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Polynomials with integral Mahler measures

Artūras Dubickas

Abstract. For each $m \in \mathbb{N}$ and each sufficiently large $d \in \mathbb{N}$, we give an upper bound for the number of integer polynomials of degree d and Mahler's measure m. We show that there are at most $\exp\left(11(md)^{2/3}\left(\log(md)\right)^{4/3}\right)$ of such polynomials. For 'small' m, i.e. $m < d^{1/2-\varepsilon}$, this estimate is better than the estimate $m^{d(1+\varepsilon)}$ that comes from a corresponding upper bound on the number of integer polynomials of degree d and Mahler's measure at most m. By the results of Zaitseva and Protasov, our estimate has applications in the theory of self-affine 2-attractors. We also show that for each integer $m \geq 3$ there is a constant $c = c(m) > 0$ such that the number of monic integer irreducible expanding polynomials of sufficiently degree d and constant coefficient m (and hence with Mahler's measure equal to m) is at least cd^{m-1}.

Contents

1. Introduction

For a degree d polynomial

$$f(x) = a_d x^d + \cdots + a_1 x + a_0 = a_d(x - \alpha_1) \dots (x - \alpha_d) \in \mathbb{C}[x], \rangle a_d \neq 0,$$

we define its *Mahler measure* by

$$M(f) = |a_d| \prod_{j=1}^{d} \max\{1, |\alpha_j|\}.$$

The Mahler measure is multiplicative, namely,

$$M(fg) = M(f)M(g) \tag{1}$$

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for any $f, g \in \mathbb{C}[x]$, and satisfies

$$M(f) = M(f^*), \tag{2}$$

where $f^*(x) = x^d f(1/x)$ for $f \in \mathbb{C}[x]$ of degree d. Throughout, we say that the polynomial f^* defined as above is *reciprocal* to the polynomial f of degree d. The Mahler measure of an algebraic number $\alpha \in \overline{\mathbb{Q}}$ with minimal polynomial $f \in \mathbb{Z}[x]$ is defined by $M(\alpha) = M(f)$.

In [7], Chern and Vaaler gave an asymptotic formula for the number of integer polynomials of degree at most d and Mahler's measure at most T as $T \to \infty$. It turns out to be asymptotic to $\varkappa_d T^{d+1}$ with some $\varkappa_d > 0$ as $T \to \infty$. The situation is much more complicated when T is bounded and d is large. The case $T = 2$ has been first considered by Mignotte [19]. Later, Mignotte's bound was improved by the author and Konyagin. In [13], it was shown that for any real $T > 1$ the number of integer polynomials of degree at most d and Mahler's measure at most T is bounded above by

$$\min\{T^{(1+\varepsilon)d}, T^{d+1}\exp(d^2/2)\} \tag{3}$$

for any $\varepsilon > 0$ and any sufficiently large d. (Throughout the paper, $\exp(x)$ stands for e^x.) For $T = 2$, this gives the upper bound $2^{(1+\varepsilon)d}$. On the other hand, the best available lower bound for the number of monic integer irreducible polynomials of degree at most d and of Mahler's measure less than 2 is only $\varkappa d^5$ with some absolute constant $\varkappa > 0$, see [11], [12].

As in [1], we say that a polynomial in $\mathbb{Z}[x]$ (or even in $\mathbb{C}[x]$) whose roots are all in $|z| > 1$ is *expanding*. Expanding polynomials also appear, for instance, in the papers of Akiyama and Zaimi [3], Brunotte [6]. Note that if $f \in \mathbb{C}[x]$ is expanding then

$$M(f) = |f(0)|. \tag{4}$$

In [23], Zaitseva and Protasov considered various questions related to so-called self-affine 2-attractors and reduced one of the problems to estimating the number of monic integer expanding polynomials of degree d with constant term ± 2. They showed that for d sufficiently large there are at least $0.06d^2$ and at most $\exp(0.7d)$ of such polynomials, the upper bound being taken from (3) with $T = 2$. Of course, such polynomials have Mahler's measure not at most 2, but exactly 2. This raises a natural question of finding a better upper bound for the number of degree d integer polynomials with Mahler's measure 2 and, more generally, with Mahler's measure m, where $m \geq 2$ is an integer.

In the case when d is fixed and $m \to \infty$ this problem has already been addressed in [2], [8], [17]. In [2, Theorem 5.2], Akiyama and Pethő proved a result which implies that the number of monic integer irreducible expanding polynomials of degree d with constant term m is asymptotic to $v_d m^{d-1}$ with some $v_d > 0$ as $m \to \infty$. By (4), such polynomials have Mahler's measure equal to m. Similar asymptotical results when the degree d is fixed and Mahler's measure tends to infinity were recently obtained by Dill [8, Section 8].

However, as in the case of the problem of estimating the number of integer polynomials with bounded Mahler's measure which we discussed above, this problem, where Mahler's measure $m \in \mathbb{N}$ of polynomials is fixed and their degree d is large, turns out to be more difficult. In this paper, we will evaluate the number of integer polynomials of degree d and Mahler's measure equal to a positive integer m. Our main result is the following upper bound which improves the bound (3) in case we count only polynomials with Mahler's measure exactly m:

Theorem 1.1. *For each positive integer m and each sufficiently large integer d there are at most*

$$\exp\Big(11(md)^{2/3}\big(\log(md)\big)^{4/3}\Big) \tag{5}$$

integer polynomials of degree d and Mahler's measure m.

We remark that $m = 1$ is the only case when a better result is known. By Kronecker's theorem (see, e.g., [20, Theorem 4.5.4]), integer polynomials with Mahler's measure 1 are products of $\pm x^k$, $k \in \mathbb{N} \cup \{0\}$, and cyclotomic polynomials. The next proposition is the main result of Boyd and Montgomery [5]:

Proposition 1.2. *The number of degree d monic integer polynomials with all roots on $|z| = 1$ is asymptototic to $\frac{c_1}{d\sqrt{\log d}}\exp(c_2\sqrt{d})$ as $d \to \infty$, with $c_1 = \sqrt{105\zeta(3)}/(4\pi^2 e^{\gamma/2})$, where γ is Euler's constant, and $c_2 = \sqrt{105\zeta(3)}/\pi$.*

Proposition 1.2 immediately implies the upper bound of the form $\exp(c_3\sqrt{d})$, where $c_3 > c_2$, on the number of integer polynomials of sufficiently large degree d and Mahler's measure $m = 1$. This is better than (5) gives for $m = 1$. Of course, the example $(x-m)f(x)$, where f runs through all monic degree $d-1$ polynomials in $\mathbb{Z}[x]$ with all roots on $|z| = 1$, shows that the exponent $2/3$ for d in (5) cannot be improved to a constant smaller than $1/2$.

On the other hand, for $m \geq 2$ fixed, and, more generally, for m in the range $2 \leq m < d^{1/2-\varepsilon}$, Theorem 1.1 gives a better bound than that $m^{(1+\varepsilon)d}$ coming from (3). In particular, for $m = 2$, Theorem 1.1 improves the upper bound in [23, Theorem 10]. Since $10.5 \cdot 2^{2/3} < 17$, Theorem 1.1, which we will prove with the better constant 10.5 (instead of 11) in (5) (see (32)), combined with [23, Corollary 6] yields the following:

Corollary 1.3. *The total number of not affinely similar 2-attractors in dimension d is less than $\exp\big(17d^{2/3}(\log d)^{4/3}\big)$ for d sufficiently large.*

We remark that in [23, Theorem 10], the bound corresponding to that of Corollary 1.3 was $\exp(0.7d)$.

It seems very likely that the main contribution in Theorem 1.1 comes from reducible polynomials, while the number of irreducible polynomials of degree d and Mahler's measure m should be much smaller. In the next theorem we will construct many monic integer irreducible polynomials with Mahler's measure $m \in \mathbb{N} \setminus \{1\}$.

Theorem 1.4. *The number of monic integer irreducible expanding polynomials of degree d with constant coefficient 2 is at least c_0d^2, where $c_0 > 0$ is an absolute constant. Furthermore, for each $m \geq 3$ there is a constant $c(m) > 0$ such that for each sufficiently large $d \in \mathbb{N}$ the number of monic integer irreducible expanding polynomials of degree d with constant coefficient m is at least $c(m)d^{m-1}$.*

Note that the gap between the bounds in Theorems 1.1 and 1.4 is large. Since we consider the situation with m small and d large, the bound in Theorem 1.4 is far from that given in the asymptotic formula $v_d m^{d-1}$ as $d \to \infty$ [2] and closer to that in [12]. For $m \geq 3$, the proof of Theorem 1.4 is based on an explicit construction. For $m = 2$, the construction is different and taken from [23]. However, for the sake of completeness, we will give a full proof of Theorem 1.4 in the case $m = 2$ too.

Earlier, somewhat unrelated results on the properties of the Mahler measure have been obtained by the author in [10]. Some of those results were recently extended by Fili, Pottmeyer and Zhang in [14], [15], but now, in the present context, a very useful result seems to be also [10, Theorem 2]. Here, in the same fashion, we will derive a result that completely characterizes all integer polynomials with integral Mahler measure. This will be a useful tool in completing the proof of Theorem 1.1:

Proposition 1.5. *Let m and d be two positive integers and let $f \in \mathbb{Z}[x]$ be a polynomial of degree d with Mahler measure equal to m. Write*

$$f(x) = ax^s \prod_{j=1}^{k} f_j(x), \tag{6}$$

where $a \in \mathbb{Z} \setminus \{0\}$, $s \in \{0, 1, \dots, d\}$ and $f_1, \dots, f_k \in \mathbb{Z}[x]$ are not necessarily distinct irreducible polynomials with positive leading coefficients satisfying $f_j(0) \neq 0$. Then, for each $j = 1, \dots, k$, the polynomial f_j either has all of its roots on $|z| = 1$ or one of the polynomials f_j, f_j^* is expanding.

By (1), (2), (4) and Proposition 1.5, it follows that with its notation we have

$$m = M(f) = |a| \prod_{j=1}^{k} M(f_j) = |a| \prod_{j=1}^{k} m_j, \tag{7}$$

where $m_j = M(f_j) = M(f_j^*) \in \mathbb{N}$. Here, $m_j = 1$ if and only if f_j is cyclotomic.

In the next section we present some auxiliary results. Then, in Section 3 we will prove Theorem 1.4 and Proposition 1.5. Finally, in Section 4 we will prove Theorem 1.1.

2. Auxiliary results

For $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{C}^d$ we put

$$\|\mathbf{x}\| = \max_{1 \leq j \leq d} |x_j|$$

for the l_∞ norm of the vector $\mathbf{x}$. For a convex closed bounded set $A \subset \mathbb{R}^d$ we put

$$F(A) = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in A, \rangle\rangle ||\mathbf{y}|| \le 1/2\} \tag{8}$$

for the 1/2–neighbourhood of the set A. Suppose that $G \subseteq \{1, 2, \dots, d\}$ and $g = |G|$. We denote by $\mathrm{Pr}_G(A)$ the orthogonal projection of the set A to the linear space $\mathbb{R}^g$ spanned by the vectors of $\mathbb{R}^d$ corresponding to the indices of G. Finally, denote by $\mathrm{Vol}\big(\mathrm{Pr}_G(A)\big)$ the volume of the g–dimensional $(1 \le g \le d)$ convex set $\mathrm{Pr}_G(A)$. With this notation we have the following lemma for $d \ge 1$:

Lemma 2.1. *We have*

$$\mathrm{Vol}\big(F(A)\big) = 1 + \sum_G \mathrm{Vol}\big(\mathrm{Pr}_G(A)\big),$$

where the sum is taken over all nonempty subsets G of $\{1, 2, \dots, d\}$.

See [13, Lemma 0] for the proof.

Lemma 2.2. *Let* $V(g, n)$ *be the maximal volume of a convex hull of n points in the parallelepiped* $\prod_{j=1}^{g}[-u_j/2, u_j/2] \subset \mathbb{R}^g$, *where* $u_1, \dots, u_g$ *are positive. Then, for* $n = g^\lambda$, *where* $\lambda > 1$ *and g is sufficiently large, we have*

$$V(g, g^\lambda) < \Big(\frac{23.22(\lambda - 1)\log g}{g}\Big)^{g/2} \prod_{j=1}^{g} u_j.$$

Proof. Let $W(g, n)$ be the maximal volume of a convex hull of n points in the unit ball W_g in $\mathbb{R}^g$. The volume of W_g equals $w_g = \frac{\pi^{g/2}}{\Gamma(g/2+1)}$; see, e.g., [22]. Next, as in [9], [13], we will need a result on the estimate of the volume of a polytope with few vertices in the style of [4], [16], [18]. Specifically, in [4, eq. (4)], it was shown that for $n = g^\lambda$, where $\lambda > 1$, and any $\varepsilon > 0$ the inequality

$$W(g, g^\lambda) < (1 + \varepsilon)^g w_g \Big(\frac{2e(\lambda - 1)\log g}{g}\Big)^{g/2} \tag{9}$$

holds for each sufficiently large g.

As observed in [13], by rescaling, it suffices to prove the inequality of the lemma for the parallepiped

$$P_g = \prod_{j=1}^{g}[-u_j/2, u_j/2] = [-1/\sqrt{g}, 1/\sqrt{g}]^g.$$

Note that P_g is inscribed into the unit ball W_g with center at the origin, hence $V(g, g^\lambda) \le W(g, g^\lambda)$. Furthermore, by Stirling's formula, $\Gamma(g/2+1) > (g/2e)^{g/2}$ for g sufficiently large, so using $u_j = 2/\sqrt{g}$ we obtain

$$w_g = \frac{\pi^{g/2}}{\Gamma(g/2 + 1)} < \Big(\frac{2\pi e}{g}\Big)^{g/2} = \Big(\frac{\pi e}{2}\Big)^{g/2} \prod_{j=1}^{g} u_j.$$

This implies the required result by (9) and $(1+\varepsilon)^2\pi e^2 < 23.22$ with appropriate choice of ε. $\square$

Lemma 2.3. *For every $k \in \mathbb{N}$ and any real numbers $m_1, \dots, m_k \geq \sqrt{2}$ and $d_1, \dots, d_k > 0$ we have*

$$(m_1 d_1)^{2/3} + \cdots + (m_k d_k)^{2/3} \leq \big(m_1 \dots m_k (d_1 + \cdots + d_k)\big)^{2/3}. \tag{10}$$

Proof. The inequality (10) is equality for $k = 1$. It is sufficient to prove (10) for $k = 2$ and then apply induction on k. Dividing both sides of (10) with $k = 2$ by $d_1^{2/3}$ and setting $y = d_2/d_1$ we see that it suffices to show that $m_1^{2/3} + (m_2 y)^{2/3}$ does not exceed $(m_1 m_2 (y+1))^{2/3}$ for $y > 0$.

Let us consider the function

$$\varphi(y) = (m_1 m_2 (y+1))^{2/3} - m_1^{2/3} - (m_2 y)^{2/3}.$$

It is positive at $y = 0$, since $m_2 > 1$. Its derivative

$$\varphi'(y) = \frac{2m_2^{2/3}}{3(y+1)^{1/3}}\Big(m_1^{2/3} - \Big(1 + \frac{1}{y}\Big)^{1/3}\Big)$$

vanishes at $y_0 = 1/(m_1^2 - 1)$. Since $\varphi'(y) < 0$ for $0 < y < y_0$ and $\varphi'(y) > 0$ for $y > y_0$, the minimum of the function $\varphi(y)$ in $[0, \infty)$ is attained at $y = y_0$. Thus, in order to prove that $\varphi(y) \geq 0$ for all $y > 0$ it remains to verify the inequality $\varphi(y_0) \geq 0$.

From

$$\begin{aligned}\varphi(y_0) = \varphi\Big(\frac{1}{m_1^2 - 1}\Big) &= (m_1 m_2)^{2/3}\Big(\frac{m_1^2}{m_1^2 - 1}\Big)^{2/3} - m_1^{2/3} - \frac{m_2^{2/3}}{(m_1^2 - 1)^{2/3}} \\ &= m_2^{2/3}(m_1^2 - 1)^{1/3} - m_1^{2/3}\end{aligned}$$

we see that $\varphi(y_0) \geq 0$ is equivalent to $(m_1^2 - 1)^{1/3} \geq (m_1/m_2)^{2/3}$. The latter inequality is equivalent to $(m_1^2 - 1)m_2^2 \geq m_1^2$, and can be also written as $(m_1^2 - 1)(m_2^2 - 1) \geq 1$, which holds due to $m_1^2 - 1 \geq 1$ and $m_2^2 - 1 \geq 1$. □

Lemma 2.4. *For every integer $m \geq 3$ there is a constant $c(m) > 0$ such that for each sufficiently large integer d there are at least $c(m)d^{m-1}$ irreducible polynomials of the form*

$$x^d - x^{b_{m-1}} - \cdots - x^{b_1} + m,$$

where $b_1, \dots, b_{m-1} \in \mathbb{Z}$ and $0 < b_1 < \cdots < b_{m-1} < d$.

Proof. Fix $m \geq 3$. For each $d > m$ let $S(d, m)$ be the set polynomials of the form $x^d - x^{b_{m-1}} - \cdots - x^{b_1} + m$ with integers $b_1, \dots, b_{m-1}$ satisfying

$$0 < b_1 < \cdots < b_{m-1} < d \tag{11}$$

and

$$\gcd(b_1, \dots, b_{m-1}) = 1. \tag{12}$$

Note that the number of vectors $(b_1, \dots, b_{m-1}) \in \mathbb{Z}^{m-1}$ satisfying (11) is $\binom{d-1}{m-1}$. It is well known that the probability of $m-1 \geq 2$ random integers being coprime is $1/\zeta(m-1)$. Thus, there is a constant $c_1(m) > 0$ such that for each sufficiently large d we have

$$|S(d,m)| \geq c_1(m)d^{m-1}. \tag{13}$$

We claim that every reciprocal monic divisor $g \in \mathbb{Z}[x]$ of $f \in S(d,m)$ must be cyclotomic. Indeed, if not, then, by Kronecker's theorem (see, e.g. [20, Theorem 4.5.4]), at least one root of g must be in $|z| > 1$. Since g is reciprocal, it must have a root α satisfying $|\alpha| < 1$. But then $f(\alpha) = 0$ implies

$$m = \alpha^{b_1} + \cdots + \alpha^{b_{m-1}} - \alpha^d \leq |\alpha^d| + \sum_{j=1}^{m-1} |\alpha^{b_j}| < m,$$

which is impossible.

By Schinzel's result [21, Theorem 4], either the nonreciprocal part of $f \in S(d,m)$ (which is equal to the noncyclotomic part by the above) is irreducible or there is a positive constant $c_2(m)$ (which given explicitly in [21] and for the polynomial of the form $x^d - x^{b_{m-1}} - \cdots - x^{b_1} + m$ depends only on m) and a vector $(\gamma_1, \dots, \gamma_m) \in \mathbb{Z}^m$ such that

$$0 < \max_{1 \leq j \leq m} |\gamma_j| \leq c_2(m)$$

and

$$\gamma_1 b_1 + \cdots + \gamma_{m-1} b_{m-1} + \gamma_m d = 0. \tag{14}$$

It is clear that at least one γ_j, $j = 1, \dots, m-1$, must be nonzero.

For each nonzero vector $(\gamma_1, \dots, \gamma_m) \in \mathbb{Z}^m$ the number of vectors

$$(b_1, \dots, b_{m-1}) \in \mathbb{Z}^{m-1}$$

satisfying both (11) and (14) is less than d^{m-2}. Furthermore, there are at most $(2c_2(m) + 1))^m$ possible vectors $(\gamma_1, \dots, \gamma_m) \in \mathbb{Z}^m$. Hence, by (13), at least

$$c_1(m)d^{m-1} - (2c_2(m) + 1)^m d^{m-2} \tag{15}$$

polynomials from $S(d,m)$ have irreducible noncyclotomic part.

Now, we will show that no polynomial in $S(d,m)$ has a cyclotomic factor. Indeed, $f \in S(d,m)$ has a cyclotomic divisor if and only if there is a root of unity ζ such that $\zeta^d = -1$ and

$$\zeta^{b_{m-1}} = \cdots = \zeta^{b_1} = 1. \tag{16}$$

Then, in view of (12) there are $u_{m-1}, \dots, u_1 \in \mathbb{Z}$ such that

$$b_{m-1}u_{m-1} + \cdots + b_1 u_1 = 1.$$

Thus, if (16) is true, we obtain

$$\zeta = \zeta^{b_{m-1}u_{m-1} + \cdots + b_1 u_1} = 1,$$

which contradicts to $\zeta^d = -1$. Hence, for every $f \in S(d,m)$, the polynomial f itself coincides with its noncyclotomic part, which is shown to be irreducible with at most $(2c_2(m) + 1)^m d^{m-2}$ exceptions. Thus, by (15), we arrive at the required result with the constant, say, $c(m) = c_1(m)/2$. □

3. Proof of Theorem 1.4 and Proposition 1.5

Proof of Theorem 1.4. To prove the theorem for $m \geq 3$ it suffices to show that each polynomial as in Lemma 2.4 is expanding. Indeed, it is clear that it has no roots in $|z| < 1$. If it has a root of unit modulus then it must be reciprocal, which is not the case. This means that all the roots of such polynomial must be in $|z| > 1$, and hence it is expanding.

In all that follows we will prove the theorem for $m = 2$. Let us consider the polynomials of the form

$$f_{a,b,c}(x) = (1 + x^a)(1 + x^b)(1 + x^c) + 1,$$

where $a \geq b \geq c$ are three positive integers satisfying $a + b + c = d$ and, for example,

$$b \not\equiv c \pmod 3). \tag{17}$$

There is an absolute constant $c_4 > 0$ such that, for d large enough, say $d \geq d_0$, there are at least $c_4 d^2$ of such polynomials. (For $d \leq d_0$ there is at least one expanding polynomial $x^d + 2$, so the lower bound $c_5 d^2$ with some other constant $c_5 > 0$ also holds for all $d \in \mathbb{N}$.)

It remains to show that $f_{a,b,c}$ has no root in $|z| \leq 1$. Indeed, then all roots of $f_{a,b,c}$ are in $|z| > 1$ and the modulus of their product is 2, so $f_{a,b,c}$ is monic integer irreducible expanding polynomial.

Suppose $\alpha \in \mathbb{C}$ satisfying $|\alpha| \leq 1$ is a root of $f_{a,b,c}$. The numbers

$$z_a = 1 + \alpha^a = |z_a| e^{i\varphi_a}, \quad z_b = 1 + \alpha^b = |z_b| e^{i\varphi_b}, \quad z_c = 1 + \alpha^c = |z_c| e^{i\varphi_c}$$

all lie in the circle $|z - 1| \leq 1$ and satisfy $z_a z_b z_c = -1$. Therefore, $|z_a z_b z_c| = 1$ and

$$\varphi_a + \varphi_b + \varphi_c = \pm\pi, \quad \varphi_a, \varphi_b, \varphi_c \in (-\pi/2, \pi/2). \tag{18}$$

Moreover, we must have $|z_a| \leq 2|\cos(\varphi_a)|$ with equality if $|\alpha| = 1$. Likewise, $|z_b| \leq 2|\cos(\varphi_b)|$ and $|z_c| \leq 2|\cos(\varphi_c)|$. Hence,

$$|\cos(\varphi_a)\cos(\varphi_b)\cos(\varphi_c)| \geq \frac{1}{8}. \tag{19}$$

Note that (18) implies that all three numbers $\varphi_a, \varphi_b, \varphi_c$ must be positive or all three negative. Replacing $(\varphi_a, \varphi_b, \varphi_c)$ by $(-\varphi_a, -\varphi_b, -\varphi_c)$ if necessary we may assume that all three numbers are positive. Then, $\varphi_a, \varphi_b, \varphi_c$ are angles of an acute triangle. It is an elementary exercise to show that the sum of their cosine angles $\cos(\varphi_a) + \cos(\varphi_b) + \cos(\varphi_c)$ attains its maximum $3/2$ only if all three angles are $\pi/3$. Thus, by (19), we obtain

$$\frac{1}{2} \leq \big(\cos(\varphi_a)\cos(\varphi_b)\cos(\varphi_c)\big)^{1/3} \leq \frac{\cos(\varphi_a) + \cos(\varphi_b) + \cos(\varphi_c)}{3} \leq \frac{1}{2}.$$

This implies that under assumption (18) inequality (19) only holds when $\varphi_a = \varphi_b = \varphi_c = \pm\pi/3$.

Then, we deduce $|z_a| = 2|\cos(\varphi_a)| = 1$ and, similarly, $|z_b| = |z_c| = 1$. Hence,

$$\alpha^a = \alpha^b = \alpha^c = -1 + e^{\pm\pi i/3} = e^{\pm 2\pi i/3}.$$

This can only happen if $|\alpha| = 1$. Moreover, α must be a root of unity, since $\alpha^{3a} = 1$. Set $\alpha = e^{2\pi k i/N}$ with some $N \in \mathbb{N}$ and $k \in \{0, \dots, N-1\}$, where $\gcd(k, N) = 1$. Then, $2\pi i k a/N = \pm 2\pi i/3 + 2\pi i s$ with $s \in \mathbb{Z}$, which implies $ka/N \mp 1/3 \in \mathbb{Z}$. Multiplying by N we see that N must be divisible by 3. Hence, k is not divisible by 3. Similarly, $kb/N \mp 1/3 \in \mathbb{Z}$ and $kc/N \mp 1/3 \in \mathbb{Z}$, which by subtracting implies $k(b-c)/N \in \mathbb{Z}$. Hence, $b - c$ must be divisible by N, and so by 3, which is impossible because of (17). This completes the proof of the theorem. □

Proof of Proposition 1.5. Assume that $k \geq 1$ and for some $j \in \{1, \dots, k\}$ the polynomial $g = f_j \in \mathbb{Z}[x]$ of positive degree is not as claimed in the proposition. If g has no roots on $|z| = 1$ and g is not as claimed in the proposition, then it must have a root in $|z| < 1$ and a root in $|z| > 1$. We will show the same is true if g has a root on $|z| = 1$. Indeed, then, as not all roots of g are on $|z| = 1$, g must have a root α of modulus distinct from 1. Note that α is reciprocal, since it has a conjugate on $|z| = 1$. So α and α^{-1} are conjugate over $\mathbb{Q}$. This implies that g has a root in $|z| < 1$ and a root in $|z| > 1$ as claimed.

Therefore, if $g = f_j$ is not as claimed in the proposition, it must have a root in $0 < |z| < 1$ and a root in $|z| > 1$. This implies $s \leq d - 2$. Assume that the nonzero roots of f defined in (6) are $\alpha_1, \dots, \alpha_{d-s}$. Without restriction of generality we can label them as

$$|\alpha_1| \geq \cdots \geq |\alpha_q| > 1 \geq |\alpha_{q+1}| \geq \cdots \geq |\alpha_{d-s}|,$$

where $1 \leq q \leq d-s-1$ because $|\alpha_1| > 1$ and $|\alpha_{d-s}| < 1$. Then, by the definition of Mahler's measure,

$$m = |a| b \alpha_1 \dots \alpha_q$$

with some (possibly negative) nonzero integer b. Here, the product $\alpha_1 \dots \alpha_q$ is a real number, because if α is a nonreal root of f then its complex conjugate $\overline{\alpha}$ is also its root with the same multiplicity.

Take an automorphism σ of the splitting field of f that maps the root α_i of g to its another root α_t, where $1 \leq i \leq q < t \leq d - s$. Then, as $\sigma(\alpha_i) = \alpha_t$, $\sigma(m) = m$ and $\sigma(|a|b) = |a|b$, we obtain

$$m = \frac{|a| b \alpha_t}{\sigma(\alpha_i)} \prod_{j=1}^{q} \sigma(\alpha_1) \dots \sigma(\alpha_q).$$

Hence,

$$|\alpha_1 \dots \alpha_q| = \frac{m}{|ab|} = \frac{|\alpha_t|}{|\sigma(\alpha_i)|} |\sigma(\alpha_1) \dots \sigma(\alpha_q)| \leq |\alpha_t| \cdot |\alpha_1 \dots \alpha_{q-1}|,$$

where the last inequality holds by the definition of $\alpha_1, \dots, \alpha_q$ (these are the only roots of f outside the unit circle). This implies $|\alpha_t| \geq |\alpha_q|$, which is impossible due to $|\alpha_t| \leq 1$ and $|\alpha_q| > 1$. □

4. Proof of Theorem 1.1

For any $\mathbf{w} = (w_1, \dots, w_d) \in \mathbb{C}^d$ and any $k \in \mathbb{N}$ we set

$$S_k(\mathbf{w}) = \sum_{j=1}^{d} w_j^k.$$

In the same fashion, for a polynomial

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 = a_d(x-\alpha_1)\dots(x-\alpha_d) \in \mathbb{R}[x],$$

where $a_d \neq 0$, we denote by

$$S_k(f) = \sum_{j=1}^{d} \alpha_j^k$$

the sum of kth powers of its d roots.

Recall that, by the Newton identities, we have

$$a_d S_k(f) + a_{d-1}S_{k-1}(f) + \cdots + a_{d-k+1}S_1(f) + a_{d-k}k = 0 \tag{20}$$

for $k = 1, 2, \dots, d$. For any $m \in \mathbb{N}$ and any two distinct polynomials

$$f(x) = mx^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$$

and

$$g(x) = mx^d + b_{d-1}x^{d-1} + \cdots + b_0 \in \mathbb{Z}[x],$$

where $a_d = b_d = m$, there is a unique index $k \in \{1, 2, \dots, d\}$ such that $a_{d-j} = b_{d-j}$ for $j = 0, 1, \dots, k-1$ and $a_{d-k} \neq b_{d-k}$. This yields $S_j(f) = S_j(g)$ for $j = 1, \dots, k-1$. Thus, by (20), we deduce

$$m(S_k(f) - S_k(g)) + k(a_{d-k} - b_{d-k}) = 0.$$

Since $a_{d-k} - b_{d-k}$ is a nonzero integer, this implies

$$|S_k(f) - S_k(g)| \geq \frac{k}{m} \tag{21}$$

for this integer $k \in \{1, 2, \dots, d\}$.

We now prove the bound

$$B = B(m,d) < \exp\Big(10.4(md)^{\frac{2}{3}}\big(\log(md)\big)^{\frac{4}{3}}\Big) \tag{22}$$

for the number $B(m,d)$ of polynomials $f \in \mathbb{Z}[x]$ of sufficiently large degree d with leading coefficient $m \in \mathbb{N}$ and all d roots in $|z| \leq 1$.

Fix

$$X := 6md. \tag{23}$$

For each complex number $z = x + iy$ satisfying $|z| \leq 1$ we define

$$\hat{z} := \frac{\lfloor X|x| \rfloor \operatorname{sign}(x) + i\lfloor X|y| \rfloor \operatorname{sign}(y)}{X}.$$

Here, $\operatorname{sign}(x) = 1$ for $x > 0$, $\operatorname{sign}(x) = -1$ for $x < 0$ and $\operatorname{sign}(0) = 0$. It is clear that $|\hat{z}| \leq 1$ and $|z - \hat{z}| < \frac{\sqrt{2}}{X}$. Hence,

$$|z^k - \hat{z}^k| = |z - \hat{z}| \cdot |z^{k-1} + \cdots + \hat{z}^{k-1}| < \frac{\sqrt{2}k}{X}. \tag{24}$$

Since each $\hat{z}$ is of the form $\frac{\mathbb{Z}+i\mathbb{Z}}{X}$, the distance between two distinct $\hat{z}$ is at least $1/X$. Consider a union of open circles at distinct $\hat{z}$ with radii $1/(2X)$. They are not intersecting and are all in the circle with radius $1 + 1/(2X)$. If there are N of them, then

$$\pi\Big(\frac{1}{2X}\Big)^2 N < \pi\Big(1 + \frac{1}{2X}\Big)^2,$$

which, by (23), for d large enough, implies

$$N < (2X + 1)^2 \leq 145m^2d^2. \tag{25}$$

Likewise, for each vector $(\alpha_1, \dots, \alpha_d) \in \mathbb{C}^d$, where $|\alpha_j| \leq 1$, we can define another vector $(\hat{\alpha}_1, \dots, \hat{\alpha}_d) \in \mathbb{C}^d$. Accordingly, for $f \in \mathbb{Z}[x]$ of degree d with leading coefficient m and roots $\alpha_1, \dots, \alpha_d$ we define

$$\hat{f}(x) = m(x - \hat{\alpha}_1) \dots (x - \hat{\alpha}_d).$$

By the definition of $\hat{z}$, the set $\{\hat{\alpha}_1, \dots, \hat{\alpha}_d\}$ is symmetric with respect to complex conjugation, since so is the initial set $\{\alpha_1, \dots, \alpha_d\}$. Hence, $\hat{f} \in \mathbb{R}[x]$ which implies $S_k(\hat{f}) \in \mathbb{R}$ for $k = 1, \dots, d$.

Assume that some two integer polynomials f, g of degree d with leading coefficient m are distinct. For each $k = 1, \dots, n$, by (23), (24), we have

$$|S_k(f) - S_k(\hat{f})| < d\frac{\sqrt{2}k}{X} < \frac{k}{4m}.$$

Choosing k as in (21) we obtain

$$\begin{aligned}|S_k(\hat{f}) - S_k(\hat{g})| &\geq |S_k(f) - S_k(g)| - |S_k(f) - S_k(\hat{f})| - |S_k(g) - S_k(\hat{g})| \\ &> \frac{k}{m} - \frac{k}{4m} - \frac{k}{4m} = \frac{k}{2m}.\end{aligned}$$

This implies that $\hat{f}$ and $\hat{g}$ are distinct and that the l_∞-distance between any distinct vectors of the form

$$\Big(2mS_1(\hat{f}), \dots, \frac{2mS_k(\hat{f})}{k}, \dots, \frac{2mS_d(\hat{f})}{d}\Big) \in \mathbb{R}^d \tag{26}$$

is at least 1. Note that there are B distinct vectors as in (26), since f runs over B distinct polynomials.

Let A be the convex hull of the vectors

$$\Big(2md\Re(u), \dots, \frac{2md\Re(u^k)}{k}, \dots, \frac{2md\Re(u^d)}{d}\Big) \in \mathbb{R}^d, \tag{27}$$

where u runs over all possible (at most N) images of the unit circle $|z| \leq 1$ under the map $z \to \hat{z}$. Each vector in (26) belongs to A, since it is the arithmetic mean of some d vectors as defined in (27). Since the distance between any vectors as in (26) is at least one, their number B is bounded above by the volume of the set $F(A)$ defined in (8), namely,

$$B \leq \mathrm{Vol}\big(F(A)\big).$$

Thus, by Lemma 2.1, we obtain

$$B \leq 1 + \sum_{G} \mathrm{Vol}\big(\mathrm{Pr}_G(A)\big). \tag{28}$$

where the sum is taken over all nonempty subsets G of $\{1, 2, \dots, d\}$. By (27) and $|\Re(u^k)| \leq 1$, the set A is contained in the parallelepiped

$$P = \prod_{j=1}^{d} [-u_j/2, u_j/2],$$

with $u_j = 4md/j$.

Fix

$$L := 10(md)^{2/3}\big(\log(md)\big)^{1/3}. \tag{29}$$

Fix a nonempty subset G of $\{1, \dots, d\}$ with $|G| = g$. Assume first that $g \geq L$, where L is defined in (29). By (25), $\mathrm{Pr}_G(A) \subseteq P$ is a convex polytope with at most

$$145m^2d^2 < L^3 \leq g^3$$

vertices. So, by Lemma 2.2 and $\lambda \leq 3$, we obtain

$$\mathrm{Vol}\big(\mathrm{Pr}_G(A)\big) < \Big(\frac{46.44 \log g}{g}\Big)^{g/2} \prod_{j \in G} u_j \leq \Big(\frac{46.44 \log L}{L}\Big)^{g/2} \prod_{j \in G} u_j.$$

Inserting L from (29) and using the fact that d is large enough we obtain

$$\mathrm{Vol}\big(\mathrm{Pr}_G(A)\big) < \left(\frac{0.568(md)^{1/3}}{\big(\log(md)\big)^{1/3}}\right)^{-g} \prod_{j \in G} u_j. \tag{30}$$

On the other hand, in case $g < L$, by (29), we have

$$\left(\frac{0.568(md)^{1/3}}{\big(\log(md)\big)^{1/3}}\right)^{g} < \exp\left(\frac{10}{3}(md)^{2/3}\big(\log(md)\big)^{4/3}\right).$$

Hence, using the trivial bound $\mathrm{Vol}\big(\mathrm{Pr}_G(A)\big) \leq \prod_{j \in G} u_j$, we derive that

$$\frac{\mathrm{Vol}\big(\mathrm{Pr}_G(A)\big)}{\exp\left(\frac{10}{3}(md)^{2/3}\big(\log(md)\big)^{4/3}\right)} < \left(\frac{\big(\log(md)\big)^{1/3}}{0.568(md)^{1/3}}\right)^{g} \prod_{j \in G} u_j. \tag{31}$$

By (30), the bound (31) is true for every nonempty $G \in \{1, \dots, d\}$, with $g = |G|$.

Also, from (28) it follows that

$$B - \exp\left(\frac{10}{3}(md)^{2/3}\big(\log(md)\big)^{4/3}\right) < B - 1 \leq \sum_{G} \mathrm{Vol}\big(\mathrm{Pr}_G(A)\big).$$

Dividing this inequality by a corresponding exponent and combining it with

$$\frac{\big(\log(md)\big)^{1/3}u_j}{0.568(md)^{1/3}} = \frac{4md\big(\log(md)\big)^{1/3}}{0.568j(md)^{1/3}} < \frac{7.05(md)^{2/3}\big(\log(md)\big)^{1/3}}{j},$$

from (31) we derive that

$$\frac{B}{\exp\Big(\frac{10}{3}(md)^{2/3}\big(\log(md)\big)^{4/3}\Big)} < \prod_{j=1}^{d}\Big(1 + \frac{7.05(md)^{2/3}\big(\log(md)\big)^{1/3}}{j}\Big).$$

Now, applying the inequalities $\prod_{j=1}^{d}(1+y_j) < \exp(y_1 + \cdots + y_d)$ and

$$\sum_{j=1}^{d}\frac{1}{j} \le \log d + 1 \le \log(md) + 1,$$

we can further bound

$$\prod_{j=1}^{d}\Big(1 + \frac{7.05(md)^{2/3}\big(\log(md)\big)^{1/3}}{j}\Big) < \exp\Big(7.06(md)^{2/3}\big(\log(md)\big)^{4/3}\Big).$$

This, by the above upper bound on B and by $10/3 + 7.06 < 10.4$, implies the upper bound on B as claimed in (22).

Now, we will estimate the number $D = D(m,d)$ of distinct integer polynomials of degree d and Mahler measure m. More precisely, we proceed to show that

$$D = D(m,d) < \exp\Big(10.5(md)^{2/3}\big(\log(md)\big)^{4/3}\Big). \tag{32}$$

In case $m = 1$ the result follows by Proposition 1.2. In the case when $m \ge d^{1/2}$, we have $d \le (md)^{2/3}$ and $\log m \le \log(md) \le \big(\log(md)\big)^{4/3}$, so in view of (3) the required bound (32) follows by

$$D(m,d) \le m^{d(1+\varepsilon)} < \exp\Big(2(md)^{2/3}\big(\log(md)\big)^{4/3}\Big).$$

So, from now on, we assume that

$$2 \le m \le d^{1/2}. \tag{33}$$

Assume that $f \in \mathbb{Z}[x]$ is a polynomial of degree d and Mahler's measure $m \ge 2$. By Proposition 1.5 (see (6) and (7)), we can write f in the form

$$f(x) = f_1(x)f_2(x),$$

where $f_1 \in \mathbb{Z}[x]$ has the leading coefficient $\pm m_1$, degree d_1 and all roots in $|z| \le 1$, and $f_2 \in \mathbb{Z}[x]$ has the constant coefficient $\pm m_2$, degree $d_2 = d - d_1$ and all roots in $|z| > 1$. Here, m_1 and m_2 are positive integers such that $m_1m_2 = m$, $M(f_1) = m_1$, $M(f_2) = M(f_2^*) = m_2$.

The number of such polynomials with $d_2 = 0$ is bounded above by $2B(m,d)$, where $B(m,d)$ has been defined in (22). If $d_2 > 0$ then $m_2 > 1$. The number of such polynomials with $d_1 = 0$ is bounded above by $2B(m,d)$ as well. If $m_1 = 1$ then f has all roots in $|z| \geq 1$, so the number of such polynomials can be bounded by $2B(m,d)$ too. Thus,

$$D(m,d) \leq 6B(m,d) + E(m,d), \tag{34}$$

where $B(m,d)$ has been defined in (22) and $E(m,d)$ stands for the number of polynomials with Mahler's measure m representable in the form $f_1 f_2$, where $f_1 \in \mathbb{Z}[x]$ of degree $d_1 \geq 1$ has all roots in $|z| \leq 1$ and Mahler measure $m_1 \geq 2$, and $f_2 \in \mathbb{Z}[x]$ of degree $d_2 = d - d_1 \geq 1$ has all roots in $|z| > 1$ and Mahler measure $m_2 \geq 2$. (Of course, the part $E(m,d)$ only appears for composite m.) Here, the leading coefficient of f_1 (with all roots in $|z| \leq 1$) is $\pm m_1$, and the leading coefficient of f_2^* (with all roots in $|z| < 1$) is $\pm m_2$. Consequently,

$$E(m,d) \leq 4 \sum_{\substack{m_1 m_2 = m,\, m_1, m_2 \geq 2 \\ d_1 + d_2 = d}} B(m_1, d_1) B(m_2, d_2).$$

Note that there are at most m pairs of positive integers (m_1, m_2) satisfying $m_1 m_2 = m$ and exactly d pairs of positive integers (d_1, d_2) for which $d_1 + d_2 = d$. Thus,

$$E(m,d) \leq 4md \max_{\substack{m_1 m_2 = m,\, m_1, m_2 \geq 2 \\ d_1 + d_2 = d}} B(m_1, d_1) B(m_2, d_2). \tag{35}$$

Take a positive integer d_0 for which the bound (22) on $B(m,d)$ is true for all $d \geq d_0$. For $d < d_0$ we will use the trivial bound

$$B(m,d) < c_6 m^{d+1}, \tag{36}$$

where c_6 is a constant depending on d_0 only (see (3)).

Now, we are ready to show the required bound (32) for d large enough. Without loss of generality, we may assume that $d \geq 2d_0$. Also, from $d_1 + d_2 = d$ we see that at least one of the numbers d_1, d_2 is greater than or equal to d_0. If both d_1 and d_2 are at least d_0 then the product $B(m_1, d_1) B(m_2, d_2)$ is less than

$$\exp\Big(10.4 (m_1 d_1)^{2/3} \big(\log(m_1 d_1)\big)^{4/3} + 10.4 (m_2 d_2)^{2/3} \big(\log(m_2 d_2)\big)^{4/3}\Big)$$

by (22). This implies the required bound (32) by Lemma 2.3 with $k = 2$ due to $m_1, m_2 \geq 2$, and (22), (34), (35).

Otherwise, we must have either $d_1 < d_0 \leq d_2$ or $d_2 < d_0 \leq d_1$. In the first case, $d_1 < d_0 \leq d_2$, by (22) and (36), we obtain

$$B(m_1, d_1) B(m_2, d_2) < c_6 m_1^{d_1 + 1} \exp\Big(10.4 (m_2 d_2)^{2/3} \big(\log(m_2 d_2)\big)^{4/3}\Big).$$

Here, the factor $c_6 m_1^{d_1+1}$ is very small, since from (33) it follows that

$$\log c_6 + (d_1 + 1) \log m_1 \leq \log c_6 + (d_1 + 1) \log m < c_7 \log d.$$

This immediately yields the desired bound (32) by $m_2 d_2 \leq md$, (22), (34) and (35). It is clear that the second case, $d_2 < d_0 \leq d_1$, is symmetric to that above and can be treated analogously. This completes the proof of Theorem 1.1.

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(Artūras Dubickas) Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania
arturas.dubickas@mif.vu.lt

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