.9. *We have*

$$\chi(S) = 4m + 5.$$

Proof. Recall that $F = G + u^2 H$, where G, H are polynomials independent of u, and $\varphi : S \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is defined by $([x : y : u], [z : w]) \mapsto ([x : y], [z : w])$.

Note that $\chi(S) = \chi(U) + \chi(A)$. Since A is a finite set of cardinality $2m + 1$, we have $\chi(A) = 2m + 1$. To compute $\chi(U)$ we notice that a fixed point $([x : y], [z : w]) \in \varphi(U) = (\mathbb{P}^1 \times \mathbb{P}^1) \setminus B$ has

- a two-element preimage if $G \neq 0$ and $H \neq 0$,
- a one-element preimage if $G = 0$ and $H \neq 0$, and
- an infinite preimage isomorphic to the affine line $\mathbb{A}^1$ if $G = 0$ and $H = 0$,

where $B = \{([x : y], [z : w]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid G \neq 0, H = 0\}$.

Recall that $C = \{([x : y], [z : w]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid G = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$. Let $L = \{([x : y], [z : w]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid G = 0, H = 0\}$. Note that

$$\begin{aligned} \{([x : y], [z : w]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid G \neq 0, H \neq 0\} &= \varphi(U) \setminus C, \\ \{([x : y], [z : w]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid G = 0, H \neq 0\} &= C \setminus L. \end{aligned}$$

Note that $\varphi(U)$ is the disjoint union of three subsets $\varphi(U) \setminus C$, $C \setminus L$ and L. Hence, $U = \varphi^{-1}(\varphi(U))$ can be written as the disjoint union of three subsets $\varphi^{-1}(\varphi(U) \setminus C)$, $\varphi^{-1}(C \setminus L)$ and $\varphi^{-1}(L)$. Since

$$\begin{aligned} \chi(\varphi^{-1}(\varphi(U) \setminus C)) &= 2\chi(\varphi(U) \setminus C), \\ \chi(\varphi^{-1}(C \setminus L)) &= \chi(C \setminus L), \\ \chi(\varphi^{-1}(L)) &= |L|\chi(\mathbb{A}^1) = |L| = \chi(L). \end{aligned}$$

we have

$$\begin{aligned} \chi(U) &= 2\chi(\varphi(U) \setminus C) + \chi(C \setminus L) + \chi(L) \\ &= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1 \setminus (B \sqcup C)) + \chi(C) \\ &= \big(2\chi(\mathbb{P}^1 \times \mathbb{P}^1) - 2\chi(B) - 2\chi(C)\big) + \chi(C) \\ &= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1) - 2\chi(B) - \chi(C) \\ &= 8 - 0 - (4 - 2m) = 2m + 4. \end{aligned}$$

Finally, since $\chi(A) = 2m + 1$ we obtain $\chi(S) = \chi(U) + \chi(A) = 4m + 5$. □

3.4. Proof of Theorem 1. Recall that $S = \mathcal{Z}(F) \subset \mathbb{P}^2 \times \mathbb{P}^1$ is the vanishing set of F. Let S_{sing} be the set of singular points of S. By Proposition 3.3, its cardinality is $|S_{\text{sing}}| = 2m + 2$.

Let $\tilde{S}$ be the smooth projective surface obtained from S by resolving all the singular points of S. By Proposition 3.4, each singular point of S requires one blow-up to resolve. Moreover, from its proof in Section 4 we see that the preimage of each singular point is locally a conic and hence locally isomorphic to $\mathbb{P}^1$. This implies that

$$\chi(\tilde{S}) = \chi(S \setminus S_{\text{sing}}) + |S_{\text{sing}}| \cdot \chi(\mathbb{P}^1) = (\chi(S) - |S_{\text{sing}}|) + 2|S_{\text{sing}}| = \chi(S) + |S_{\text{sing}}|.$$

Hence,

$$\chi(\tilde{S}) = \chi(S) + |S_{\text{sing}}| = (4m + 5) + (2m + 2) = 6m + 7.$$

Since S is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, $\tilde{S}$ is a smooth projective surface birational to $\mathbb{P}^1 \times \mathbb{P}^1$. It is known that $\mathbb{P}^1 \times \mathbb{P}^1$ is a minimal smooth projective surface, namely, it is not a blow-up of any smooth projective surface (see e.g. [3] and [4]). Hence, we can blow down $\tilde{S}$ over $\mathbb{P}^1$ some number of times so that it becomes a fiber bundle $\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{P}^1$.

Let N be such that $\tilde{S}$ is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by N one-point blow-ups. Then

$$\chi(\tilde{S}) = (\chi(\mathbb{P}^1 \times \mathbb{P}^1) - N) + N \cdot \chi(\mathbb{P}^1) = 4 + N.$$

Hence, $N = \chi(\tilde{S}) - 4 = 6m + 3$. This proves Theorem 1.

4. Blow-ups at singular points

In this section, we prove Proposition 3.4 and therefore complete the proof of Theorem 1. We will show that each of the singular points s_1 and $s_3^{(k)}$ of the projective model S requires only one blow-up to resolve. Namely, the blow-up of S at $p = s_1$ (or $p = s_3^{(k)}$) is smooth everywhere except at the preimages of the singular points $q \neq p$ of S. The proofs for s_2 and $s_4^{(k)}$ are similar.

Recall that the defining equation for S in $\mathbb{P}^2 \times \mathbb{P}^1 = \{([x : y : u], [z : w])\}$ is

$$F = (xyw - 2u^2z)w^{2m} - (x^2w^2 + y^2w^2 + u^2z^2 - xyzw - 4u^2w^2)T_mT_{m-1},$$

where $T_k = T_k(z, w) = w^k S_k(\frac{z}{w})$.

4.1. Singular point s_1. To perform the blow-up of S at $s_1 = ([0 : 1 : 0], [1 : 0])$, we consider the affine open set A_1' such that $y \neq 0$ and $z \neq 0$. Since A_1' contains the singular points $s_3^{(k)}$ and $s_4^{(k)}$ where $1 \leq k \leq m$, we actually look at the blow-up of S at s_1 in the affine open set $A_1 = A_1' \setminus \bigcup_{1 \leq k \leq m}\{s_3^{(k)}, s_4^{(k)}\}$. The local affine coordinates for $A_1 \cong \mathbb{A}^3$ are x, u, w. So to blow up S at s_1, we blow up $X_1 = \mathcal{Z}(F|_{y=1,z=1})$ at the point $(x, u, w) = (0, 0, 0)$ in A_1. Using coordinates a, b, c for $\mathbb{P}^2$, the blow-up Y_1 of X_1 at $(0, 0, 0)$ is the closed subset in $A_1 \times \mathbb{P}^2$ defined as the zero set of the following polynomials:

$$\begin{aligned}
F_1 &= F|_{y=1,z=1} \\
&= (xw - 2u^2)w^{2m} - (x^2w^2 + w^2 + u^2 - xw - 4u^2w^2)T_m(1, w)T_{m-1}(1, w), \\
e_1 &= xb - ua, \\
e_2 &= xc - wa, \\
e_3 &= wb - uc.
\end{aligned}$$

We will determine the local model of Y_1 and check for smoothness by looking at Y_1 in the affine open sets defined by $a \neq 0$, $b \neq 0$, and $c \neq 0$.

Let $D(w) = T_m(1, w)T_{m-1}(1, w)$. Note that $D(0) = 1$ (by Lemma 3.1(1)).

4.1.1. $a \neq 0$. First we look at Y_1 in the affine open set defined by $a \neq 0$ (we can assume $a = 1$). In this open set, the defining equations for Y_1 become

$$\begin{aligned}
F_1 &= (xw - 2u^2)w^{2m} - (x^2w^2 + w^2 + u^2 - xw - 4u^2w^2)D(w), \\
e_1 &= xb - u, \\
e_2 &= xc - w, \\
e_3 &= wb - uc.
\end{aligned}$$

From equations $e_1 = 0$ and $e_2 = 0$, we have $u = xb$ and $w = xc$. By replacing u with xb and w with xc in F_1, we obtain

$$F_1 = x^2\left[(c-2b^2)(xc)^{2m} - (x^2c^2+c^2+b^2-c-4x^2b^2c^2)D(xc)\right].$$

The first factor corresponds to the exceptional plane E_1 and the other factor is the defining equation for the local model of Y_1. Note that the preimage of s_1 is exactly the intersection of E_1 and Y_1 which is equal to the smooth conic $c^2+b^2-c=0$. This local model of Y_1 is smooth in $A_1 \times \mathbb{P}^2$ if we can show that

$$R(b,c,x) := (c-2b^2)(xc)^{2m} - (x^2c^2+c^2+b^2-c-4x^2b^2c^2)D(xc)$$

is smooth. We now prove that the system $R = R_b = R_c = R_x = 0$ has no solutions.

By direct calculations, we have

$$\begin{aligned}
R_b &= -2b\left(2x^{2m}c^{2m} + (1-4x^2c^2)D\right),\\
R_c &= (xc)^{2m} + 2m(c-2b^2)x^{2m}c^{2m-1} - (2x^2c+2c-1-8x^2b^2c)D\\
&\quad - (x^2c^2+c^2+b^2-c-4x^2b^2c^2)xD_w,\\
R_x &= 2m(c-2b^2)x^{2m-1}c^{2m} - (2c^2x-8xb^2c^2)D\\
&\quad -(x^2c^2+c^2+b^2-c-4x^2b^2c^2)cD_w.
\end{aligned}$$

Note that

$$\begin{aligned}
R - bR_b/2 &= c\left(x^{2m}c^{2m} - (x^2c+c-1)D\right),\\
xR_x - cR_c &= c\left(-x^{2m}c^{2m} + (2c-1)D\right).
\end{aligned}$$

Assume that $R = R_b = R_c = R_x = 0$ at some point (b,c,x). We will consider the two cases $b = 0$ and $b \neq 0$ separately.

Suppose $b = 0$. We claim that $xc \neq 0$. Indeed, if $c = 0$ then $R_c = D(0) = 1 \neq 0$. If $c \neq 0$ and $x = 0$, then $R - bR_b/2 = 0$ implies that $(c-1)D(0) = 1$. So $c = 1$ and $R_c = -D(0) = -1 \neq 0$. Hence, $xc \neq 0$. From $R - bR_b/2 = 0$ and $xR_x - cR_c = 0$, we have $x^{2m}c^{2m} - (x^2c+c-1)D = 0$ and $-x^{2m}c^{2m} + (2c-1)D = 0$. So $x^2c + c - 1 = 2c - 1$, i.e. $x = \pm 1$. Then $D = \frac{x^{2m}c^{2m}}{2c-1} = \frac{w^{2m}}{\pm 2w-1}$. Since $D = T_m(1,w)T_{m-1}(w) = w^{2m-1}S_m(\frac{1}{w})S_{m-1}(\frac{1}{w})$, we obtain $S_m(\frac{1}{w})S_{m-1}(\frac{1}{w}) = \frac{w}{\pm 2w-1}$. This is equivalent to $(\pm 2 - \frac{1}{w})S_m(\frac{1}{w})S_{m-1}(\frac{1}{w}) = 1$, i.e. $(S_m(\frac{1}{w}) \mp S_{m-1}(\frac{1}{w}))^2 = 0$ (by Lemma 2.3). Hence,

$$\begin{aligned}
([x : y : u], [z : w]) &= ([x : 1 : u], [1 : w])\\
&= ([\pm 1 : 1 : 0], [1 : w])\\
&= ([1 : \pm 1 : 0], [\frac{1}{w} : 1]),
\end{aligned}$$

which is equal to either $s_3^{(k)}$ or $s_4^{(k)}$. This point is not in A_1, since it has already been removed from A_1.

Suppose $b \neq 0$. Then $R_b = 0$ implies that $2x^{2m}c^{2m} + (1-4x^2c^2)D = 0$. Note that $xc \neq 0$. (Otherwise $2x^{2m}c^{2m} + (1-4x^2c^2)D = D(0) = 1 \neq 0$.) From

$R - bR_b/2 = 0$ and $xR_x - cR_c = 0$, we also have $x^{2m}c^{2m} - (x^2c + c - 1)D = 0$ and $-x^{2m}c^{2m} + (2c-1)D = 0$. This implies that $x^2c + c - 1 = 2c - 1 = \frac{1}{2}(4x^2c^2 - 1)$. Hence, $x^2 = 1$ and $2c - 1 = \frac{1}{2}(4c^2-1)$, so $c = 1/2$. But then $2x^{2m}c^{2m} + (1 - 4x^2c^2)D = 2x^{2m}c^{2m} \neq 0$, a contradiction.

4.1.2. $b \neq 0$. Now we look at Y_1 in the affine open set defined by $b \neq 0$ (we can assume $b = 1$). In this open set, the defining equations for Y_1 become

$$\begin{aligned} F_1 &= (xw - 2u^2)w^{2m} - (x^2w^2 + w^2 + u^2 - xw - 4u^2w^2)D(w), \\ e_1 &= x - ua, \\ e_2 &= xc - wa, \\ e_3 &= w - uc. \end{aligned}$$

From equations $e_1 = 0$ and $e_3 = 0$, we have $x = ua$ and $w = uc$. By replacing x with ua and w with uc in F_1, we obtain

$$F_1 = u^2\left[(ac-2)(uc)^{2m} - (a^2c^2u^2 + c^2 + 1 - ac - 4u^2c^2)D(uc)\right].$$

The first factor corresponds to the exceptional plane E_1 and the other factor is the defining equation for the local model of Y_1. Note that the preimage of s_1 is exactly the intersection of E_1 and Y_1 which is equal to the smooth conic $c^2 + 1 - ac = 0$. This local model of Y_1 is smooth in $A_1 \times \mathbb{P}^2$ if we can show that

$$R(a,c,u) : = (ac-2)(uc)^{2m} - (a^2c^2u^2 + c^2 + 1 - ac - 4u^2c^2)D(uc)$$

is smooth. We now prove that the system $R = R_a = R_c = R_u = 0$ has no solutions.

By direct calculations, we have

$$\begin{aligned} R_a &= c\left(u^{2m}c^{2m} - (2au^2c - 1)D\right), \\ R_c &= a(uc)^{2m} + 2m(ac-2)u^{2m}c^{2m-1} - (2a^2cu^2 + 2c - a - 8u^2c)D \\ &\quad - (a^2c^2u^2 + c^2 + 1 - ac - 4u^2c^2)uD_w, \\ R_u &= 2m(ac-2)u^{2m-1}c^{2m} - (2a^2c^2u - 8uc^2)D \\ &\quad -(a^2c^2u^2 + c^2 + 1 - ac - 4u^2c^2)cD_w. \end{aligned}$$

Note that

$$uR_u - cR_c = c\left(-au^{2m}c^{2m} + (2c-a)D\right).$$

Assume that $R = R_a = R_c = R_u = 0$ at some point (a,c,u). If $c = 0$, then $R = -D(0) = -1 \neq 0$, a contradiction. Hence, $c \neq 0$. Then $R_a = 0$ implies that $u^{2m}c^{2m} - (2au^2c - 1)D = 0$. Note that $u \neq 0$. (Otherwise $u^{2m}c^{2m} - (2au^2c - 1)D = D(0) = 1 \neq 0$.) Hence, $2au^2c - 1 \neq 0$ and $D = \frac{u^{2m}c^{2m}}{2au^2c-1}$. From $uR_u - cR_c = 0$, we get $-au^{2m}c^{2m} + (2c-a)\frac{u^{2m}c^{2m}}{2au^2c-1} = 0$. This implies that $-a + \frac{2c-a}{2au^2c-1} = 0$, i.e. $a^2u^2 = 1$.

Similarly, from $R = (ac-2)(uc)^{2m} - (a^2c^2u^2 + c^2 + 1 - ac - 4u^2c^2)\frac{u^{2m}c^{2m}}{2au^2c-1} = 0$ we have $ac - 2 - \frac{a^2c^2u^2 + c^2 + 1 - ac - 4u^2c^2}{2au^2c-1} = 0$. Since $u^2 = 1/a^2$, we obtain $ac - 2 -$

$\frac{2c^2+1-ac-4c^2/a^2}{2c/a-1} = 0$. This is equivalent to $(\frac{2c}{a}-1)^2 = 0$, i.e. $2c = a$. But then $2au^2c - 1 = a^2u^2 - 1 = 0$, a contradiction.

4.1.3. $c \neq 0$. Finally we look at Y_1 in the affine open set defined by $c \neq 0$ (we can assume $c = 1$). In this open set, the defining equations for Y_1 become

$$\begin{aligned} F_1 &= (xw - 2u^2)w^{2m} - (x^2w^2 + w^2 + u^2 - xw - 4u^2w^2)D(w), \\ e_1 &= xb - ua, \\ e_2 &= x - wa, \\ e_3 &= wb - u. \end{aligned}$$

From equations $e_2 = 0$ and $e_2 = 0$, we have $x = wa$ and $u = wb$. By replacing x with wa and u with wb in F_1, we obtain

$$F_1 = w^2\left[(a-2b^2)w^{2m} - (a^2w^2 + 1 + b^2 - a - 4b^2w^2)D(w)\right].$$

The first factor corresponds to the exceptional plane E_1 and the other factor is the defining equation for the local model of Y_1. Note that the preimage of s_1 is exactly the intersection of E_1 and Y_1 which is equal to the smooth conic $1 + b^2 - a = 0$. This local model of Y_1 is smooth in $A_1 \times \mathbb{P}^2$ if we can show that

$$R(a,b,w) :\ = (a-2b^2)w^{2m} - (a^2w^2 + 1 + b^2 - a - 4b^2w^2)D(w),$$

is smooth. We now prove that the system $R = R_a = R_b = R_w = 0$ has no solutions.

By direct calculations, we have

$$\begin{aligned} R_a &= w^{2m} - (2aw^2 - 1)D, \\ R_b &= -2b\left(2w^{2m} + (1-4w^2)D\right), \\ R_w &= 2m(a-2b^2)w^{2m-1} - (2a^2w - 8b^2w)D \\ &\quad -(a^2w^2 + 1 + b^2 - a - 4b^2w^2)D_w. \end{aligned}$$

Note that

$$R - (a-2b^2)R_a = (a^2w^2 - 1 + b^2 + 4b^2w^2 - 4ab^2w^2)D.$$

Assume that $R = R_a = R_b = R_w = 0$ at some point (a, b, w). We will consider the two cases $b = 0$ and $b \neq 0$ separately.

Suppose $b = 0$. Then $R - (a-2b^2)R_a = 0$ implies that $(a^2w^2 - 1)D = 0$. If $D = 0$, then from $R_a = 0$ we have $w = 0$. This implies that $D = D(0) = 1 \neq 0$, a contradiction. Hence, $a^2w^2 - 1 = 0$, i.e. $a = \pm 1/w$. From $R_a = 0$, we have $D = \frac{w^{2m}}{\pm 2w - 1}$. This is equivalent to $(S_m(\frac{1}{w}) \mp S_{m-1}(\frac{1}{w}))^2 = 0$. Hence,

$$\begin{aligned} ([x:\ y:\ u],[z:\ w]) &= ([aw:\ 1:\ bw],[1:\ w]) \\ &= ([\pm 1:\ 1:\ 0],[1:\ w]) \\ &= ([1:\ \pm 1:\ 0],[\frac{1}{w}:\ 1]), \end{aligned}$$

which corresponds to either $s_3^{(k)}$ or $s_4^{(k)}$. This point is not in A_1, since it has already been removed from A_1.

Suppose $b \neq 0$. From $R_b = 0$, we have $2w^{2m} + (1-4w^2)D = 0$. This implies that $w \neq 0$ (otherwise $2w^{2m} + (1-4w^2)D = D(0) = 1 \neq 0$), so $4w^2 - 1 \neq 0$ and $D = \frac{2w^{2m}}{4w^2-1} \neq 0$. Then $R_a = 0$ becomes $1 - \frac{2(2aw^2-1)}{4w^2-1} = 0$, which means that $a = 1 + \frac{1}{4w^2}$. From $R - (a-2b^2)R_a = 0$ and $D \neq 0$, we have $a^2w^2 - 1 + b^2 + 4b^2w^2 - 4ab^2w^2 = 0$. But $b^2 + 4b^2w^2 - 4ab^2w^2 = b^2(1 + 4w^2 - 4aw^2) = 0$, so $a^2w^2 - 1 = 0$. Hence, $a = 1 + \frac{1}{4w^2} = 1 + \frac{a^2}{4}$, i.e. $a = 2$. This implies that $4w^2 - 1 = 0$, which contradicts $4w^2 - 1 \neq 0$.

4.1.4. Conclusion. From the cases $a \neq 0$, $b \neq 0$, and $c \neq 0$ considered above, we conclude that the singular point s_1 requires only one blow-up to resolve.

4.2. Singular points $s_3^{(k)}$. To perform the blow-up of S at

$$s_3^{(k)} = (1 : 1 : 0, z_3^{(k)} : 1),$$

we consider the affine open set A_3' such that $x \neq 0$ and $z \neq 0$. Since A_3' contains all other singularities except s_1, we actually look at the blow-up of S at s_1 in the affine open set $A_3 = A_3' \setminus (S_{\text{sing}} \setminus \{s_1, s_3^{(k)}\})$. The local affine coordinates for $A_3 \cong \mathbb{A}^3$ are y, u, w. So to blow up S at $s_3^{(k)}$, we blow up $X_3 = \mathcal{Z}(F|_{x=1,\, z=z_3^{(k)}})$ at the point $(y, u, w) = (1, 0, 1)$ in A_3. For short, we write z_0 for $z_3^{(k)}$. Note that $S_m(z_0) - S_{m-1}(z_0) = 0$. Using coordinates a, b, c for $\mathbb{P}^2$, the blow-up Y_3 of X_3 at $(1, 0, 1)$ is the closed subset in $A_3 \times \mathbb{P}^2$ defined as the zero set of the following polynomials:

$$\begin{aligned}
F_3 &= F|_{x=1,\, z=z_0} \\
&= (yw - 2u^2 z_0)w^{2m} + (yz_0w + 4u^2w^2 - w^2 - y^2w^2 - u^2z_0^2)P(w), \\
e_1 &= ua - (y-1)b, \\
e_2 &= (w-1)a - (y-1)c, \\
e_3 &= (w-1)b - uc,
\end{aligned}$$

where $P(w) = T_m(z_0, w)T_{m-1}(z_0, w)$. Note that $P(0) = z_0^{2m-1}$ (by Lemma 3.1(1)).

We will determine the local model of Y_3 and check for smoothness by looking at Y_3 in the affine open sets defined by $a \neq 0$, $b \neq 0$, and $c \neq 0$.

By Lemma 3.1(3), we have $w^{2m} + (z-2w)T_mT_{m-1} = (T_m - w\,T_{m-1})^2$. Hence,

$$\begin{aligned}
F_3 &= yw\left(w^{2m} + (z_0 - 2w)P\right) - 2u^2z_0w^{2m} + (4u^2w^2 - (y-1)^2w^2 - u^2z_0^2)P \\
&= yw(T_m(z_0, w) - T_{m-1}(z_0, w))^2 - 2u^2z_0w^{2m} + (4u^2w^2 - (y-1)^2w^2 - u^2z_0^2)P.
\end{aligned}$$

Let

$$Q = Q(w) = \frac{T_m(z_0, w) - wT_{m-1}(z_0, w)}{w-1}.$$

Note that $Q \in \mathbb{C}[w]$, since $T_m(z_0,1) - T_{m-1}(z_0,1) = S_m(z_0) - S_{m-1}(z_0) = 0$. Then

$$F_3 = yw(w-1)^2Q^2 - 2u^2z_0w^{2m} + (4u^2w^2 - (y-1)^2w^2 - u^2z_0^2)P.$$

Lemma 4.1. *We have* $S_m^2(z_0) = \frac{1}{2-z_0}$ *and*

$$Q(1) = -\frac{(2m+1)z_0}{z_0+2}S_m(z_0).$$

Proof. Since $S_m^2(z_0) + S_{m-1}^2(z_0) - z_0S_m(z_0)S_{m-1}(z_0) = 1$ (by Lemma 2.3) and $S_m(z_0) - S_{m-1}(z_0) = 0$, we get $S_m^2 = \frac{1}{2-z_0}$. By L'Hospital rule, we have

$$\begin{aligned} Q(1) &= w^m\frac{S_m(\frac{z_0}{w}) - S_{m-1}(\frac{z_0}{w})}{w-1}|_{w=1} \\ &= \frac{-z_0}{w^2}(S'_m(\frac{z_0}{w}) - S'_{m-1}(\frac{z_0}{w}))|_{w=1} \\ &= -z_0(S'_m(z_0) - S'_{m-1}(z_0)). \end{aligned}$$

Since $S_m(z_0) = S_{m-1}(z_0)$, we have $S_{m+1}(z) = (z_0-1)S_m(z_0)$ and $S_{m-2}(z) = (z_0-1)S_m(z_0)$. Lemma 2.5 then implies that

$$\begin{aligned} S'_m(z_0) &= \frac{mS_{m+1}(z_0) - (m+2)S_{m-1}(z_0)}{z_0^2-4} \\ &= \frac{m(z_0-1)-(m+2)}{z_0^2-4}S_m(z_0), \\ S'_{m-1}(z_0) &= \frac{(m-1)S_m(z_0) - (m+1)S_{m-2}(z_0)}{z_0^2-4} \\ &= \frac{m-1-(m+1)(z_0-1)}{z_0^2-4}S_m(z_0). \end{aligned}$$

Hence, $Q(1) = -z_0(S'_m(z_0) - S'_{m-1}(z_0)) = -\frac{(2m+1)z_0}{z_0+2}S_m(z_0)$. □

4.2.1. $a \neq 0$. First we look at Y_3 in the affine open set defined by $a \neq 0$ (we can assume $a = 1$). In this open set, the defining equations for Y_3 become

$$\begin{aligned} F_3 &= (yw - 2u^2z_0)w^{2m} + (yz_0w + 4u^2w^2 - w^2 - y^2w^2 - u^2z_0^2)P(w), \\ e_1 &= u - (y-1)b, \\ e_2 &= (w-1) - (y-1)c, \\ e_3 &= (w-1)b - uc. \end{aligned}$$

From equations $e_1 = 0$ and $e_2 = 0$, we have $u = (y-1)b$ and $w = (y-1)c+1$. By replacing u with $(y-1)b$ and w with $(y-1)c+1$ in F_3, we obtain

$$\begin{aligned} F_3 &= yw(w-1)^2Q^2 - 2u^2z_0w^{2m} + (4u^2w^2 - (y-1)^2w^2 - u^2z_0^2)P \\ &= (y-1)^2\left[ywc^2Q^2 - 2b^2z_0w^{2m} + (4b^2w^2 - w^2 - b^2z_0^2)P\right]. \end{aligned}$$

Let

$$R(b,c,y) = ywc^2Q^2 - 2b^2z_0w^{2m} + (4b^2w^2 - w^2 - b^2z_0^2)P,$$

where $w = (y-1)c+1$. Then

$$\begin{aligned} R|_{y=1} &= c^2Q^2(1) - 2b^2z_0 + (4b^2 - 1 - b^2z_0^2)P(1) \\ &= c^2\frac{(2m+1)^2z_0^2}{(z_0+2)^2}S_m^2(z_0) - 2b^2z_0 + (4b^2-1-b^2z_0^2)S_m(z_0)S_{m-1}(z_0) \\ &= \frac{1}{2-z_0}\left(c^2\frac{(2m+1)^2z_0^2}{(z_0+2)^2} - 2b^2z_0(2-z_0) + (4b^2-1-b^2z_0^2)\right) \\ &= \frac{1}{2-z_0}\left(c^2\frac{(2m+1)^2z_0^2}{(z_0+2)^2} + b^2(z_0-2)^2 - 1\right). \end{aligned}$$

We have $F_3 = (y-1)^2R$. The first factor corresponds to the exceptional plane E_3 and the other factor is the defining equation for the local model of Y_3. Note that the preimage of $s_3^{(k)}$ is exactly the intersection of E_3 and Y_3 which is equal to the smooth conic $c^2\frac{(2m+1)^2z_0^2}{(z_0+2)^2} + b^2(z_0-2)^2 - 1 = 0$. This local model of Y_3 is smooth in $A_3 \times \mathbb{P}^2$ if we can show that $R(b,c,y)$ is smooth.

We now prove that the system $R = R_b = R_c = R_y = 0$ has no solutions. By direct calculations, we have

$$\begin{aligned} R_b &= 2b\left(-2z_0w^{2m} + (4w^2 - z_0^2)P\right), \\ R_c &= y(y-1)c^2Q^2 + 2ywcQ^2 + ywc^2(y-1)(Q^2)_w - 4mb^2z_0(y-1)w^{2m-1} \\ &\quad + (8b^2w - 2w)(y-1)P + (4b^2w^2 - w^2 - b^2z_0^2)(y-1)P_w, \\ R_y &= wc^2Q^2 + yc^3Q^2 + ywc^3(Q^2)_w - 4mb^2z_0cw^{2m-1} \\ &\quad + (8b^2w - 2w)cP + (4b^2w^2 - w^2 - b^2z_0^2)cP_w. \end{aligned}$$

Note that

$$\begin{aligned} R - bR_b/2 &= w(yc^2Q^2 - wP), \\ cR_c - (y-1)R_y &= (y+1)wc^2Q^2. \end{aligned}$$

Assume that $R = R_b = R_c = R_y = 0$ at some point (b,c,y). We first claim that $w \neq 0$. Indeed, if $w = 0$ then $R = 0$ implies that $-b^2z_0^2P(0) = 0$. Since $P(0) = z_0^{2m-1} \neq 0$, we get $b = 0$. Then $R_y = 0$ implies that $yc^3Q^2(0) = 0$. Note that $c \neq 0$ (since $w = (y-1)c+1 = 0$) and $Q(0) = T_m^2(z_0,0) = z_0^{2m} \neq 0$. Hence, $y = 0$. Then $([x:y:u],[z:w]) = ([1:0:0],[z_0:0]) = s_2$ which has been removed from A_3. This proves that $w \neq 0$.

Now $cR_c - (y-1)R_y = 0$ implies $y = -1$ or $c^2Q^2 = 0$. If $c^2Q^2 = 0$ then $w^{2m} + (z_0 - 2w)P = (y-1)^2c^2Q^2 = 0$, which implies that $P \neq 0$. Then $R - bR_b/2 = -w^2P \neq 0$, a contradiction. Hence, $y = -1$.

Since $w^{2m} + (z_0 - 2w)P = (w-1)^2Q^2 = (y-1)^2c^2Q^2 = 4c^2Q^2$, we have $c^2Q^2 = \frac{w^{2m}+(z_0-2w)P}{4}$. From $R - bR_b/2 = 0$, we get $-\frac{w^{2m}+(z_0-2w)P}{4} - wP = 0$, which implies that $w^{2m} + (z_0 + 2w)P = 0$. By Lemma 3.1(3), this is equivalent to $T_m(z_0, w) + wT_{m-1}(z_0, w) = 0$, i.e. $S_m(\frac{z_0}{w}) + S_{m-1}(\frac{z_0}{w}) = 0$. So

$$([x : y : u], [z : w]) = ([1 : -1 : 0], [z_0 : w]) = ([1 : -1 : 0], [\frac{z_0}{w} : 1]) = s_4^{(l)}$$

which has been removed from A_3.

4.2.2. $b \neq 0$. Now we look at Y_3 in the affine open set defined by $b \neq 0$ (we can assume $b = 1$). In this open set, the defining equations for Y_3 become

$$\begin{aligned}
F_3 &= (yw - 2u^2z_0)w^{2m} + (yz_0w + 4u^2w^2 - w^2 - y^2w^2 - u^2z_0^2)P(w),\\
e_1 &= ua - (y-1),\\
e_2 &= (w-1)a - (y-1)c,\\
e_3 &= (w-1) - uc.
\end{aligned}$$

From equations $e_1 = 0$ and $e_3 = 0$, we have $y = au + 1$ and $w = uc + 1$. By replacing y with $au + 1$ and w with $uc + 1$ in F_3, we obtain

$$\begin{aligned}
F_3 &= yw(w-1)^2Q^2 - 2u^2z_0w^{2m} + \left(4u^2w^2 - (y-1)^2w^2 - u^2z_0^2\right)P\\
&= u^2\left[(au+1)wc^2Q^2 - 2z_0w^{2m} + (4w^2 - a^2w^2 - z_0^2)P\right].
\end{aligned}$$

Let

$$R(a, c, u) = (au + 1)wc^2Q^2(w) - 2z_0w^{2m} + (4w^2 - a^2w^2 - z_0^2)P(w),$$

where $w = uc + 1$. Then

$$\begin{aligned}
R|_{u=0} &= c^2Q^2(1) - 2z_0 + (4 - a^2 - z_0^2)P(1),\\
&= c^2\frac{(2m+1)^2z_0^2}{(z_0+2)^2}S_m^2(z_0) - 2z_0 + (4 - a^2 - z_0^2)S_m(z_0)S_{m-1}(z_0)\\
&= \frac{1}{2 - z_0}\left(c^2\frac{(2m+1)^2z_0^2}{(z_0+2)^2} - 2z_0(2 - z_0) + (4 - a^2 - z_0^2)\right)\\
&= \frac{1}{2 - z_0}\left(c^2\frac{(2m+1)^2z_0^2}{(z_0+2)^2} - a^2 + (z_0 - 2)^2\right).
\end{aligned}$$

We have $F_3 = u^2R$. The first factor corresponds to the exceptional plane E_3 and the other factor is the defining equation for the local model of Y_3. Note that the preimage of $s_3^{(k)}$ is exactly the intersection of E_3 and Y_3 which is equal to the smooth conic $c^2\frac{(2m+1)^2z_0^2}{(z_0+2)^2} - a^2 + (z_0 - 2)^2 = 0$. This local model of Y_3 is smooth in $A_3 \times \mathbb{P}^2$ if we can show that $R(a, c, u)$ is smooth.

We now prove that the system $R = R_a = R_c = R_u = 0$ has no solutions. By direct calculations, we have

$$\begin{aligned}
R_a &= w(uc^2Q^2 - 2awP),\\
R_c &= (au+1)uc^2Q^2 + 2(au+1)wcQ^2 + (au+1)wc^2u(Q^2)_w - 4mz_0uw^{2m-1}\\
&\quad + 2(4-a^2)uwP + (4w^2 - a^2w^2 - z_0^2)uP_w,\\
R_u &= awc^2Q^2 + (au+1)c^3Q^2 + (au+1)wc^3(Q^2)_w - 4mz_0cw^{2m-1}\\
&\quad + 2(4-a^2)cwP + (4w^2 - a^2w^2 - z_0^2)cP_w.
\end{aligned}$$

Note that

$$\begin{aligned}
R - aR_a/2 &= (au/2+1)wc^2Q^2 - 2z_0w^{2m} + (4w^2 - z_0^2)P,\\
cR_c - uR_u &= (au+2)wc^2Q^2.
\end{aligned}$$

We first claim that $w \neq 0$. Indeed, if $w = 0$ then $R = 0$ implies that $-z_0^2P(0) = 0$. But $P(0) = z_0^{2m-1} \neq 0$, a contradiction. Hence, $w \neq 0$.

From $cR_c - uR_u = 0$ and $R - aR_a/2 = 0$, we have $(au+2)wc^2Q^2 = 0$ and $-2z_0w^{2m} + (4w^2 - z_0^2)P = 0$. Since $z_0w^{2m} \neq 0$, we get $4w^2 - z_0^2 \neq 0$ and $P = \frac{2z_0w^{2m}}{4w^2 - z_0^2}$.

If $c^2Q^2 = 0$, then $w^{2m} + (z_0 - 2w)P = (w-1)^2Q^2 = u^2c^2Q^2 = 0$. This implies that $2w - z_0 \neq 0$ and $P = \frac{w^{2m}}{2w - z_0}$. Together with $P = \frac{2z_0w^{2m}}{4w^2 - z_0^2}$, we get $\frac{2z_0}{2w+z_0} = 1$. So $z_0 = 2w$, which contradicts $z_0 - 2w \neq 0$.

If $au + 2 = 0$, then $a = -2/u$. From $R_a = 0$, we have $u^2c^2Q + 4wP = 0$, i.e. $(w-1)^2Q^2 + 4wP = 0$. This is equivalent to $w^{2m} + (z_0 - 2w)P + 4wP = 0$. So $2w + z_0 \neq 0$ and $P = -\frac{w^{2m}}{2w+z_0}$. Together with $P = \frac{2z_0w^{2m}}{4w^2 - z_0^2}$, we get $\frac{2z_0}{2w - z_0} = -1$. So $z_0 = -2w$, which contradicts $2w + z_0 \neq 0$.

4.2.3. $\boldsymbol{c \neq 0}$**.** Finally we look at Y_3 in the affine open set defined by $c \neq 0$ (we can assume $b = 1$). In this open set, the defining equations for Y_3 become

$$\begin{aligned}
F_3 &= (yw - 2u^2z_0)w^{2m} + (yz_0w + 4u^2w^2 - w^2 - y^2w^2 - u^2z_0^2)P(w),\\
e_1 &= ua - (y-1)b,\\
e_2 &= (w-1)a - (y-1),\\
e_3 &= (w-1)b - u.
\end{aligned}$$

From equations $e_2 = 0$ and $e_3 = 0$, we have $y = a(w-1)+1$ and $u = b(w-1)$. By replacing y with $a(w-1)+1$ and u with $b(w-1)$ in F_3, we obtain

$$\begin{aligned}
F_3 &= yw(w-1)^2Q^2 - 2u^2z_0w^{2m} + \left(4u^2w^2 - (y-1)^2w^2 - u^2z_0^2\right)P\\
&= (w-1)^2\left[(a(w-1)+1)wQ^2 - 2b^2z_0w^{2m} + (4b^2w^2 - a^2w^2 - b^2z_0^2)P\right].
\end{aligned}$$

Let

$$R(a,b,w) = (a(w-1)+1)wQ^2(w) - 2b^2z_0w^{2m} + (4b^2w^2 - a^2w^2 - b^2z_0^2)P(w).$$

Then

$$\begin{aligned} R|_{w=1} &= Q^2(1) - 2b^2z_0 + (4b^2 - a^2 - b^2z_0^2)P(1), \\ &= \frac{(2m+1)^2z_0^2}{(z_0+2)^2}S_m^2(z_0) - 2b^2z_0 + (4b^2 - a^2 - b^2z_0^2)S_m(z_0)S_{m-1}(z_0) \\ &= \frac{1}{2-z_0}\left(\frac{(2m+1)^2z_0^2}{(z_0+2)^2} - 2b^2z_0(2-z_0) + (4b^2 - a^2 - b^2z_0^2)\right) \\ &= \frac{1}{2-z_0}\left(\frac{(2m+1)^2z_0^2}{(z_0+2)^2} - a^2 + b^2(z_0-2)^2\right). \end{aligned}$$

We have $F_3 = (w-1)^2R$. The first factor corresponds to the exceptional plane E_3 and the other factor is the defining equation for the local model of Y_3. Note that the preimage of $s_3^{(k)}$ is exactly the intersection of E_3 and Y_3 which is equal to the smooth conic $\frac{(2m+1)^2z_0^2}{(z_0+2)^2} - a^2 + b^2(z_0-2)^2 = 0$. This local model of Y_3 is smooth in $A_3 \times \mathbb{P}^2$ if we can show that $R(a,b,w)$ is smooth.

We now prove that the system $R = R_a = R_b = R_w = 0$ has no solutions. By direct calculations, we have

$$\begin{aligned} R_a &= (w-1)wQ^2 - 2aw^2P, \\ R_b &= 2b\left(-2z_0w^{2m} + (4w^2 - z_0^2)P\right), \\ R_w &= awQ^2 + (a(w-1)+1)Q^2 + (a(w-1)+1)w(Q^2)_w - 4mb^2z_0w^{2m-1} \\ &\quad + 2(4b^2-a^2)wP + (4b^2w^2 - a^2w^2 - b^2z_0^2)P_w. \end{aligned}$$

Note that

$$2R - bR_b - aR_a = (a(w-1)+2)wQ^2.$$

We first claim that $w \neq 0$. Indeed, if $w = 0$ then $R = 0$ implies that $b^2z_0^2P(0) = 0$. Since $z_0 \neq 0$ and $P(0) = 1$, we have $b = 0$. Then $R_w = 0$ becomes $(a(w-1)+1)Q^2 = 0$. Note that $Q(0) = z_0^{2m} \neq 0$, hence $a(w-1)+1 = 0$. Then $([x:y:u],[z:w]) = ([1:0:0],[z_0:0]) = s_2$ which has been removed from A_3. Hence, $w \neq 0$.

From $2R - bR_b - aR_a = 0$, we have $a(w-1)+2$ or $Q = 0$. Similarly, $R_b = 0$ implies that $b = 0$ or $-2z_0w^{2m} + (4w^2 - z_0^2)P = 0$. There are four cases to consider.

<u>Case 1</u>: Suppose $b = 0$ and $Q = 0$. Then $R_a = 0$ implies that $aP = 0$. Note that $P \neq 0$, since $w^{2m} + (z_0 - 2w)P = (w-1)^2Q^2 = 0$. Hence, $a = 0$. From $Q = 0$, we have $T_m(z_0,w) - wT_{m-1}(z_0,w) = 0$, which is equivalent to $S_m(\frac{z_0}{w}) - S_{m-1}(\frac{z_0}{w}) = 0$, so $\frac{z_0}{w} = z_3^{(l)}$ for some l. Note that $Q(1) = \frac{1}{2-z_0}\frac{(2m+1)^2z_0^2}{(z_0+2)^2} \neq 0$, so $w \neq 1$. This implies that $z_3^{(l)} = \frac{z_0}{w} \neq z_3^{(k)}$. Since $([x:y:u],[z:w]) = ([1:1:0],[z_3^{(l)}:1]) = s_3^{(l)}$ has been removed from A_3, we obtain a contradiction.

Case 2: Suppose $b = 0$ and $a(w-1)+2 = 0$. Then $a = -2/(w-1)$ and $y = a(w-1)+1 = -1$. From $R = 0$, we have $(w-1)^2Q^2 + 4wP = 0$, i.e. $w^{2m} + (z_0 - 2w)P + 4wP = 0$. By Lemma 3.1(3), this is equivalent to $S_m(\frac{z_0}{w}) + S_{m-1}(\frac{z_0}{w}) = 0$, so $\frac{z_0}{w} = z_4^{(l)}$ for some l. Then $([x : y : u], [z : w]) = ([1 : -1 : 0], [z_4^{(l)} : 1]) = s_4^{(l)}$ which has been removed from A_3.

Case 3: Suppose $-2z_0w^{2m} + (4w^2 - z_0^2)P = 0$ and $Q = 0$. Then $4w^2 - z_0^2 \neq 0$ and $P = \frac{2z_0w^{2m}}{4w^2-z_0^2}$. From $Q = 0$, we have $w^{2m} + (z_0 - 2w)P = (w-1)^2Q^2 = 0$. Hence, $1 + (z_0 - 2w)\frac{2z_0}{4w^2-z_0^2} = 0$, i.e. $1 - \frac{2z_0}{z_0+2w} = 0$. This implies that $z_0 = 2w$, which contradicts $4w^2 - z_0^2 \neq 0$.

Case 4: Suppose $-2z_0w^{2m} + (4w^2 - z_0^2)P = 0$ and $a(w-1)+2 = 0$. From $R_a = 0$, we have $(w-1)^2Q^2 + 4wP = 0$, which is equivalent to $w^{2m} + (z_0 - 2w)P + 4wP = 0$. So $1 + (z_0 + 2w)\frac{2z_0}{4w^2-z_0^2} = 0$, i.e. $1 - \frac{2z_0}{z_0-2w} = 0$. This implies that $z_0 = -2w$, which contradicts $4w^2 - z_0^2 \neq 0$.

4.2.4. Conclusion. From the cases $a \neq 0$, $b \neq 0$, and $c \neq 0$ considered above, we conclude that the singular point $s_3^{(k)}$ requires only one blow-up to resolve.

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