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Feynman's operational calculus in topological algebras

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Abstract. We develop a rather general version of Feynman's operational calculus. Instead of using operator-valued (or $\mathcal{L}(X)$-valued) functions in the operational calculus, we use functions taking values in a topological algebra E. While this complicates some aspects of the operational calculus, it supplies a quite general framework in which the time-ordering calculations required for the operational calculus can be carried out, allowing the operational calculus to be used with functions taking values in any algebra which satisfies the conditions which are imposed on the topological algebras considered in this paper.

Contents

1. Introduction

In this paper, we develop the abstract formulation of Feynman's operational calculus (see [17] for the abstract version of the operational calculus in $\mathcal{L}(X)$) using functions which take their values in a topological algebra. The reason we address the operational calculus in a topological algebra E instead of in $\mathcal{L}(X)$ (X a Banach space) is that having Feynman's operational calculus in the more general setting of a topological algebra allows us to immediately access the operational calculus in any space with the same underlying structure. (The

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author has received questions along these lines more than a few times over the years). For instance, if we wish to carry out the operational calculus using C^* algebra-valued functions, the knowledge of the operational calculus in a topological algebra E can be carried over to a topological algebra with involution and we then have Feynman's calculus ready for use in the C^* algebra.

It is worthwhile at this point to describe, in informal terms, the abstract approach to Feynman's operational calculus using either operators in $\mathcal{L}(X)$ or $\mathcal{L}(X)$-valued functions. (A more detailed introduction to the operational calculus is found in Section 2.) The starting point is found in [7], where Feynman supplies heuristic rules for the calculus:

(1) *Attach time indices to the operators to specify the order of operators in products.*
(2) *With time indices attached, form functions of these operators by treating them as though they were commuting.*
(3) *Finally, "disentangle" the resulting expressions; that is, restore the conventional ordering of the operators.*

We note that rule (3) above means that we write the result of rule (2) in terms of time-ordered products (in practice, a sum of time-ordered products of the operators or operator-valued functions).

To be able to carry out the computations necessary for the operational calculus in a mathematically rigorous fashion, we need a commutative environment (or "commutative world") in which we can form the functions of the operators (or operator-valued functions). The paper [11] by B. Jefferies and G. W. Johnson supplied the necessary commutative world in which Feynman's rules can be applied in a rigorous fashion. Indeed, given operators $A_1, \dots, A_n \in \mathcal{L}(X)$ (X a Banach space), we use Borel probability measures $\mu_1, \dots, \mu_n$ on $[0,T]$ to attach time indices to the operators via

$$A_j = \int_{[0,T]} A_j(s)\, \mu_j(ds),$$

where $A_j(s) \equiv A_j$ for every $s \in [0,T]$. In order to form the desired function of the operators, we begin by computing the operator norms $\|A_1\|_{\mathcal{L}(X)}, \dots, \|A_n\|_{\mathcal{L}(X)}$ and defining $\mathbb{A}\left(\|A_1\|_{\mathcal{L}(X)}, \dots, \|A_n\|_{\mathcal{L}(X)}\right)$ to be the family of all $\mathbb{C}$-valued functions f which are analytic at the origin and are such that their power series expansion

$$f(z_1, \dots, z_n) = \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n} z_1^{m_1} \cdots z_n^{m_n}$$

converges absolutely at least on the closed polydisk

$$\left\{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1| \le \|A_1\|_{\mathcal{L}(X)}, \dots, |z_n| \le \|A_n\|_{\mathcal{L}(X)}\right\}.$$

Such functions are analytic at least in the open polydisk

$$\left\{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1| < \|A_1\|_{\mathcal{L}(X)}, \dots, |z_n| < \|A_n\|_{\mathcal{L}(X)}\right\}.$$

Note that any entire function of $z_1, \dots, z_n$ is an element of this algebra. It turns out that $\mathbb{A}$ is a commutative Banach algebra (see Section 2 below and also [17, Chapter 2 and 6]). With the algebra $\mathbb{A}$ in hand, we define the *disentangling algebra* $\mathbb{D}$. To do this, we discard all operator-theoretic properties of $A_1, \dots, A_n$ except for the operator norms $\|A_1\|_{\mathcal{L}(X)}, \dots, \|A_n\|_{\mathcal{L}(X)}$ and introduce a commutative Banach algebra $\mathbb{D}\left(\tilde{A}_1, \dots, \tilde{A}_n\right)$ consisting of "analytic functions" $f\left(\tilde{A}_1, \dots, \tilde{A}_n\right)$ where $\tilde{A}_1, \dots, \tilde{A}_n$ are treated as purely formal commuting objects. While it is natural that $A_1, \dots, A_n$ should be linearly independent, we do not require them to be distinct. However, if $A_i = A_j$ we still regard $\tilde{A}_i$ and $\tilde{A}_j$ as distinct in $\mathbb{D}\left(\tilde{A}_1, \dots, \tilde{A}_n\right)$. It is $\mathbb{D}$ that supplies the commutative setting in which we can carry out the disentangling calculations. Furthermore, we note that $\mathbb{A}$ and $\mathbb{D}$ are isometrically isomorphic (see [17, Chapter 2] or [11]). With $f \in \mathbb{D}$ written as

$$f\left(\tilde{A}_1, \dots, \tilde{A}_n\right) = \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n} \left(\tilde{A}_1\right)^{m_1} \cdots \left(\tilde{A}_n\right)^{m_n},$$

we disentangle f by first disentangling

$$\begin{aligned} P^{m_1,\dots,m_n}\left(\tilde{A}_1, \dots, \tilde{A}_n\right) &= \left(\tilde{A}_1\right)^{m_1} \cdots \left(\tilde{A}_n\right)^{m_n} \\ &= \left(\int_{[0,T]} \tilde{A}_1(s)\, \mu_1(ds)\right)^{m_1} \\ &\qquad \cdots \left(\int_{[0,T]} \tilde{A}_n(s) \mu_n(ds)\right)^{m_n} \end{aligned}$$

in $\mathbb{D}$, where commutativity allows us to do our computations rigorously. With the time-ordering of $P^{m_1,\dots,m_n}\left(\tilde{A}_1, \dots, \tilde{A}_n\right)$ in hand, we map the time-ordered expression (really a sum of time-ordered operator products) using the *disentangling map*

$$\mathcal{T}_{\mu_1,\dots,\mu_n} : \mathbb{D} \to \mathcal{L}(X)$$

by "erasing the tildes" on the operators. We obtain the *disentangled operator*

$$P^{m_1,\dots,m_n}_{\mu_1,\dots,\mu_n}\left(A_1, \dots, A_n\right) := \mathcal{T}_{\mu_1,\dots,\mu_n} P^{m_1,\dots,m_n}\left(\tilde{A}_1, \dots, \tilde{A}_n\right).$$

Once this operator is determined, we obtain the disentangled operator

$$f_{\mu_1,\dots,\mu_n}\left(A_1, \dots, A_n\right) := \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n} P^{m_1,\dots,m_n}_{\mu_1,\dots,\mu_n}\left(A_1, \dots, A_n\right);$$

i.e. we apply the disentangling map term-by-term in the series expansion of f. It turns out that, with this definition, the series above converges in norm on $\mathcal{L}(X)$ (see [17, Chapter 2].) The series (or disentangled operator) just above is the *disentangling of f under the time-ordering directions supplied by the measures* $\mu_1, \dots, \mu_n$ or more simply the *disentangling of f indexed by the measures* $\mu_1, \dots, \mu_n$.

Remark 1.1. We make a couple of simple observations.

1) Generally speaking, given a function $f \in \mathbb{D}$, changing the time-ordering measures will change the disentangled operator, though one can come up with simple examples for which this does not happen.

2) As the reader will likely have noticed, the "heart" of the disentangling process is the time-ordering (or disentangling) of the monomial

$$P^{m_1,\dots,m_n}\left(\tilde{A}_1,\dots,\tilde{A}_n\right).$$

Knowing the disentangled monomial enables us to find the disentangling of any function $f \in \mathbb{D}$, via its Taylor series at the origin.

For the time-dependent setting mentioned above, instead of fixed operators, we use operator-valued functions $A_j : [0,T] \to \mathcal{L}(X)$. In the time-dependent setting, the ideas are very much the same as those for the time independent setting. Indeed, the definition of the commutative Banach algebras $\mathbb{A}$ and $\mathbb{D}$ are essentially the same as above. However, the fact that we have operator-valued functions changes, a bit, how the algebra $\mathbb{A}$ is defined – the radii of the polydisk change – and the formal commuting objects are, in fact a bit different due to the role that the time-ordering measures play in the time-dependent setting; see [21] and [17], for instance. (However, one can still think of the formal objects as $\tilde{A}_1,\dots,\tilde{A}_n$ and, in practice, this is exactly what is done.)

The purpose of this paper is, as mentioned at the start, the development of the operational calculus in the setting of a topological algebra. The idea is to develop a version of Feynman's operational calculus using functions on $[0,T]$ taking their values in a topological algebra. So, with E a topological algebra, we use functions $A_j : [0,T] \to E$, $j = 1,\dots,n$ and, to each $A_j(\cdot)$ we associate a Borel probability measure μ_j on $[0,T]$. The questions which immediately come to mind are: (1) What notion of measurability are we to use?; (2) How much structure must E have to enable a satisfactory theory of integration? Of course, this must include, at least, a Fubini theorem.

The answers to the questions above are found in the very interesting paper "Totally summable functions with values in locally convex spaces," by Thomas [31]. We will leave the topic of measurability to Section 4. It is the idea of totally summable functions that will be of most use to us below. An example, cited by Thomas on page 117 of [31] is that of a function with values in the space of bounded linear operators on a Hilbert space endowed with the strong operator topology. If this function is measurable with respect to this topology, in a sense to be made precise in Section 4 below, the composition of this function with the operator norm is measurable, and if this composition is an integrable function, the function will be totally summable. However, before the consideration of totally summable functions, summable functions will be defined and here the locally convex space in which functions take their values will need to be assumed to be Hausdorff and quasi-complete (closed and bounded subsets are complete). Once summable functions are defined, totally summable functions are defined to be measurable functions which satisfy an integrability condition (with respect to the measure one uses) which involves the gauge functional of a closed, bounded and absolutely convex set. (Every totally summable function will be summable.) As shown in [31] totally summable functions allow for a Fubini theorem as well as a dominated convergence theorem; two primary tools needed when studying the operational calculus. As mentioned earlier, [31] considers integration of functions taking values in a locally convex space.

For the operational calculus we obviously require an algebra in which to work. We will, therefore, use functions which take their values in a locally convex topological algebra (a locally convex topological vector space which is also an algebra). While we will outline the necessary properties of topological algebras in Section 3 below, we will mention here that the topology on the algebra will be taken to be generated by a family of "submultiplicative" seminorms (see Definition 3.1 below). We will also take the algebra to be a complete locally (multiplicatively) convex algebra, that is, an Arens-Michael algebra. (The reader will note that, strictly speaking, we only need to take our algebra to be quasi-complete, i.e. closed subsets are complete. However, it is convenient to take our algebras to be complete.)

To end this introduction, we take the time to outline the contents of this paper. Section 2 is a brief outline of Feynman's operational calculus in the usual setting where our functions take their values in $\mathcal{L}(X)$, the Banach space of bounded linear operators on the Banach space X. No detailed time-ordering calculations are carried out, but one can find all the details in [17], [11, 10, 12, 13], [14], [18] and [27]. The main purpose of Section 2 is to outline the construction of the necessary commutative Banach algebras $\mathbb{A}$ and $\mathbb{D}$, the definition of the so-called disentangling map which takes us from the commutative setting of $\mathbb{D}$ to the noncommutative setting of $\mathcal{L}(X)$, and the time-orderings of the monomial

$$P^{m_1,\ldots,m_n}\left(\tilde{A}_1(\cdot),\ldots,\tilde{A}_n(\cdot)\right) := \left(\tilde{A}_1(\cdot)\right)^{m_1}\cdots\left(\tilde{A}_n(\cdot)\right)^{m_n}$$

in the cases where, respectively, the time-ordering measures are continuous, have finitely supported discrete parts and where the measures are arbitrary (i.e., have arbitrary discrete parts).

Remark 1.2. As the reader will notice below, when we discuss the operational calculus in a topological algebra E, we will use continuous measures. (We note here that we will be working with finite Borel measures μ on intervals $[0,T]$ for $T>0$ and so, by "continuous measure" we will mean that $\mu(\{x\})=0$ for every $x\in[0,T]$ since $\{x\}$ is a Borel set in $[0,T]$. See page 11 of [4], for example.) This is mostly for convenience for if we use measures with nonzero discrete parts, we gain relatively little with regard to developing the essential ideas at the expense of much more combinatorial complexity. However, comments are made from time to time concerning how things would change if we use more general measures.

The third section of the paper is rather brief but supplies the necessary background on topological algebras. The fourth section contains a discussion of the more important definitions and results of the paper [31]. Here the reader will find the definitions of measurability which we will use as well as definitions and theorems concerning summability and, more importantly, total summability.

Sections 5 and 6 detail the development of Feynman's operational calculus using functions taking values in a topological algebras. We do take the time in Section 6 to develop the operational calculus in the presence of a strongly continuous semigroup on a Banach space which is introduced via total summability.

Finally, Section 7 is devoted to a stability result for the operational calculus in the setting of a topological algebra.

2. Feynman's operational calculus – overview

Here we give an overview of Feynman's operational calculus in the usual setting where we have $\mathcal{L}(X)$-valued functions on $[0,T]$. Detailed discussions of the ideas presented in this section can be found in [17] as well as [11, 10, 12, 13], [14], [18], [27].

Before we delve into the details of the commutative Banach algebras which are used in the abstract approach to the operational calculus, it may be useful, especially for the nonspecialist, to present a couple of simple examples which serve to illustrate the types of computations one needs to carry out in order to obtain the "disentangled operator." In each of the two examples presented below we will carry out our time-ordering (or *disentangling*) calculations heuristically, following Feynman's 'rules' stated in the introduction. Indeed, this is the approach taken by Feynman in [7] as well as in many books and papers since.

Example 1: We consider here the function $f(x,y) = xy$ and let A and B be bounded linear operators on some Banach space X. We do not assume that $AB = BA$. We wish to compute the function $f(A,B)$. However, since these operators are not assumed to commute, there is ambiguity present. Indeed, we could take $f(A,B)$ to be AB, BA, $\frac{1}{2}AB+\frac{1}{2}BA$ or any one of infinitely many other possibilities. We follow Feynman's 'rules' stated above in the introduction. First, we will attach time indices to the operators A and B. To attach time indices to operators, Feynman nearly always used Lebesgue measure ℓ and wrote

$$A = \frac{1}{T}\int_0^T A(s)\,ds$$

where $A(s) \equiv A$ for all $s \in [0,T]$. Instead of using Lebesgue measure to attach time-indices to our operators, we will use Borel probability measures on $[0,T]$. Specifically, associate to the operator A the continuous Borel probability measure μ on $[0,T]$ and associate to the operator B the continuous Borel probability measure ν on $[0,T]$. We may then write

$$A = \int_{[0,T]} A(s)\,\mu(ds) \text{ and } B = \int_{[0,T]} B(s)\,\nu(ds)$$

where $A(s) \equiv A$ and $B(s) \equiv B$ for all $s \in [0,T]$. We can then compute

$$\begin{aligned} AB &= \left(\int_{[0,T]} A(s)\,\mu(ds)\right)\left(\int_{[0,T]} B(s)\,\nu(ds)\right) \\ &= \int_{[0,T]^2} A(s_1)B(s_2)\ (\mu\times\nu)\,(ds_1, ds_2) \\ &= \int_{\{s_1<s_2\}} B(s_2)A(s_1)\ (\mu\times\nu)\,(ds_1, ds_2) \\ &\qquad + \int_{\{s_2<s_1\}} A(s_1)B(s_2)\ (\mu\times\nu)\,(ds_1, ds_2) \\ &= (\mu\times\nu)\left(\{(s_1,s_2)\in[0,T]^2 \ :\ s_1<s_2\}\right) BA \\ &\qquad + (\mu\times\nu)\left(\{(s_1,s_2)\in[0,T]^2 \ :\ s_2<s_1\}\right) AB. \end{aligned}$$

It"s worth commenting briefly on the computation above. The first equality uses the fact that $A(s) \equiv A$ and $B(s) \equiv B$ and the fact that μ and ν are probability measures on $[0,T]$. The second line writes the product of integrals as a double integral over $[0,T]^2$. The third equality is obtained by writing

$$\begin{aligned} [0,T]^2 = \{(s_1,s_2)\in[0,T]^2 \ :\ s_1<s_2\} \cup \{(s_1,s_2)\in[0,T]^2 \ :\ s_1=s_2\} \\ \cup \{(s_1,s_2)\in[0,T]^2 \ :\ s_2<s_1\} \end{aligned}$$

and then using the fact that both μ and ν are continuous Borel measures so that

$$(\mu\times\nu)\left(\{(s_1,s_2)\in[0,T]^2 \ :\ s_1=s_2\}\right) = 0.$$

We also carry out the *time-ordering* here using the convention that operators with an *earlier* time index appear to the right (or before) operators with a *later* time index. Finally, we obtain the last equality by using the fact that $A(s) \equiv A$ and $B(s) \equiv B$. We have therefore obtained the *disentangled operator*

$$\begin{aligned} f_{\mu,\nu}(A,B) &= (\mu\times\nu)\left(\{(s_1,s_2)\in[0,T]^2 \ :\ s_1<s_2\}\right) BA \\ &\qquad + (\mu\times\nu)\left(\{(s_1,s_2)\in[0,T]^2 \ :\ s_2<s_1\}\right) AB, \end{aligned}$$

where we've used $f_{\mu,\nu}$ to denote that the function f was computed using time-ordering directions supplied by the time-ordering measures μ and ν.

Example 2: In this example we continue with $f(x,y) = xy$ and the operators A and B. We will also continue to associate to A the continuous Borel probability measure μ on $[0,T]$. To B we now associate the Borel probability measure $\nu = \lambda + \omega\delta_\tau$ where λ is a continuous Borel measure on $[0,T]$ and where $\tau \in (0,T)$. As in Example 1, we use these measures to attach time indices to our operators. With these choices of measures, see more clearly the role that the measures play in the disentangling process.

Proceeding as in example 1, we compute, successively,

$$
\begin{aligned}
AB &\overset{(1)}{=} \left(\int_{[0,T]} A(s)\mu(ds)\right)\left(\int_{[0,T]} B(s)\lambda(ds) + \omega\int_{[0,T]} B(s)\delta_\tau(ds)\right)\\
&\overset{(2)}{=} \left(\int_{[0,T]} A(s)\mu(ds)\right)\left(\int_{[0,T]} B(s)\lambda(ds)\right)\\
&\qquad + \left(\int_{[0,T]} A(s)\mu(ds)\right)\left(\omega\int_{[0,T]} B(s)\delta_\tau(ds)\right)\\
&\overset{(3)}{=} \int_{[0,T]^2} A(s_1)B(s_2)\,(\mu\times\lambda)\,(ds_1, ds_2)\\
&\quad + \left(\int_{[0,\tau]} A(s)\mu(ds) + \int_{[\tau,T]} A(s)\mu(ds)\right)\left(\omega\int_{\{\tau\}} B(s)\delta_\tau(ds)\right)\\
&\overset{(4)}{=} \int_{\{s_1<s_2\}} B(s_2)A(s_1)\,(\mu\times\lambda)\,(ds_1, ds_2)\\
&\quad + \int_{\{s_2<s_1\}} A(s_1)B(s_2)\,(\mu\times\lambda)\,(ds_1, ds_2)\\
&\quad + \left(\omega\int_{\{\tau\}} B(s)\delta_\tau(ds)\right)\left(\int_{[0,\tau]} A(s)\mu(ds)\right)\\
&\qquad + \left(\int_{[\tau,T]} A(s)\mu(ds)\right)\left(\omega\int_{\{\tau\}} B(s)\delta_\tau(ds)\right)\\
&\overset{(5)}{=} (\mu\times\lambda)\left(\{(s_1,s_2)\in[0,T]^2 \;:\; s_1<s_2\}\right)BA\\
&\quad + (\mu\times\lambda)\left(\{(s_1,s_2)\in[0,T]^2 \;:\; s_2<s_1\}\right)AB\\
&\quad + \omega\cdot\mu\,([0,\tau])\,BA + \omega\cdot\mu\,([\tau,T])\,AB.
\end{aligned}
$$

We comment on each of the numbered equalities:

(1) Equality (1) uses the fact that the measures μ and ν are probability measures as well as $A(s)\equiv A$ and $B(s)\equiv B$.

(2) Equality (2) distributes the integral of A.

(3) Equality (3) first writes the product of the integral of A against μ with the integral of B against λ as a double integral over $[0,T]^2$ and writes the integral of A against λ as a sum of integrals over $[0,\tau]$ and $[\tau,T]$, respectively. This is done so that the time ordering can be carried out; see the next item for details.

(4) For equality (4), the first two lines after this equality use the fact that $[0,T]$ is a disjoint union $\{s_1<s_2\}\cup\{s_2<s_1\}$ (up to a set of measure zero), allowing the integral to be written as the sum shown. The third and fourth lines deal with the relation between the support of the Dirac point mass δ_τ and the time indices in $[0,T]$. The integral of A over $[0,T]$

is written as the sum of an integral over $[0, \tau]$ and an integral over $[\tau, T]$. The interval $[0, \tau]$ consists of time indices *earlier* than τ and, because B acts at time τ and the time indices for A are earlier, A will act before B in this situation. The interval $[\tau, T]$ consists of time indices *later* than τ and so, B will act before A here. These are the reasons that we see lines (3) and (4) written as shown. Note that the time-ordering is carried out here.

(5) Finally, equality (5) is obtained by evaluating the integrals in each of the four lines after equality (4), using the fact that $A(s) \equiv A$ and $B(s) \equiv B$.

The end result of the calculations above is written as

$$\begin{aligned} f_{\mu,\nu}(A,B) &= (\mu \times \lambda)\left(\left\{(s_1,s_2) \in [0,T]^2 \ : \ s_1 < s_2\right\}\right) BA \\ &+ (\mu \times \lambda)\left(\left\{(s_1,s_2) \in [0,T]^2 \ : \ s_2 < s_1\right\}\right) AB \\ &+ \omega \cdot \mu([0,\tau]) BA + \omega \cdot \mu([\tau,T]) AB. \end{aligned}$$

If one wishes to stress the structure of ν, we would write $f_{\mu,\lambda+\omega\delta_\tau}(A,B)$.

While the examples above are quite simple, they serve to illustrate the essential ideas of the disentangling process. More generally, given bounded linear operators $A_1, \ldots, A_n$, we associate to each operator A_j a Borel probability measure μ_j on $[0,T]$. Given nonnegative integers $m_1, \ldots, m_n$ we begin by writing

$$A_1^{m_1} \cdots A_n^{m_n} = \left(\int_{[0,T]} A_1(s)\mu_1(ds)\right)^{m_1} \cdots \left(\int_{[0,T]} A_n(s)\mu_n(ds)\right)^{m_n}.$$

If each of our measures are continuous (i.e., $\mu_j(\{x\}) = 0$ for $x \in [0,T]$) we continue by (heuristically) computing

$$\begin{aligned} &\left(\int_{[0,T]} A_1(s)\mu_1(ds)\right)^{m_1} \cdots \left(\int_{[0,T]} A_n(s)\mu_n(ds)\right)^{m_n} \\ &= \int_{[0,T]^{m_1+\cdots+m_n}} A_1(s_1)\cdots A_1(s_{m_1})A_2(s_{m_1+1})\cdots A_2(s_{m_1+m_2}) \\ &A_3(s_{m_1+m_2+1})\cdots A_3(s_{m_1+m_2+m_3})\cdots A_n(s_{m_1+\cdots+m_{n-1}+1})\cdots \\ &A_n(s_{m_1+\cdots+m_n})\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\ldots,ds_{m_1+\cdots+m_n}) \end{aligned}$$

where the time indices $s_1, \ldots, s_{m_1}$ are used with A_1, the indices $s_{m_1+1}, \ldots, s_{m_1+m_2}$ are used with A_2, etc, until we reach the last block of time indices

$$s_{m_1+\cdots+m_{n-1}+1}, \ldots, s_{m_1+\cdots+m_n}$$

which are used with A_n. The reader will observe that the operators in the product in the integral above are in no particular order. However, the disentangling process requires that we time order our operator products. To this end, define $m := m_1 + \cdots + m_n$ and for $\pi \in S_m$ (S_m is the group of permutations of m objects) define

$$\Delta_m(\pi) := \left\{(s_1, \ldots, s_m) \in [0,T]^m \ : \ 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < T\right\}.$$

Now, observe that because our measures are continuous, we may write

$$[0,T]^m = \bigcup_{\pi \in S_m} \Delta_m(\pi)$$

where the union is a disjoint union. Therefore

$$\begin{aligned}
&\left(\int_{[0,T]} A_1(s)\mu_1(ds)\right)^{m_1} \cdots \left(\int_{[0,T]} A_n(s)\mu_n(ds)\right)^{m_n} \\
&= \int_{[0,T]^{m_1+\cdots+m_n}} A_1(s_1)\cdots A_1(s_{m_1})A_2(s_{m_1+1})\cdots A_2(s_{m_1+m_2}) \\
&A_3(s_{m_1+m_2+1})\cdots A_3(s_{m_1+m_2+m_3})\cdots A_n(s_{m_1+\cdots+m_{n-1}+1})\cdots \\
&A_n(s_{m_1+\cdots+m_n})\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\dots,ds_{m_1+\cdots+m_n}) \\
&= \sum_{\pi\in S_m}\int_{\Delta_m(\pi)} C_{\pi(m)}\left(s_{\pi(m)}\right)\cdots C_{\pi(1)}\left(s_{\pi(1)}\right)\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right) \\
&\qquad (ds_1,\dots,ds_m)
\end{aligned}$$

where we identify the operators $C_j(s)$ via

$$C_j(s) = \begin{cases} A_1(s) & \text{if } j \in \{1,\dots,m_1\}, \\ A_2(s) & \text{if } j \in \{m_1+1,\dots,m_1+m_2\}, \\ & \vdots \\ A_n(s) & \text{if } j \in \{m_1+\cdots+m_{n-1}+1,\dots,m\}. \end{cases}$$

We have, then, deduced the disentangling of the monomial

$$\left(\int_{[0,T]} A_1(s)\mu_1(ds)\right)^{m_1} \cdots \left(\int_{[0,T]} A_n(s)\mu_n(ds)\right)^{m_n}$$

to be

$$\sum_{\pi\in S_m}\int_{\Delta_m(\pi)} C_{\pi(m)}\left(s_{\pi(m)}\right)\cdots C_{\pi(1)}\left(s_{\pi(1)}\right)\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\dots,ds_m).$$

As the reader will note, this is the disentangled operator obtained from the monomial

$$P^{m_1,\dots,m_n}(z_1,\dots,z_n) := z_1^{m_1}\cdots z_n^{m_n}$$

where our bounded linear operators $A_1,\dots,A_n$ have associated to them the continuous Borel probability measures $\mu_1,\dots,\mu_n$ on $[0,T]$, respectively. (See (2.13) below.)

The ideas outlined above also apply to the cases where we allow non-zero discrete parts to our time-ordering measures. For example, when we allow our time-ordering measures to have finitely supported discrete parts, our monomial will be, writing

$$\mu_j = \sigma_j + \sum_{i=1}^{h} \rho_{j,i}\delta_{\tau_i}$$

where, for each $j = 1, \dots, n$, σ_j is continuous and $0 < \tau_1 < \cdots < \tau_h < T$. We then write

$$\left(\int_{[0,T]} A_1(s)\mu_1(ds)\right)^{m_1} \cdots \left(\int_{[0,T]} A_n(s)\mu_n(ds)\right)^{m_n}$$
$$= \left(\int_{[0,T]} A_1(s)\sigma_1(ds) + \sum_{i=1}^{h} \rho_{1,i} A_1(\tau_i)\right)^{m_1}$$
$$\cdots \left(\int_{[0,T]} A_n(s)\sigma_n(ds) + \sum_{i=1}^{h} \rho_{n,i} A_n(\tau_i)\right)^{m_n}.$$

At this point, we carry out the time-ordering calculations (i.e., the *disentangling*). The end result is seen below in (2.15). (See also [17] or [18].)

Finally, if we allow the supports of the discrete parts of our time-ordering measures to be arbitrary, our monomial will look like

$$\left(\int_{[0,T]} A_1(s)\mu_1(ds)\right)^{m_1} \cdots \left(\int_{[0,T]} A_n(s)\mu_n(ds)\right)^{m_n}$$
$$= \left(\int_{[0,T]} A_1(s)\sigma_1(ds) + \sum_{i=1}^{\infty} \rho_{1,i} A_1(\tau_i)\right)^{m_1}$$
$$\cdots \left(\int_{[0,T]} A_n(s)\sigma_n(ds) + \sum_{i=1}^{\infty} \rho_{n,i} A_n(\tau_i)\right)^{m_n}.$$

The disentangling calculations necessary to obtain the fully time-ordered result are quite involved and the reader is referred first to the end of Section 2 below and for the detailed time-ordering calculations see the paper [27].

We finally remark that, even though the disentangling sketched out above is done only for monomials, the determination of the disentangled monomial is the crucial element of our approach to Feynman's operational calculus. Indeed, given a function $f(z_1, \dots, z_n)$ which is analytic on the appropriate polydisk in $\mathbb{C}^n$, we write out its Taylor series at $0 \in \mathbb{C}^n$ as

$$\sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n} P^{m_1,\dots,m_n}(z_1, \dots, z_n)$$

and we apply the disentangling of the monomial term-by-term in the infinite series to obtain the disentangled operator

$$f_{\mu_1,\dots,\mu_n}(A_1, \dots, A_n).$$

As the reader will see below (and also in [17] and [11]), the series obtained by disentangling each monomial $P^{m_1,\dots,m_n}$ for $m_1, \dots, m_n \in \mathbb{N} \cup \{0\}$ leads to an infinite series in $\mathcal{L}(X)$ which converges in operator norm.

To make the ideas summarized above mathematically rigorous, see the following section. For a much more detailed and complete exposition, see [17] and [27].

2.1. The commutative Banach algebras. To make the discussion above mathematically rigorous, we begin by constructing two commutative Banach algebras, often denoted by $\mathbb{A}$ and $\mathbb{D}$. Let $r_1, \ldots, r_n$ be positive numbers and let $\mathbb{A}(r_1, \ldots, r_n)$ be the space of complex-valued functions f of n complex variables which are analytic at $(0, \ldots, 0)$ and are such that their power series expansion

$$f(z_1, \ldots, z_n) = \sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} z_1^{m_1} \cdots z_n^{m_n} \tag{2.1}$$

converges absolutely, at least on the closed polydisk

$$\{(z_1, \ldots, z_n) : |z_1| \le r_1, \ldots, |z_n| \le r_n\}.$$

Such functions are, of course, analytic at least in the open polydisk

$$\{(z_1, \ldots, z_n) : |z_1| < r_1, \ldots, |z_n| < r_n\}.$$

Observe that, for all n-tuples $(r_1, \ldots, r_n)$ of positive numbers, entire functions of $z_1, \ldots, z_n$ are elements of $\mathbb{A}(r_1, \ldots, r_n)$.

For $f \in \mathbb{A}(r_1, \ldots, r_n)$ given by (2.1), we define the norm

$$\|f\| = \|f\|_{\mathbb{A}(r_1, \ldots, r_n)} := \sum_{m_1, \ldots, m_n = 0}^{\infty} |c_{m_1, \ldots, m_n}| r_1^{m_1} \cdots r_n^{m_n}. \tag{2.2}$$

The norm defined by (2.2) makes $\mathbb{A}(r_1, \ldots, r_n)$ into a commutative Banach algebra. Perhaps the easiest way to see this is to realize that $\mathbb{A}(r_1, \ldots, r_n)$ can be identified with a weighted ℓ_1-space where the weight on the index $(m_1, \ldots, m_n)$ is $r_1^{m_1} \cdots r_n^{m_n}$. Indeed, $\left(\mathbb{A}(r_1, \ldots, r_n), \|\cdot\|_{\mathbb{A}(r_1, \ldots, r_n)}\right)$ is a commutative Banach algebra with identity under pointwise operations (see [11], [17]).

We now turn to the Banach algebra $\mathbb{D}$. Let X be a Banach space and let $A_j : [0, T] \to \mathcal{L}(X)$ ($\mathcal{L}(X)$ is the Banach space of bounded linear operators on X) be strongly measurable in the sense that $A_j^{-1}(V)$ is a Borel subset of $[0, T]$ for every strongly open $V \subseteq \mathcal{L}(X)$. For each $j \in \{1, \ldots, n\}$, let μ_j be a Borel probability measure on $[0, T]$ and associate μ_j to $A_j(\cdot)$. Except for the operator norms $\|A_j(\cdot)\|_{\mathcal{L}(X)}$, $j = 1, \ldots, n$, we ignore for the moment the nature of $A_j(\cdot)$ as operators (really, operator-valued functions). We make use of the operator norms $\|A_j(\cdot)\|_{\mathcal{L}(X)}$ by assuming that $A_j \in L^1([0, T], \mu_j; \mathcal{L}(X))$ and defining nonnegative real numbers $r_1, \ldots, r_n$ by

$$r_j = \int_{[0,T]} \|A_j(s)\|_{\mathcal{L}(X)} \, \mu_j(ds) \tag{2.3}$$

for $j = 1, \ldots, n$. Now introduce a commutative Banach algebra consisting of "analytic functions" $f\left(\tilde{A}_1(\cdot), \ldots, \tilde{A}_n(\cdot)\right)$ where $\tilde{A}_1(\cdot), \ldots, \tilde{A}_n(\cdot)$ are treated as purely formal commuting objects. It is natural to assume that $A_1(s_1), \ldots, A_n(s_n)$, for $s_1, \dots, s_n \in [0, T]$, are linearly independent, but we do not require them to be distinct (or even nonzero). We will also regard $\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)$ to be distinct in $\mathbb{D}$ even if there are equalities present for the actual operator-valued functions. We let $\mathbb{D}\left((A_1(\cdot), \mu_1)^{\sim}, \dots, (A_n(\cdot), \mu_n)^{\sim}\right)$, or more conveniently $\mathbb{D}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right)$, to be the collection of all expressions of the form

$$f\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right) = \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n} \tilde{A}_1(\cdot)^{m_1} \cdots \tilde{A}_n(\cdot)^{m_n} \tag{2.4}$$

where $a_{m_1,\dots,m_n} \in \mathbb{C}$ for all $m_1, \dots, m_n \in \mathbb{N} \cup \{0\}$ and

$$\begin{aligned} &\left\| f\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right) \right\| \\ &= \left\| f\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right) \right\|_{\mathbb{D}} \\ &:= \sum_{m_1,\dots,m_n=0}^{\infty} |a_{m_1,\dots,m_n}| r_1^{m_1} \cdots r_n^{m_n} < \infty \end{aligned} \tag{2.5}$$

by (2.3) and the assumption that each $A_j(\cdot) \in L^1([0, T], \mu_j; \mathcal{L}(X))$. Adding and scalar multiplying such expressions coordinatewise, it is easily seen that $\mathbb{D}$ is a vector space and that $\|\cdot\|_{\mathbb{D}}$ is a norm. The normed linear space

$$\left(\mathbb{D}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right), \|\cdot\|_{\mathbb{D}}\right)$$

can be readily identified with the weighted ℓ_1-space where the weight at the index $(m_1, \dots, m_n)$ is

$$\left(\int_{[0,T]} \|A_1(s)\|_{\mathcal{L}(X)} \, \mu_1(ds)\right)^{m_1} \cdots \left(\int_{[0,T]} \|A_n(s)\|_{\mathcal{L}(X)} \, \mu_n(ds)\right)^{m_n}.$$

It follows that $\mathbb{D}$ is a Banach space.

We can also introduce a (point-wise) product in $\mathbb{D}$ which makes $\mathbb{D}$ into a commutative and unital Banach algebra. See page 36 and Proposition 2.1.2 of [17] for details. Finally, with

$$r_j = \int_{[0,T]} \|A_j(s)\|_{\mathcal{L}(X)} \, \mu_j(ds), \tag{2.6}$$

$j = 1, \dots, n$, the Banach algebras $\mathbb{A}(r_1, \dots, r_n)$ and $\mathbb{D}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right)$ are isometrically isomorphic as unital, commutative Banach algebras. (See Proposition 2.1.3 of [17].)

2.2. The disentangling map. In this subsection, we will largely follow [17]. Take X to be a separable Banach space and consider $\mathcal{L}(X)$-valued maps $A_j : [0, T] \to \mathcal{L}(X)$, $j = 1, \dots, n$ and associated Borel probability measures μ_j, $j = 1, \dots, n$. It will be assumed throughout this subsection that each $A_j(\cdot)$ is strongly measurable in the sense that $A_j^{-1}(V)$ is a Borel set in $[0, T]$ whenever V is a strongly open subset of $\mathcal{L}(X)$. We wish to define the disentangling map

$$\mathcal{T}_{\mu_1,\dots,\mu_n} : \mathbb{D}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right) \to \mathcal{L}(X) \tag{2.7}$$

according to the time-ordering directions supplied by the measures $\mu_1, \ldots, \mu_n$. Said differently, given an analytic function $f \in \mathbb{A}(r_1, \ldots, r_n)$, where r_j is given by (2.6), we wish to form the function

$$f_{\mu_1,\ldots,\mu_n}(A_1(\cdot), \ldots, A_n(\cdot)) := \mathcal{T}_{\mu_1,\ldots,\mu_n} f\left(\tilde{A}_1(\cdot), \ldots, \tilde{A}_n(\cdot)\right)$$

of the not necessarily commuting operator-valued functions $A_1(\cdot), \ldots, A_n(\cdot)$ as directed by the time-ordering measures $\mu_1, \ldots, \mu_n$. As becomes apparent, what is a unique procedure when the operators commute is far from unique when the operators do not commute.

In what follows, the product of the measures $\mu_1, \ldots, \mu_k$ will be denoted by $\mu_1 \times \cdots \times \mu_k$. If $\mu_1 = \cdots = \mu_k = \mu$, say, we will write μ^k for the product instead. Hence the symbol $\mu_1^{m_1} \times \cdots \times \mu_k^{m_k}$ will denote the product (in that order) of m_1 copies of $\mu_1, \ldots, m_n$ copies of μ_n.

We begin by stating a measure-theoretic lemma which we shall often use, mostly without explicit mention. This result and its proof can be found in [16] as well as in [17].

Lemma 2.1. *Let I be an interval in $\mathbb{R}$ and let $\nu_1, \ldots, \nu_l$ be continuous σ-finite measures on the Borel class $\mathcal{B}(I)$ of I. Then the following sets have $\nu_1 \times \cdots \times \nu_l$-measure zero*:

(i) *The subsets of I^l where two or more coordinates are equal.*

(ii) *The subsets of I^l where one or more coordinates have a fixed value.*

Given nonnegative integers $m_1, \ldots, m_n$ we will let

$$P^{m_1,\ldots,m_n}(z_1, \ldots, z_n) := z_1^{m_1} \cdots z_n^{m_n}, \tag{2.8}$$

so that

$$P^{m_1,\ldots,m_n}\left(\tilde{A}_1(\cdot), \ldots, \tilde{A}_n(\cdot)\right) := \tilde{A}_1(\cdot)^{m_1} \cdots \tilde{A}_n(\cdot)^{m_n}. \tag{2.9}$$

We will carry out our time-ordering (or, *disentangling*) calculations in the Banach algebra $\mathbb{D}$ which will end by showing, following Feynman's ideas, how to define

$$\mathcal{T}_{\mu_1,\ldots,\mu_n} P^{m_1,\ldots,m_n}\left(\tilde{A}_1(\cdot), \ldots, \tilde{A}_n(\cdot)\right).$$

Since we will want $\mathcal{T}_{\mu_1,\ldots,\mu_n}$ to be linear and continuous, it will be clear from (2.4) how to define the operator $\mathcal{T}_{\mu_1,\ldots,\mu_n} f\left(\tilde{A}_1(\cdot), \ldots, \tilde{A}_n(\cdot)\right)$, for any $f \in \mathbb{D}$.

At this point, it may be worthwhile to review Feynman's heuristic rules since they are followed explicitly in our approach to the operational calculus, but in a mathematically rigorous way. (See [7] for the Feynman's discussion of his heuristic rules.) The first of Feynman's 'rules' is to attach time indices to the operators in question, in order to specify the order of operation in operator products. (Operators occasionally come with time indices attached, especially in evolution problems though we will not consider such problems here.) For us, the measures associated with the operators will determine the ordering of the operators and we can do this in a variety of ways. Feynman himself did not think in terms of measures, but in effect, his choice was virtually always Lebesgue measure, writing

$$A = \frac{1}{t}\int_0^t A(s)\,ds$$

where $A(s) \equiv A$ for all $s \in [0,t]$. If, more generally, μ is a probability measure on $[0,t]$, we can write

$$A = \int_{[0,t]} A(s)\,\mu(ds).$$

Feynman's next 'rule' was to form the desired function of the operators just as if they were commuting and then "disentangle" the result, that is, bring the expression to a sum of time-ordered products. The disentangling will be carried out in the commutative environment of the disentangling algebra

$$\mathbb{D}\left(\tilde{A}_1(\cdot),\dots,\tilde{A}_n(\cdot)\right)$$

and once we have the time-ordering finished, Feynman tells us to return from the commutative framework to the operators themselves. It is at this point that we will define the disentangling map.

For each $m \in \mathbb{N}$, let S_m denote the group of permutations of the integers $\{1,\dots,m\}$, and given $\pi \in S_m$, we let

$$\Delta_m(\pi) := \left\{(s_1,\dots,s_m) \in [0,T]^m \,:\, 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < T\right\}. \tag{2.10}$$

Note that (using Lemma 2.1), if our time-ordering measures $\mu_1,\dots,\mu_n$ are continuous, we can write, to a set of $\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}$-measure zero,

$$[0,T]^m = \bigcup_{\pi\in S_m} \Delta_m(\pi),$$

where the union is a disjoint union. For $j \in \{1,\dots,n\}$ and all $s \in [0,T]$, let

$$\tilde{A}_j(s) \equiv \tilde{A}_j. \tag{2.11}$$

Now, for nonnegative integers $m_1,\dots,m_n$ and with $m := m_1+\cdots+m_n$ we define

$$\tilde{C}_i(s) := \begin{cases} \tilde{A}_1(s) & \text{if } i \in \{1,\dots,m_1\}, \\ \tilde{A}_2(s) & \text{if } i \in \{m_1+1,\dots,m_1+m_2\}, \\ \quad\vdots & \\ \tilde{A}_n(s) & \text{if } i \in \{m_1+\cdots+m_{n-1}+1,\dots,m\}, \end{cases} \tag{2.12}$$

for $i = 1,\dots,m$ and $s \in [0,T]$. As the reader will notice, expressions like (2.12) will be used frequently in the sequel.

We are now ready to begin recording the time-orderings of $P^{m_1,\dots,m_n}\left(\tilde{A}_1(\cdot),\dots,\tilde{A}_n(\cdot)\right)$ in the disentangling algebra $\mathbb{D}$ when the time-ordering measures are continuous, have finitely supported discrete parts and when the time-ordering measures have arbitrary discrete parts.

First, when $\mu_1, \dots, \mu_n$ are continuous, we obtain (see [14], [17], for example)

$$P^{m_1,\dots,m_n}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right) = \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} \tilde{C}_{\pi(m)}\left(s_{\pi(m)}\right) \cdots \tilde{C}_{\pi(1)}\left(s_{\pi(1)}\right)\left(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}\right)(ds_1, \dots, ds_m). \tag{2.13}$$

Next, before we record the time-ordered monomial when our measures have finitely supported discrete parts, we need a few preliminaries. First, we need a refined version of the time-ordered sets $\Delta_m(\pi)$ given in (2.10). Let $\tau_1, \dots, \tau_h \in [0, T]$ be such that $0 < \tau_1 < \cdots < \tau_h < T$. Given $m \in \mathbb{N}$ and $\pi \in S_m$, along with nonnegative integers $\theta_1, \dots, \theta_{h+1}$ with $\theta_1 + \cdots + \theta_{h+1} = m$, we define

$$\begin{aligned}\Delta_{m;\theta_1,\dots,\theta_{h+1}}(\pi) := \big\{&(s_1, \dots, s_m) \in [0, T]^m : 0 < s_{\pi(1)} < \cdots < s_{\pi(\theta_1)} < \tau_1 < \\ &s_{\pi(\theta_1+1)} < \cdots < s_{\pi(\theta_1+\theta_2)} < \tau_2 < s_{\pi(\theta_1+\theta_2+1)} < \cdots < \\ &s_{\pi(\theta_1+\cdots+\theta_h)} < \tau_h < s_{\pi(\theta_1+\cdots+\theta_h+1)} < \cdots < s_{\pi(m)} < T\big\}.\end{aligned} \tag{2.14}$$

Now let $\mu_1, \dots, \mu_n$ be Borel probability measures on $[0, T]$ written as

$$\mu_j = \gamma_j + \eta_j$$

for $j = 1, \dots, n$, where γ_j is a continuous measure for each $j = 1, \dots, n$ and η_j is a finitely supported discrete measure for each $j = 1, \dots, n$. We let $\{\tau_1, \dots, \tau_h\}$ be the set obtained by taking the union of the supports of the discrete measures $\eta_1, \dots, \eta_n$ and write

$$\eta_l = \sum_{j=1}^{h} p_{lj} \delta_{\tau_j}$$

for each $l = 1, \dots, n$. Note that, with this notation it may well be that many of the nonnegative numbers p_{lj} are equal to zero. We compute the time-ordering of the monomial in $\mathbb{D}$ to be

$$\begin{aligned}&P^{m_1,\dots,m_n}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right) \\ &= \sum_{q_{11}+q_{12}=m_1} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left(\frac{m_1! \cdots m_n!}{q_{11}! q_{12}! \cdots q_{n1}! q_{n2}!}\right) \sum_{\pi \in S_{q_{11}+\cdots+q_{n1}}} \\ &\sum_{\theta_1+\cdots+\theta_{h+1}=q_{11}+\cdots+q_{n1}} \sum_{j_{11}+\cdots+j_{1h}=q_{12}} \cdots \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \\ &\left(\frac{q_{12}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! \cdots j_{n1}! \cdots j_{nh}!}\right) \int_{\Delta_{q_{11}+\cdots+q_{n1};\theta_1,\dots,\theta_{h+1}}(\pi)} \\ &\tilde{C}_{\pi(q_{11}+\cdots+q_{n1})}\left(s_{\pi(q_{11}+\cdots+q_{n1})}\right) \cdots \tilde{C}_{\pi(\theta_1+\cdots+\theta_h+1)}\left(s_{\pi(\theta_1+\cdots+\theta_h+1)}\right)\end{aligned}$$

$$\left\{\prod_{\beta=0}^{n-1}\left[p_{n-\beta,h}\tilde{A}_{n-\beta}\left(\tau_h\right)\right]^{j_{n-\beta,h}}\right\}\tilde{C}_{\pi(\theta_1+\cdots+\theta_h)}\left(s_{\pi(\theta_1+\cdots+\theta_h)}\right)\cdots$$
$$\tilde{C}_{\pi(\theta_1+1)}\left(s_{\pi(\theta_1+1)}\right)\left\{\prod_{\beta=0}^{n-1}\left[p_{n-\beta,1}\tilde{A}_{n-\beta}\left(\tau_1\right)\right]^{j_{n-\beta,1}}\right\}\tilde{C}_{\pi(\theta_1)}\left(s_{\pi(\theta_1)}\right) \tag{2.15}$$
$$\cdots\tilde{C}_{\pi(1)}\left(s_{\pi(1)}\right)\left(\gamma_1^{q_{11}}\times\cdots\times\gamma_n^{q_{n1}}\right)\left(ds_1,\ldots,ds_{q_{11}+\cdots+q_{n1}}\right)$$

where we take

$$\prod_{\beta=0}^{n-1}a_{n-\beta} := a_n a_{n-1}\cdots a_1$$

in this order. (See [18], [17].)

Finally, we record the time-ordering of the monomial

$$P^{m_1,\ldots,m_n}\left(\tilde{A}_1(\cdot),\ldots,\tilde{A}_n(\cdot)\right)$$

when the time-ordering measures are arbitrary Borel probability measures on $[0,T]$. (Details can be found in [27].) It is in this setting that the time-ordering becomes quite complicated to carry out. Just below we sketch out some details of how we carry out the time-ordering of the monomial. Recall that we use strongly measurable $\mathcal{L}(X)$-valued maps $A_j : [0,T] \to \mathcal{L}(X)$ for $j = 1,\ldots,n$ and, to each $A_j(\cdot)$, we associate the Borel probability measure μ_j on $[0,T]$,

As above, we use the unique decomposition

$$\mu_j = \gamma_j + \eta_j$$

where γ_j is a continuous measure on $[0,T]$ and where η_j is purely discrete with arbitrary support in $[0,T]$. (See, for instance, [4].) For each $j = 1,\ldots,n$, write

$$\eta_j = \sum_{p=1}^{\infty}\omega_{p;j}\delta_{t_{p;j}}$$

where $\{t_{p;j}\}_{p=1}^{\infty}$ is a sequence from $[0,T]$ and $\{\omega_{p;j}\}_{p=1}^{\infty}$ is a sequence of real numbers such that

$$\|\eta_j\| = \sum_{p=1}^{\infty}|\omega_{p;j}| < \infty.$$

We will assume that

$$R_j := \int_{[0,T]}\|A_j(s)\|_{\mathcal{L}(X)}\mu_j(ds) < \infty$$

for each $j = 1,\ldots,n$. With the nonnegative real numbers $R_1,\ldots,R_n$ in hand, we construct the commutative Banach algebra $\mathbb{A}\left(R_1,\ldots,R_n\right)$ and the associated disentangling algebra $\mathbb{D}\left(\tilde{A}_1(\cdot),\ldots,\tilde{A}_n(\cdot)\right)$.

Let $m_1, \dots, m_n$ be positive integers and consider the monomial

$$
\begin{aligned}
&\left(\int_{[0,T]} \tilde{A}_1(s)\,\mu_1(ds)\right)^{m_1} \cdots \left(\int_{[0,T]} \tilde{A}_n(s)\,\mu_n(ds)\right)^{m_n} \\
&= \left(\int_{[0,T]} \tilde{A}_1(s)\,\gamma_1(ds) + \sum_{p=0}^{\infty} \omega_{p;1}\tilde{A}_1(t_{p;1})\right)^{m_1} \cdots \qquad (2.16)\\
&\quad \left(\int_{[0,T]} \tilde{A}_n(s)\,\gamma_n(ds) + \sum_{p=0}^{\infty} \omega_{p;n}\tilde{A}_n(t_{p;n})\right)^{m_n}.
\end{aligned}
$$

What is done at this point is to apply the ordinary binomial expansion followed by using the "$\aleph_0$-nomial" formula (see [15, p. 41])

$$
\left(\sum_{p=0}^{\infty} b_p\right)^q = \sum_{h=1}^{\infty} \sum_{\substack{u_0+\cdots+u_h=q \\ u_h \neq 0}} \frac{q!}{u_0! \cdots u_h!} b_0^{u_0} \cdots b_h^{u_h}.
$$

Of course, we have one of these sums for each factor in (2.16); label the h used in each factor as h_j for $j = 1, \dots, n$ and, correspondingly, label the $u_0, \dots, u_h$ as $u_{0,j}, \dots, u_{h_j,j}$ for $j = 1, \dots, n$. Now note that, in the $\aleph_0$-nomial formula above, we only take the b_p's in finite sets for each h. (See [27, pp. 53-57] for details.) Given $h_1, \dots, h_j \in \mathbb{N}$ and fixed $u_{0,1}, \dots, u_{h_1,1}, \dots, u_{0,n}, \dots, u_{h_n,n}$, we have finite sequences $\{t_{p_1;1}\}_{p_1=0}^{h_1}, \dots, \{t_{p_n;n}\}_{p_n=0}^{h_n}$ of time indices. Note that these time indices "come with" corresponding coefficients of the discrete measures involved. We relabel these time indices as well as the $u_{i,j}$ and the $\omega_{i,j}$ as

$$
\begin{aligned}
&\tau_0 := t_{0;1}, \dots, \tau_{h_1} := t_{h_1;1}, \dots, \tau_{h_1+\cdots+h_{n-1}+n+-1} := t_{0;n}, \dots, \\
&\quad \tau_{h_1+\cdots+h_n+n} := t_{h_n;n},
\end{aligned}
$$

and

$$
\begin{aligned}
&v_0 := u_{0,1}, \dots, v_{h_1} := u_{h_1,1}, \dots, v_{h_1+\cdots+h_{n-1}+n-1} := u_{0,n}, \dots, \\
&\quad v_{h_1+\cdots+h_n+n} := u_{h_n,n},
\end{aligned}
$$

and finally

$$
\begin{aligned}
&a_0 := \omega_{0;1}, \dots, a_{h_1} := \omega_{h_1;1}, \dots, a_{h_1+\cdots+h_{n-1}+n-1} := \omega_{0;n}, \dots, \\
&\quad a_{h_1+\cdots+h_n+n} := \omega_{h_n;n}.
\end{aligned}
$$

Using the τ's, we note that our finite sequences of time indices allow us to choose a permutation σ of $\{0, \dots, h_1 + \cdots + h_n + n\}$ for which

$$
\tau_{\sigma(0)} \leq \cdots \leq \tau_{\sigma(h_1+\cdots+h_n+n)}.
$$

If all of the τ's are distinct, the permutation σ is unique.

With

$$\tilde{A}^{\sigma}(s) := \begin{cases} 1 & \text{if } s \notin \operatorname{supp}(\eta_j), \\ \tilde{A}_j(s) & \text{if } s \in \operatorname{supp}(\eta_j), \end{cases} \tag{2.17}$$

and with

$$a_j^{\sigma}(s) := \begin{cases} 1 & \text{if } s \notin \operatorname{supp}(\eta_j), \\ a_j & \text{if } s \in \operatorname{supp}(\eta_j), \end{cases} \tag{2.18}$$

we can write the time-ordered monomial in $\mathbb{D}$ as

$$P^{m_1,\dots,m_n}\left(\tilde{A}_1(\cdot),\dots,\tilde{A}_n(\cdot)\right) \tag{2.19}$$

$$= \sum_{q_{11}+q_{12}=m_1} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left[\frac{m_1!\cdots m_n!}{q_{11}!q_{12}!\cdots q_{n1}!q_{n2}!}\right] \sum_{h_1,\dots,h_n=0}^{\infty} \sum_{\substack{v_0+\cdots+v_{h_1}=q_{12} \\ v_{h_1}\neq 0}}$$

$$\cdots \sum_{\substack{v_{h_1+\cdots h_{n-1}+n-1}+\cdots+v_{h+n}=q_{n2} \\ v_{h+n}\neq 0}} \left[\frac{q_{12}!\cdots q_{n2}!}{v_0!\cdots v_{h+n}!}\right] \sum_{\pi\in S_q} \sum_{\theta_1+\cdots+\theta_{h+n+1}=q}$$

$$\int_{\Delta_{q;\theta_1,\dots,\theta_{h+n+1}}(\pi)} \tilde{C}_{\pi(q)}\left(s_{\pi(q)}\right)\cdots \tilde{C}_{\pi(\theta_1+\cdots+\theta_{h+n}+1)}\left(s_{\pi(\theta_1+\cdots+\theta_{h+n}+1)}\right)$$

$$\left[a^{\sigma}_{\sigma(h+n)}\left(\tau_{\sigma(h+n)}\right)\tilde{A}^{\sigma}\left(\tau_{\sigma(h+n)}\right)\right]^{v_{\sigma(h+n)}} \tilde{C}_{\pi(\theta_1+\cdots+\theta_{h+n})}\left(s_{\pi(\theta_1+\cdots+\theta_{h+n})}\right)\cdots$$

$$\tilde{C}_{\pi(\theta_1+1)}\left(s_{\pi(\theta_1+1)}\right)\left[a^{\sigma}_{\sigma(0)}\left(\tau_{\sigma(0)}\right)\tilde{A}^{\sigma}\left(\tau_{\sigma(0)}\right)\right]^{v_{\sigma(0)}} \tilde{C}_{\pi(\theta_1)}\left(s_{\pi(\theta_1)}\right)\cdots$$

$$\tilde{C}_{\pi(1)}\left(s_{\pi(1)}\right)\left(\gamma_1^{q_{11}}\times\cdots\times\gamma_n^{q_{n1}}\right)(ds_1,\dots,ds_q),$$

where $q := q_{11}+\cdots+q_{n1}$ and $h := h_1+\cdots+h_n$.

Remark 2.2. The reader likely notices that the evaluations at the support points of the discrete measures are written quite differently in (2.19) than in (2.15). In particular, it appears that, in (2.19) we have evaluation at only one of the operators (really, formal objects here). However, it may well be that several of the $\tau_{\sigma(j)}$'s may be equal and, in this case, we would obtain a time-ordering that would more closely resemble (2.15).

In each case, that is, when the time-ordering measures are continuous, when the time-ordering measures have finitely supported discrete parts and when the time-ordering measures are arbitrary, we define the disentangling map

$$\mathcal{T}_{\mu_1,\dots,\mu_n} : \mathbb{D} \to \mathcal{L}(X)$$

by effectively "erasing the tildes" in the time-ordering of the monomials and then, given $f \in \mathbb{D}$ written as

$$f\left(\tilde{A}_1(\cdot),\dots,\tilde{A}_n(\cdot)\right) = \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n} P^{m_1,\dots,m_n}\left(\tilde{A}_1(\cdot),\dots,\tilde{A}_n(\cdot)\right),$$

define

$$\begin{aligned} & f_{\mu_1,\ldots,\mu_n}\left(\tilde{A}_1(\cdot),\ldots,\tilde{A}_n(\cdot)\right) \\ & := \mathcal{T}_{\mu_1,\ldots,\mu_n} f\left(\tilde{A}_1(\cdot),\ldots,\tilde{A}_n(\cdot)\right) \\ & \quad = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}\left(\tilde{A}_1(\cdot),\ldots,\tilde{A}_n(\cdot)\right), \end{aligned} \tag{2.20}$$

where

$$P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}\left(A_1(\cdot),\ldots,A_n(\cdot)\right) := \mathcal{T}_{\mu_1,\ldots,\mu_n} P^{m_1,\ldots,m_n}\left(\tilde{A}_1(\cdot),\ldots,\tilde{A}_n(\cdot)\right). \tag{2.21}$$

With this definition in hand, one shows that the disentangling map $\mathcal{T}_{\mu_1,\ldots,\mu_n}$ is a bounded linear map. See Proposition 6.1.9 of [17] for the continuous measure case, Theorem 8.3.2 of [17] for the case where the time-ordering measures have finitely supported discrete parts and Proposition 30 of [27] for the case where the time-ordering measures are arbitrary.

3. Brief background on topological algebras

We now present a brief outline of the basic facts about topological algebras. (More to the point, we are interested in the basic facts of the algebras we will consider in the current paper.) We will follow the notation and terminology of the monograph *Topological Algebras with Involution* by M. Fragoulopoulou ([8]).

Algebraically, we are working with a vector space E which is endowed with the extra structure of a ring. As is usual, we will refer to E as an algebra. It will be assumed that E is unital throughout.

Definition 3.1. Let E be an algebra. A seminorm ρ on E compatible with the multiplication in E in the sense that

$$\rho(xy) \leq \rho(x)\rho(y)$$

for all $x, y \in E$ (we say that ρ is *submultiplicative*) is called a m-seminorm. An m-norm is defined similarly and we refer to E as a *normed algebra* if there is a m-norm on E (if complete, E is a Banach algebra).

We also note:

Definition 3.2. Let E be an algebra. A subset U of E is called *multiplicative* if $UU \subseteq U$.

If p is a seminorm on E, the unit semiball $U_p(1)$ corresponding to p defined by

$$U_p(1) := \{x \in E : p(x) \leq 1\}$$

is multiplicative. Moreover, $U_p(1)$ is absolutely convex and absorbing. Recall that, given an absolutely convex absorbing and multiplicative set V in E, the *gauge* functional is defined by

$$p_V(x) := \inf\{\lambda > 0 : x \in \lambda V\}.$$

We have the following proposition concerning the *gauge* or *Minkowski* functional.

Proposition 3.3. *Let E be an algebra and let $\rho : E \to \mathbb{R}$. The following are equivalent*:

(*1*) *ρ is an m-seminorm.*

(*2*) *$\rho = \rho_U$ with U an absorbing absolutely convex and multiplicative subset of E where $\rho_U(x) = \inf\{\lambda > 0 : x \in \lambda U\}$.*

We now state a basic definition.

Definition 3.4. A topological vector space (TVS) which is also an algebra such that multiplication is separately continuous is called a *topological algebra*. A *locally convex algebra* is a topological algebra whose underlying TVS is a locally convex space. A topological algebra whose underlying TVS is metrizable and complete is called a *Frechet topological algebra*.

The topology of a locally convex algebra is defined by a fundamental zero-neighborhood system consisting of closed absolutely convex sets. Equivalently, the topology is determined by a family of nonzero seminorms. This family of seminorms will typically be denoted by Γ (or Γ_E if we need to refer to the algebra E) and, without loss of generality, will always be assumed to be *saturated*; i.e. for any finite subset $\mathcal{F}$ of Γ the seminorm

$$p(x) = \max_{p \in \mathcal{F}} p(x),\ x \in E,$$

is an element of Γ. Given such a defining family Γ of seminorms for the topology, a fundamental system of basic neighborhoods of zero is given by the sets

$$U_p(\epsilon) := \{x \in E : p(x) \leq \epsilon\}$$

for $p \in \Gamma$ and $\epsilon > 0$.

Taking the completion of a topological algebra E (that is, taking the completion of the underlying TVS) may not result in a topological algebra unless the multiplication is jointly continuous. However, we will be assuming that the topological algebras considered in this paper are complete. (We will also be assuming that our topological algebras are Hausdorff or, equivalently, that Γ is a separating family of seminorms.)

Next, if (E, Γ_E) and (F, Γ_F) are locally convex algebras, a linear map $T : E \to F$ is continuous if and only if T is continuous at zero; i.e. for every $q \in \Gamma_F$ there is a $\rho \in \Gamma_E$ and a $C > 0$ such that

$$q(Tx) \leq C\rho(x)$$

for all $x \in E$.

The main definition for our purposes is the following.

Definition 3.5. A topological algebra (E, Γ_E), where Γ_E consists of m-seminorms, is called a *locally m-convex algebra*. A complete locally m-convex algebra is called an Arens-Michael algebra.

4. Measurability and summability

We discuss in this section the measurability/summability definitions and properties from E. Thomas' paper [31] which supplies much of the structure we will use. The definitions and theorems we will record here are for functions taking values in a locally convex topological space E over $\mathbb{C}$. We will assume here that E is Hausdorff and quasi-complete (all closed and bounded subsets of E are complete). Let S be a metric space and denote by $\mathcal{B}(S)$ the Borel class of S. We also let $\mathcal{K}(S)$ denote the family of compact subsets of S.

Let μ be a Radon measure on S; i.e. let μ be a measure which satisfies

$$(1)\, \mu(A) = \sup\{\mu(K) : K \subseteq A, K \in \mathcal{K}(S)\}\ (A \in \mathcal{B}(S))$$

and

$$(2)\, S \text{ is the union of open sets of finite } \mu\text{-measure.}$$

(See also [1, Vol. 2, p. 68].)

Remark 4.1. (1) It will be assumed that S is, in fact, σ-finite.
(2) Denote by B^μ the Lebesgue completion (see [1, Vol. 1, p. 18]) of $\mathcal{B}(S)$ with respect to μ which is extended to B^μ in the standard fashion. As is well-known, B^μ is the family of "μ-measurable sets." In the main part of this paper, our measures will be complete.

We now state some definitions and theorems from Thomas' paper [31] which will be important in this paper.

Definition 4.2. We say that $C \subseteq \mathcal{K}(S)$ is μ-dense when the following conditions are satisfied:
(1) If $K_1, K_2 \in C$, then $K_1 \cup K_2 \in C$;
(2) If $K' \subseteq K$ for $K \in C$ and if $K' \in \mathcal{K}(S)$, then $K' \in C$;
(3) For all $K \in \mathcal{K}(S)$ and every $\epsilon > 0$, there is a $K' \in C$ with $K' \subseteq K$ and $\mu\left(K \backslash K'\right) \le \epsilon$.

We let C_f be the set of all $K \in \mathcal{K}(S)$ for which $f\big|_K$ is continuous (where $f : S \to \mathbb{C}$). Then C_f possesses properties (1) and (2) of Definition 4.2. It is easy to see that f is μ-measurable if and only if C_f is μ-dense; i.e. $f^{-1}(V) \in B^\mu$ for every $V \in \mathcal{B}(S)$ if and only if C_f is μ-dense.

Remark 4.3. We observe, under part (1) of the definition of Radon measure and under the assumption that S is σ-finite, that (1), (2) and (3) of Definition 4.2 are equivalent to properties (1), (2) and

(4) Every $A \in B^\mu$ has a partition $A = N \cup \left[\bigcup_{n=1}^\infty K_n\right]$ with $K_n \in C$ for every $n \in \mathbb{N}$ and $\mu(N) = 0$.

Next, on page 120 of [31] we find the following proposition.

Proposition 4.4. *Let $f : S \to \mathbb{C}$ be μ-measurable and let $C \subseteq C_f$ be any μ-dense class. The following conditions are equivalent*:

(1) $f \in \mathscr{L}^1(\mu)$ *($\mathscr{L}^1(\mu)$ consists of functions, not equivalence classes).*
(2) The net defined by $K \in C \mapsto \int_K f\, d\mu$ *converges. (C is a directed set); i.e.,*

$$\lim_{K \in C} \int_K f\, d\mu$$

exists.
(3) $\sum_{n=1}^{\infty} \int_{K_n} f\, d\mu$ *converges for any countable disjoint family* $\{K_n\}_{n=1}^{\infty}$ *from* C.

We now introduce summable functions. Let E be a locally convex topological vector space over $\mathbb{C}$ and assume that E is Hausdorff and quasi-complete. We first define, for $f : S \to E$, what it means for f to be μ-measurable.

Definition 4.5. We define $f : S \to E$ to be μ-measurable if C_f is μ-dense.

We note the following properties:

(1) The sum of μ-measurable functions f, g is again μ-measurable since $C_f \cap C_g \subseteq C_{f+g}$.
(2) The product of a μ-measurable E-valued function f and a μ-measurable scalar-valued function ρ is μ-measurable since $C_f \cap C_\rho \subseteq C_{\rho f}$.
(3) If E is a Banach space, μ-measurability is the same as strong measurability in the sense of Bochner. (For strong measurability in the sense of Bochner, see for instance [9, Section 1.1].)

Before we get to the next proposition seen in [31], we very briefly remind the reader of the Pettis integral. A vector function $\psi : S \to E$ is called *Pettis integrable* if for every $\ell \in E'$ (the dual of E) the function $\langle \ell, \psi \rangle$ is integrable and its integral $\ell(m)$, where $m \in E$ does not depend on ℓ; then m is called the Pettis integral of ψ. The proposition of interest is Proposition 2, p. 121 of [31].

Proposition 4.6. *Let* $f : S \to E$ *be μ-measurable and let* $C \subseteq C_f$ *be any μ-dense class. The following are equivalent*:

(1) $\lim_{K \in C} \int_K f\, d\mu$ *exists.*
(2) *If* $\{K_n\}_{n=1}^{\infty}$ *is a disjoint family from* C*, then* $\sum_{n=1}^{\infty} \int_{K_n} f\, d\mu$ *converges unconditionally.*
(3) f *is Pettis integrable.*

Note that if one (and so all) of the conditions of Proposition 4.6 are satisfied, then

$$\int_S f\, d\mu = \lim_{K \in C} \int_K f\, d\mu \text{ and } \int_A f\, d\mu = \lim_{\substack{K \subseteq A \\ K \in C}} \int_K f\, d\mu.$$

With this proposition in hand, we state the following definition.

Definition 4.7. We will say that a μ-measurable $f : S \to E$ which is Pettis integrable is *μ-summable*.

We now take the time to define the seminorms which will play a crucial role in what follows. Let $\beta \subseteq E$ be a closed and absolutely convex set. Define

$$|x|_\beta := \inf \{\lambda : x \in \lambda \beta\}.$$

As is well-known, $x \mapsto |x|_\beta$ is a lower semicontinuous, positively homogeneous symmetric function into $[0, \infty]$. This map is a continuous seminorm if β is a neighborhood of zero. It follows that, if β is a neighborhood of zero and $f : S \to E$ is μ-measurable, $s \mapsto |f(s)|_\beta$ is μ-measurable.

The next proposition, Proposition 3 of [31] gives us an inequality we use frequently in the sequel.

Proposition 4.8. *Suppose that $f : S \to E$ is μ-summable. Then*

$$\left| \int_A f \, d\mu \right|_\beta \leq \int_A |f|_\beta \, d\mu. \tag{4.1}$$

We now move on to the definition of totally summable functions, namely, the functions which we will use in our development of Feynman's calculus on E. Let $B \subseteq E$ be closed, bounded and absolutely convex in E.

Lemma 4.9. (*a*) *Define*

$$E_B := \{x \in E : |x|_B < \infty\} = \bigcup_{\lambda > 0} \lambda B. \tag{4.2}$$

Then E_B is a linear subspace of E.

(*b*) *The map $x \mapsto |x|_B$ is a norm on E_B. With this norm, E_B is a Banach space.* (*See* [2, p. 119], *for instance.*)

(*c*) *The inclusion $i : E_B \hookrightarrow E$ is continuous.*

The next definition (see page 123 of [31]) plays a prominent role in the current paper.

Definition 4.10. A μ-measurable $f : S \to E$ is said to be *totally summable* if there is a closed, bounded and absolutely convex subset B of E for which

$$\int_S |f|_B \, d\mu < \infty. \tag{4.3}$$

For completeness, we state Proposition 4 of [31], which states an expected relation between "totally summable" and "summable."

Proposition 4.11. *Every totally summable function is summable.*

Without stating a formal result, we make the observation that because

$$\left| \int_A f \, d\mu \right|_B \leq \int_A |f|_B \, d\mu < \infty,$$

we have

$$\int_A f \, d\mu \in E_B$$

for all $A \in B^\mu$. A rather standard argument then shows that $A \mapsto \int_A f \, d\mu$ is countably additive in E_B.

We will denote by $\mathscr{L}^1(\mu; E_B; E)$ the family of μ-measurable functions $f : S \to E$ for which (4.3) holds. We also let $L^1(\mu; E_B; E)$ the set of equivalence classes of functions equal μ-a.e.

Remark 4.12. If E is a Banach space, a function f is totally summable if and only if f is Bochner integrable.

Note that for $f \in L^1(\mu; E_B; E)$, (4.3) tells us that $|f|_B$ is finite μ-a.e. on S and so $f(s) \in E_B$ μ-a.e. We can characterize totally summable functions as follows (Proposition 5 of [31]).

Proposition 4.13. *A function $f : S \to E$ is totally summable if and only if there is a $\rho : E \to \mathbb{C}$, $\rho \in L^1(\mu)$, and a $g : S \to E$ which is bounded and μ-measurable such that $f(s) = \rho(s)g(s)$, μ-a.e.*

We find a remark on page 123 of [31] which will be very helpful to us.

Remark 4.14. If $\mu(S) < \infty$, Proposition 4.13 tells us that every bounded μ-measurable function is totally μ-summable.

To finish our outline of definitions and results from [31], we state a dominated convergence theorem for $\mathscr{L}^1$ and two corollaries. Theorem 4.15 is Theorem 1 on page 124 of [31] and the corollaries are Corollary 1 and Corollary 2 on page 124 of [31]. (It is worth noting that Theorem 4.15 may remain valid when the dominated convergence is replaced by pointwise convergence combined with uniform integrability, but this will not be investigated here.)

Theorem 4.15. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence from $\mathscr{L}^1(\mu; E_B; E)$ be such that $f_n \to f$ in E_B μ-a.e. and suppose that there is a scalar-valued function $g \in \mathscr{L}^1(\mu)$ with $|f_n|_B \le g$ μ-a.e. (Note that we are taking $g \ge 0$.)* Then $f \in \mathscr{L}^1(\mu; E_B; E)$ and

$$\lim_{n\to\infty} \int_S |f_n - f|_B \, d\mu = 0; \tag{4.4}$$

in particular

$$\lim_{n\to\infty} \int_S f_n \, d\mu = \int_S f \, d\mu \tag{4.5}$$

in E_B.

Corollary 4.16. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence from $\mathscr{L}^1(\mu; E_B; E)$ for which*

$$\sum_{n=1}^{\infty} \int_S |f_n|_B \, d\mu < \infty,$$

then

$$f(s) := \sum_{n=1}^{\infty} f_n(s)$$

exists μ-a.e., $f \in \mathscr{L}^1(\mu; E_B; E)$ and

$$\int_S f \, d\mu = \sum_{n=1}^{\infty} \int_S f_n \, d\mu.$$

Corollary 4.17. *$L^1(\mu; E_B; E)$ is a Banach space.*

We next turn to the Fubini theorem that was stated in [31]. Of course, a Fubini theorem is a necessity for Feynman's calculus. As stated in [31] the theorem is Theorem 2.

Theorem 4.18. *Let μ be a Radon measure on S and let ν be a Radon measure on T where S and T are metric spaces. There is a unique Radon measure τ on $S \times T$ for which $\tau(U \times V) = \mu(U)\nu(V)$ for all $U \in \mathcal{B}(S)$ and $V \in \mathcal{B}(T)$. (If we assume that μ and ν are σ-finite, then τ is also σ-finite.) If $f : S \times T \to E$ is totally τ-summable, then $s \to f(s,t)$ is totally μ-summable for almost all $t \in T$ and $s \to \int_T f(s,t)\, d\nu(t)$ is totally μ-summable and*

$$\int_{S\times T} f\, d(\mu \times \nu) = \int_S \int_T f(s,t)\, d\nu d\mu.$$

5. Developing Feynman's operational calculus on topological algebras

Throughout what follows, we take E to be a topological algebra. We will assume that E is a locally convex topological algebra whose topology is determined by a (saturated) family Γ_E of m-seminorms. (See Section 3 above.) Moreover, we will take the algebra E to be unital, Hausdorff and complete (so E is an Arens-Michael algebra).

The first step in developing Feynman's operational calculus on E is to look at products of measurable functions with values in E. To this end, let μ and ν be Borel probability measures on $[0,T]$. Since $[0,T]$ is separable and complete, μ and ν are Radon measures. It is also the case that μ and ν are tight (see [1, Vol. 2, pp. 69, 70].

Now suppose that $f : [0,T] \to E$ is μ-measurable and that $g : [0,T] \to E$ is ν-measurable. (See Definition 4.5.) Then C_f and C_g are, respectively, μ-dense and ν-dense. We define

$$F : [0,T] \times [0,T] \to E \times E$$

by

$$F(s,t) = (f(s), g(t))\,.$$

It is easy to see that $C_F = C_f \times C_g$. Now let $K_1, K_2 \subseteq [0,T]$ be compact. Given $\epsilon > 0$, there is a $K_1' \in C_f$ for which $\mu\left(K_1 \backslash K_1'\right) < \epsilon/2$ and there is a $K_2' \in C_g$ for which $\nu\left(K_2 \backslash K_2'\right) < \epsilon/2$. We then have, using $K := K_1 \times K_2$ and $K' := K_1' \times K_2'$,

$$\begin{aligned}(\mu \times \nu)\left(K \backslash K'\right) &= \mu\left(K_1 \backslash K_1'\right)\nu\left(K_2\right) + \mu\left(K_1\right)\nu\left(K_2 \backslash K_2'\right)\\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

So F is $\mu \times \nu$-measurable into $E \times E$. (Note that we do not need the algebraic structure of E here, just the underlying TVS structure.) Now, denote by $\mathfrak{m}$ the multiplication on E. From Proposition 1.6 of [19], $\mathfrak{m}$ is jointly continuous (as opposed to separately continuous). Hence the composition $\mathfrak{m} \circ F$ is $\mu \times \nu$-measurable. We have established:

Proposition 5.1. *Under the assumptions and notation above, if $f : [0,T] \to E$ and $g : [0,T] \to E$ are μ-measurable and ν-measurable respectively, then the product $f(s)g(t)$ is $\mu \times \nu$-measurable.*

We now assume that the μ-measurable function $f : [0,T] \to E$ is totally μ-summable (see Definition 4.10) and that the ν-measurable function $g : [0,T] \to E$ is totally ν-summable. There are then closed, bounded and absolutely convex $B_f, B_g \subseteq E$ for which

$$\int_{[0,T]} |f(s)|_{B_f} \mu(ds) < \infty \text{ and } \int_{[0,T]} |g(s)|_{B_g} \nu(ds) < \infty. \tag{5.1}$$

Define $F : [0,T]^2 \to E$ by

$$F(s,t) = f(s)g(t). \tag{5.2}$$

From Proposition 5.1, F is $\mu \times \nu$-measurable. Define $B_F \subseteq E$ to be the closed absolutely convex hull of $B_f \cup B_g$. Then B_F is also bounded and $B_f \cup B_g \subseteq B_F$ and so $|\cdot|_{B_F} \le |\cdot|_{B_f}$ and $|\cdot|_{B_F} \le |\cdot|_{B_g}$. It follows from part (4) of Lemma 5.50 in [3] that, because B_F is closed, absolutely convex and bounded, $|\cdot|_{B_F}$ is a continuous seminorm and so, by Proposition 1.4.11 of [2], $|\cdot|_{B_F}$ has bounded image in $[0,\infty)$. (We will apply this discussion a number of times below, mostly without comment.) Because μ and ν are finite measures on $\mathcal{B}([0,T])$, it then follows that

$$\int_{[0,T]^2} |F(s,t)|_{B_F} (\mu \times \nu)(ds, dt) < \infty.$$

Hence F is $\mu\times\nu$-totally summable and Theorem 2 of [31] (Theorem 4.18 above) tells us that

$$\int_{[0,T]^2} F\, d(\mu \times \nu) = \int_{[0,T]} \int_{[0,T]} F\, d\mu d\nu = \int_{[0,T]} \int_{[0,T]} F\, d\nu d\mu.$$

We can now proceed with the necessary constructions to prepare for Feynman's operational calculus in the topological algebra E.

For each $j \in \{1, \dots, n\}$, let μ_j be a Borel probability measure on $[0,T]$. Also, for each $j \in \{1, \dots, n\}$, let $A_j : [0,T] \to E$ be μ_j-measurable and assume that $A_j(\cdot)$ is totally μ_j-summable. Under this assumption, for each $j \in \{1, \dots, n\}$ there is a closed, bounded and absolutely convex set $B_j \subseteq E$ for which

$$\int_{[0,T]} |A_j(s)|_{B_j} \mu_j(ds) < \infty.$$

Remark 5.2. We can, instead of simply assuming that each $A_j(\cdot)$ is totally μ_j-summable, assume instead that each $A_j(\cdot)$ is μ_j-measurable and that $A_j([0,T])$ is bounded in E. Since each μ_j is finite, it follows that $A_j(\cdot)$ is then totally μ_j-summable ([31, p. 123]).

Define B_0 to be the closed, absolutely convex hull of $B_1 \cup \cdots \cup B_n$. Since $B_j \subseteq B_0$ for every $j = 1, \dots, n$ we have $|\cdot|_{B_0} \leq |\cdot|_{B_j}$ for each j. (As we've seen above, $|\cdot|_{B_0}$ is a continuous seminorm.) It follows that

$$\int_{[0,T]} |A_j(s)|_{B_0} \mu_j(ds) \leq \int_{[0,T]} |A_j(s)|_{B_j} \mu_j(ds) < \infty$$

for every $j \in \{1, \dots, n\}$. We will therefore take $A_j \in L^1\left(\mu_j; E_{B_0}; E\right)$. For each $j \in \{1, \dots, n\}$, define

$$R_j := \int_{[0,T]} |A_j(s)|_{B_0} \mu_j(ds) \tag{5.3}$$

and construct the commutative Banach algebra (see Section 2) $\mathbb{A}_{B_0}(R_1, \dots, R_n)$ (where we've indexed $\mathbb{A}$ with B_0 as the algebra clearly depends on B_0). With the algebra $\mathbb{A}_{B_0}$ in hand, construct the associated commutative Banach algebra $\mathbb{D}_{B_0}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right)$ (the *disentangling algebra*; see Section 2). It is at this stage that we are prepared to carry out the time-ordering calculations required by Feynman's 'rules.' As noted previously, the time-ordering calculations depend very much on the type of time-ordering measures we have. The most straightforward case occurs when we have continuous time-ordering measures (see [11, 10, 12, 13], [21], [14], [17]). When our time-ordering measures have non-zero discrete parts, the calculations become much more involved, with the most difficult case being the general case where the time-ordering measures are allowed to have arbitrary discrete parts (See [18], [17, Chapter 8], [27].) All of this being said, we will use continuous time-ordering measures in the explicit calculations below. This is for brevity but we will make some comments along the way concerning how more general measures change affect calculations and conclusions.

Given $f \in \mathbb{D}_{B_0}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right)$ written as

$$f\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right) = \sum_{m_1, \dots, m_n = 0}^{\infty} a_{m_1, \dots, m_n} P^{m_1, \dots, m_n}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right), \tag{5.4}$$

we obtained a disentangled element of the topological algebra E by first carrying out time-ordering calculations in $\mathbb{D}_{B_0}$. We will begin by carrying out the time-ordering of the monomial $P^{m_1, \dots, m_n}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right)$ in $\mathbb{D}_{B_0}$. Because we will be interested in showing that the disentangling map is continuous, we will need to compute a norm estimate (in E_{B_0}) for

$$\mathcal{T}_{\mu_1, \dots, \mu_n} P^{m_1, \dots, m_n}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right)$$

using the norm $|\cdot|_{B_0}$. However, because each $A_j(\cdot)$ is totally μ_j-summable for each $j = 1, \dots, n$, $|A_j(s)|_{B_0}$ is finite only μ_j-a.e. in $[0, T]$ (this is different from using $\mathcal{L}(X)$-valued functions where $\|A_j(s)\|_{\mathcal{L}(X)} < \infty$ for all $s \in [0, T]$). We must carry out the time-ordering with this restriction in mind. To this end, for each $j \in \{1, \dots, n\}$, there is a Borel $U_j \subseteq [0, T]$ with $\mu_j\left(U_j\right) = 0$ and $|A_j(s)|_{B_0} < \infty$ for all $s \in [0, T] \backslash U_j$. We can now carry out the time-ordering in the disentangling algebra $\mathbb{D}_{B_0}$ using the sets $[0,T]\backslash U_1, \dots, [0,T]\backslash U_n$. We compute, successively,

$$
\begin{aligned}
&P^{m_1,\dots,m_n}\left(\tilde{A}_1(\cdot),\dots,\tilde{A}_n(\cdot)\right) \\
&= \left(\int_{[0,T]} \tilde{A}_1(s)\,\mu_1(ds)\right)^{m_1} \cdots \left(\int_{[0,T]} \tilde{A}_n(s)\,\mu_n(ds)\right)^{m_n}
\end{aligned} \tag{5.5}
$$

$$
\begin{aligned}
&= \left(\int_{[0,T]\backslash U_1} \tilde{A}_1(s)\,\mu_1(ds)\right)^{m_1} \cdots \left(\int_{[0,T]\backslash U_n} \tilde{A}_n(s)\,\mu_n(ds)\right)^{m_n} \\
&= \left\{\int_{([0,T]\backslash U_1)^{m_1}} \tilde{A}_1\left(s_{1,m_1}\right)\cdots\tilde{A}_1\left(s_{1,1}\right)\mu_1^{m_1}\left(ds_{1,1},\dots,ds_{1,m_1}\right)\right\} \\
&\cdots\left\{\int_{([0,T]\backslash U_n)^{m_n}} \tilde{A}_n\left(s_{n,m_n}\right)\cdots\tilde{A}_n\left(s_{n,1}\right)\mu_n^{m_n}\left(ds_{n,1},\dots,ds_{n,m_n}\right)\right\} \\
&= \int_{([0,T]\backslash U_1)^{m_1}\times\cdots\times([0,T]\backslash U_n)^{m_n}} \tilde{A}_1\left(s_1\right)\cdots\tilde{A}_1\left(s_{m_1}\right)\tilde{A}_2\left(s_{m_1+1}\right)\cdots\tilde{A}_2\left(s_{m_1+m_2}\right) \\
&\cdots\tilde{A}_n\left(s_{m_1+\cdots+m_{n-1}+1}\right)\cdots\tilde{A}_n\left(s_m\right)\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)\left(ds_1,\dots,ds_m\right) \\
&=: J_{m_1,\dots,m_n},
\end{aligned}
$$

where $m := m_1 + \cdots + m_n$ and where, after the second to last equality above, we've relabeled the time indices. To continue we define, given $\pi \in S_m$,

$$
\Delta'_m(\pi) := \left\{(s_1,\dots,s_m) \in \prod_{j=1}^{n}\left(\left\{[0,T]\backslash U_j\right\}^{m_j}\right) : 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < T\right\}. \tag{5.6}
$$

It is apparent that

$$
\bigcup_{\pi\in S_m} \Delta'_m(\pi) = \prod_{j=1}^{n}\left(\left\{[0,T]\backslash U_j\right\}^{m_j}\right)
$$

up to a set of measure zero and also that this union is a disjoint union. We can then write

$$
\begin{aligned}
&J_{m_1,\dots,m_n} \\
&= \sum_{\pi\in S_m}\int_{\Delta'_m(\pi)} \tilde{C}_{\pi(m)}\left(s_{\pi(m)}\right)\cdots\tilde{C}_{\pi(1)}\left(s_{\pi(1)}\right)\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)\left(ds_1,\dots,ds_m\right)
\end{aligned}
$$

where

$$\tilde{C}_j(s) := \begin{cases} \tilde{A}_1(s) & \text{if } j \in \{1, \dots, m_1\}, \\ \tilde{A}_2(s) & \text{if } j \in \{m_1+1, \dots, m_1+m_2\}, \\ \vdots & \\ \tilde{A}_n(s) & \text{if } j \in \{m_1+\cdots+m_{n-1}+1, \dots, m\}. \end{cases} \tag{5.7}$$

The time-ordering of the monomial in $\mathbb{D}_{B_0}$ is therefore

$$\begin{aligned} &P^{m_1,\dots,m_n}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right) \\ &= \sum_{\pi \in S_m} \int_{\Delta'_m(\pi)} \tilde{C}_{\pi(m)}\left(s_{\pi(m)}\right) \cdots \tilde{C}_{\pi(1)}\left(s_{\pi(1)}\right) \left(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}\right)(ds_1, \dots, ds_m). \end{aligned} \tag{5.8}$$

Definition 5.3. The disentangling map

$$\mathcal{T}_{\mu_1,\dots,\mu_n} : \mathbb{D}_{B_0}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right) \longrightarrow E$$

is defined by first taking

$$\begin{aligned} &\mathcal{T}_{\mu_1,\dots,\mu_n} P^{m_1,\dots,m_n}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right) \\ &:= \sum_{\pi \in S_m} \int_{\Delta'_m(\pi)} C_{\pi(m)}\left(s_{\pi(m)}\right) \cdots C_{\pi(1)}\left(s_{\pi(1)}\right) \left(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}\right)(ds_1, \dots, ds_m) \end{aligned} \tag{5.9}$$

where

$$C_j(s) := \begin{cases} A_1(s) & \text{if } j \in \{1, \dots, m_1\}, \\ A_2(s) & \text{if } j \in \{m_1+1, \dots, m_1+m_2\}, \\ \vdots & \\ A_n(s) & \text{if } j \in \{m_1+\cdots+m_{n-1}+1, \dots, m\}. \end{cases} \tag{5.10}$$

We then define, for $f \in \mathbb{D}_{B_0}$, written as in (5.4) above,

$$\begin{aligned} &\mathcal{T}_{\mu_1,\dots,\mu_n} f\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right) \\ &= \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n} \mathcal{T}_{\mu_1,\dots,\mu_n} P^{m_1,\dots,m_n}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right); \end{aligned} \tag{5.11}$$

that is, we apply $\mathcal{T}_{\mu_1,\dots,\mu_n}$ term-by-term in the series expansion for f.

Remark 5.4. (1) Of course, the definition above is essentially the same when our time-ordering measures have non-trivial discrete parts. One first time-orders the monomial $P^{m_1,\dots,m_n}$ in the disentangling algebra and then defines the action of the disentangling map on the time-ordered monomial. Finally, with an element f of the disentangling algebra given, the disentangling map applied to f is a term-by-term application of the disentangling map for monomials in the series representation of f.

(2) We also note that, by Proposition (5.1) and the discussion just following this proposition, the integrand in (5.9) is totally $\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}$-summable.

We now take some time to establish a basic (and expected) theorem.

Theorem 5.5. *The disentangling map* $\mathcal{T}_{\mu_1,\ldots,\mu_n} : \mathbb{D}_{B_0} \longrightarrow E$ *is a bounded linear map.*

Proof. We will use the closed, bounded and absolutely convex $B_0 \subseteq E$ defined above (just after Remark 5.2). We start with a norm bound for

$$\left|\mathcal{T}_{\mu_1,\ldots,\mu_n} P^{m_1,\ldots,m_n}\left(\tilde{A}_1(\cdot),\ldots,\tilde{A}_n(\cdot)\right)\right|_{B_0}$$

in E_{B_0}. First, observe that we may write

$$\begin{aligned}
&\left|\mathcal{T}_{\mu_1,\ldots,\mu_n} P^{m_1,\ldots,m_n}\left(\tilde{A}_1(\cdot),\ldots,\tilde{A}_n(\cdot)\right)\right|_{B_0} \\
&\leq \sum_{\pi\in S_m}\left|\int_{\Delta'_m(\pi)} C_{\pi(m)}\left(s_{\pi(m)}\right)\cdots\right. \\
&\qquad \left. C_{\pi(1)}\left(s_{\pi(1)}\right)\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\ldots,ds_m)\right|_{B_0} \\
&\leq \sum_{\pi\in S_m}\int_{\Delta'_m(\pi)}\left|C_{\pi(m)}\left(s_{\pi(m)}\right)\cdots C_{\pi(1)}\left(s_{\pi(1)}\right)\right|_{B_0}\cdot \\
&\qquad \left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\ldots,ds_m).
\end{aligned}$$

Now, we are able to write

$$\left|C_{\pi(m)}\left(s_{\pi(m)}\right)\cdots C_{\pi(1)}\left(s_{\pi(1)}\right)\right|_{B_0} \leq \left|C_{\pi(m)}\left(s_{\pi(m)}\right)\right|_{B_0}\cdots\left|C_{\pi(1)}\left(s_{\pi(1)}\right)\right|_{B_0}$$

due to the fact that multiplication is continuous which gives us, for the norm $|\cdot|_{B_0}$,

$$|xy|_{B_0} \leq |x|_{B_0}|y|_{B_0}$$

for $x, y \in E_{B_0}$ (see, for instance, the remark on p. 192 of [5]). It is above that we need to know that, for each j, $|A_j(s)|_{B_0} < \infty$; since we're integrating over the sets $\Delta'_m(\pi)$, we know this to be the case. Therefore,

$$\begin{aligned}
&\left|\mathcal{T}_{\mu_1,\ldots,\mu_n} P^{m_1,\ldots,m_n}\left(\tilde{A}_1(\cdot),\ldots,\tilde{A}_n(\cdot)\right)\right|_{B_0} \\
&\leq \sum_{\pi\in S_m}\int_{\Delta'_m(\pi)}\left|C_{\pi(m)}\left(s_{\pi(m)}\right)\right|_{B_0}\cdots\left|C_{\pi(1)}\left(s_{\pi(1)}\right)\right|_{B_0} \\
&\qquad\qquad \left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\ldots,ds_m) \qquad (5.12) \\
&= \left(\int_{[0,T]\setminus U_1}\left|A_1(s)\right|_{B_0}\mu_1(ds)\right)^{m_1}\cdots\left(\int_{[0,T]\setminus U_n}\left|A_n(s)\right|_{B_0}\mu_n(ds)\right)^{m_n} \\
&= R_1^{m_1}\cdots R_n^{m_n} \\
&= \left\|\mathcal{T}_{\mu_1,\ldots,\mu_n} P^{m_1,\ldots,m_n}\left(\tilde{A}_1(\cdot),\ldots,\tilde{A}_n(\cdot)\right)\right\|_{\mathbb{D}_{B_0}}.
\end{aligned}$$

The first equality above follows from the fact that the expression on the previous line contains scalar-valued factors in the integrands and so we can "unravel" the disentangling process back to the starting point only with the integrals having the scalar-valued functions $\left|A_j(\cdot)\right|_{B_0}$, $j = 1, \dots, n$. The second equality is, of course, the definition of the radii $R_1, \dots, R_n$ and the last equality is simply the definition of the norm in a disentangling algebra (see Section 2, above).

To finish the proof, we note that, given $f \in \mathbb{D}_{B_0}$ written as in (5.4),

$$\begin{aligned}
&\left|\mathcal{T}_{\mu_1,\dots,\mu_n} f\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right)\right|_{B_0} \\
&\quad\leq \sum_{m_1,\dots,m_n=0}^{\infty} |a_{m_1,\dots,m_n}| \left|\mathcal{T}_{\mu_1,\dots,\mu_n} P^{m_1,\dots,m_n}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right)\right|_{B_0} \\
&\quad\leq \sum_{m_1,\dots,m_n=0}^{\infty} |a_{m_1,\dots,m_n}| R_1^{m_1} \cdots R_n^{m_n} \\
&\quad= \|f\|_{\mathbb{D}_{B_0}}.
\end{aligned} \tag{5.13}$$

Therefore $\mathcal{T}_{\mu_1,\dots,\mu_n}$ is continuous from $\mathbb{D}_{B_0}$ into E. $\square$

We now make some remarks concerning the disentangling map when non-continuous time-ordering measures are used. As the reader has seen in Section 2.2, when we have time ordering measures $\lambda_1, \dots, \lambda_n$ with finitely supported discrete parts, we write $\lambda_j = \mu_j + \eta_j$ with μ_j continuous and η_j purely discrete and write the union of the supports of the purely discrete measures as $\{\tau_1, \dots, \tau_h\}$ where $0 < \tau_1 < \tau_2 < \cdots < \tau_h < T$. It then follows that we may write, for each $j \in \{1, \dots, n\}$,

$$\eta_j = \sum_{i=1}^{h} p_{ji} \delta_{\tau_i}$$

where, of course, many of the coefficients p_{ji}, $i = 1, \dots, h$, may be zero. Referring to Section 2 (Equation (2.15) in particular), we recall that when we have $\mathcal{L}(X)$-valued functions $A_1(\cdot), \dots, A_n(\cdot)$ the disentangled monomial is

$$\begin{aligned}
&P_{\lambda_1,\dots,\lambda_n}^{m_1,\dots,m_n}\left(A_1(\cdot), \dots, A_n(\cdot)\right) \\
&\quad= \sum_{q_{11}+q_{12}=m_1} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left(\frac{m_1! \cdots m_n!}{q_{11}! q_{12}! \cdots q_{n1}! q_{n2}!}\right) \sum_{j_{11}+\cdots+j_{1h}=q_{12}} \cdots \\
&\qquad \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \left(\frac{q_{12}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! \cdots j_{n1}! \cdots j_{nh}!}\right) \sum_{\pi \in S_{q_{11}+\cdots+q_{n1}}} \sum_{\theta_1+\cdots+\theta_{h+1}=q_{11}+\cdots+q_{n1}} \\
&\qquad \int_{\Delta_{q_{11}+\cdots+q_{n1};\theta_1,\dots,\theta_{h+1}}(\pi)} C_{\pi(q_{11}+\cdots+q_{n1})}\left(s_{\pi(q_{11}+\cdots+q_{n1})}\right) \cdots \\
&\qquad C_{\pi(\theta_1+\cdots+\theta_h+1)}\left(s_{\pi(\theta_1+\cdots+\theta_h+1)}\right) \left[\prod_{\beta=0}^{n} \left(p_{n-\beta,h} A_{n-\beta}\left(\tau_h\right)\right)^{j_{n-\beta,h}}\right]
\end{aligned}$$

$$
\begin{aligned}
&C_{\pi(\theta_1+\cdots+\theta_h)}\left(s_{\pi(\theta_1+\cdots+\theta_h)}\right)\cdots C_{\pi(\theta_1+1)}\left(s_{\pi(\theta_1+1)}\right)\\
&\left[\prod_{\beta=0}^{n}\left(p_{n-\beta,1}A_{n-\beta}\left(\tau_1\right)\right)^{j_{n-\beta,1}}\right]C_{\pi(\theta_1)}\left(s_{\pi(\theta_1)}\right)\cdots C_{\pi(1)}\left(s_{\pi(1)}\right)\cdot\\
&\left(\mu_1^{q_{11}}\times\cdots\times\mu_n^{q_{n1}}\right)\left(ds_1,\ldots,ds_{q_{11}+\cdots+q_{n1}}\right),
\end{aligned}
\tag{5.14}
$$

where the ordered sets $\Delta_{q_{11}+\cdots+q_{n1};\theta_1,\ldots,\theta_{h+1}}(\pi)$ of time indices are defined above in (2.14). Note that, in addition to the evaluation of the operator-valued functions at the time indices $s_1,\ldots,s_{q_{11}+\cdots+q_{n1}}$ we have evaluation of the operator-valued functions at the support points of the discrete parts of the time-ordering measures. Of course, we have to do exactly this when we are using our E-valued functions and would therefore have to adjust the sets we integrate over just as we did above when we used continuous time-ordering measures by exploiting the fact that $A_j(\cdot)$ takes values in E_{B_0} μ_j-a.e. The added detail here is that we only use the support points of the discrete measures that fall outside the union of the null sets $U_1,\ldots,U_n$ (we will assume that the union $U_1\cup\cdots\cup U_n$ does not contain all of the support points of the discrete measures). Writing this set of support points as $\{\tau_1,\ldots,\tau_h\}$, $0<\tau_1<\cdots<\tau_h<T$, just as above, we can use the notation introduced above to write, in the disentangling algebra $\mathbb{D}_{B_0}$,

$$
\begin{aligned}
&P^{m_1,\ldots,m_n}\left(\tilde{A}_1(\cdot),\ldots,\tilde{A}_n(\cdot)\right)\\
&=\left(\int_{[0,T]}\tilde{A}_1(s)\,\lambda_1(ds)\right)^{m_1}\cdots\left(\int_{[0,T]}\tilde{A}_n(s)\,\lambda_n(ds)\right)^{m_n}\\
&=\left(\int_{[0,T]\setminus U_1}\tilde{A}_1(s)\,\lambda_1(ds)\right)^{m_1}\cdots\left(\int_{[0,T]\setminus U_n}\tilde{A}_n(s)\,\lambda_n(ds)\right)^{m_n}\\
&=\left(\int_{[0,T]\setminus U_1}\tilde{A}_1(s)\mu_1(ds)+\sum_{i=1}^{h}p_{1i}\tilde{A}_1\left(\tau_i\right)\right)^{m_1}\cdots\\
&\qquad\left(\int_{[0,T]\setminus U_n}\tilde{A}_n(s)\mu_n(ds)+\sum_{i=1}^{h}p_{ni}\tilde{A}_n\left(\tau_i\right)\right)^{m_n}.
\end{aligned}
$$

The next step in this calculation is an application of the binomial theorem in each of the factors followed by applications of the multinomial formula for the powers of the sums. At this point, for each factor, we have an integral to an integer power and an integer power of a sum of evaluations at the support points of the discrete measures. To take care of the power of the integral, we proceed essentially as in (5.5) above except that we change the sets $\Delta_{q_{11}+\cdots+q_{n1};\theta_1,\ldots,\theta_{h+1}}(\pi)$ in (5.14) to

$$
\Delta'_{q_{11}+\cdots+q_{n1};\theta_1,\ldots,\theta_{h+1}}(\pi)=\left\{\left(s_1,\ldots,s_{q_{11}+\cdots+q_{n1}}\right)\in\prod_{j=1}^{n}\left([0,T]\setminus U_j\right)^{q_{j1}}\ :\right.
\tag{5.15}
$$

$$\left.\begin{aligned}&0 < s_{\pi(1)} < \cdots < s_{\pi(\theta_1)} < \tau_1 < s_{\pi(\theta_1+1)} < \cdots < \\ &s_{\pi(\theta_1+\cdots+\theta_h)} < \tau_h < s_{\pi(\theta_1+\cdots+\theta_h+1)} < \cdots \\ &\qquad < s_{\pi(q_{11}+\cdots+q_{n1})} < T\end{aligned}\right\}$$

and use the fact that, to a set of $\mu_1^{q_{11}} \times \cdots \times \mu_n^{q_{n1}}$- measure zero,

$$\bigcup_{\pi \in S_{q_{11}+\cdots+q_{n1}}} \Delta'_{q_{11}+\cdots+q_{n1};\theta_1,\ldots,\theta_{h+1}}(\pi) = \prod_{j=1}^{n} \left([0,T]\backslash U_j\right)^{q_{j1}}.$$

See Chapter 8 of [17] for details of the time-ordering using time-ordering measures with finitely supported discrete parts. We obtain (5.5) with the sets of ordered time-indices given in (5.15) instead of $\Delta_{q_{11}+\cdots+q_{n1};\theta_1,\ldots,\theta_{h+1}}(\pi)$ as the fully disentangled expression in E. When we set about to establish the continuity of the disentangling map in this case, it turns out that we obtain the same norm bound as obtained in (5) for $\left|P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}\left(\tilde{A}_1(\cdot),\ldots,\tilde{A}_n(\cdot)\right)\right|_{B_0}$.

Finally, we remark that the ideas outlined above are applicable when we use arbitrary time-ordering measures, but the combinatorial details are much more complex. See [27] for details when $\mathcal{L}(X)$-valued functions are used. When working in a topological algbra, one proceeds much as in the $\mathcal{L}(X)$ setting making adjustments as we did above.

6. F.O.C. in the presence of a (C_0)-semigroup

In this section we address the use of a strongly continuous semigroup of linear contractions on the Banach space $\left(E_{B_0}, |\cdot|_{B_0}\right)$ (notation as above) in the operational calculus. The use of these types of semigroups in the operational calculus have allowed the development of integral equations and evolution equations for Feynman's operational calculus; see [6], [16], [14], [22], [17], [26], [27].

We will continue using the notation introduced previously. In particular, we will use time-ordering measures which are continuous and will denote them by μ_j, $j = 1,\ldots,n$, as before. We will also continue to work in the Banach space $(E_{B_0}, |\cdot|_{B_0})$. On the Banach space E_{B_0}, let $\{T(t)\}_{t\geq 0}$ be a (C_0)-semigroup of linear contractions with infinitesimal generator $-\alpha_{B_0}$. As is customary, we will write

$$T(t) = e^{-t\alpha_{B_0}},\ t \geq 0.$$

In order to insert this semigroup into the operational calculus, we will need a new Banach algebra of functions. Following [20] (and [26]), we begin by denoting by F_k, $k \in \mathbb{N}$, the family of entire functions on $\mathbb{C}^k$ and we let $\mathscr{F}_k$ denote the collection of all $f \in F_k$ for which

$$\|f\|^2_{\mathscr{F}_k} := \pi^{-k} \int_{\mathbb{C}^k} |f(z)|^2 e^{-|z|^2} dV < \infty,$$

where

$$dV = \prod_{m=1}^{k} dx_m dy_m,\ z_m = x_m + iy_m.$$

Given $f, g \in \mathscr{F}_k$ we define their inner product by (with $\bar{g}$ denoting the complex conjugate of g)

$$\langle f, g\rangle_{\mathscr{F}_k} := \pi^{-k} \int_{\mathbb{C}^k} f(z)\bar{g}(z)e^{-|z|^2} dV.$$

With this inner product, $\mathscr{F}_k$ becomes a Hilbert space. Note that, via pointwise operations, $\mathscr{F}_k$ is also an algebra. In our setting we will take $k = 1$ so that

$$\|f\|^2_{\mathscr{F}_1} = \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dxdy$$

and

$$\langle f, g\rangle_{\mathscr{F}_1} = \frac{1}{\pi} \int_{\mathbb{C}} f(z)\bar{g}(z)e^{-|z|^2} dxdy.$$

We now take $\widetilde{-\alpha_{B_0}}$ to be the formal object corresponding to $-\alpha_{B_0}$ (see Section 2 above for a brief discussion about formal objects in Feynman's calculus) and take $\mathscr{F}_1\left(\widetilde{-\alpha_{B_0}}\right)$ to be the family of all $f\left(\widetilde{-\alpha_{B_0}}\right)$ for $f \in \mathscr{F}_1$ – we are merely changing the name of the indeterminate z. Note that for any real constant a, $f_1(z) := e^{az}$ is in $\mathscr{F}_1$. The norm of f_1 is easily computed to be

$$\|f_1\|_{\mathscr{F}_1} = e^{a^2}. \tag{6.1}$$

Define (see, for example, [29])

$$\mathbb{D}_{B_0,\mathscr{F}_1} := \mathscr{F}_1\left(\widetilde{-\alpha_{B_0}}\right) \hat{\otimes}_\pi \mathbb{D}_{B_0}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right)$$

(the projective tensor product). It is in this algebra that we will carry out our time-ordering in a rigorous fashion. For $e^{-Tz} \in \mathscr{F}_1$ and for $f \in \mathbb{D}_{B_0}$ (with series representation in (5.4)) we consider first the time-ordering of the monomial $P^{m_1,\dots,m_n}\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right)$ in $\mathbb{D}_{B_0,\mathscr{F}_1}$. We compute, successively, associating

Lebesgue measure ℓ with $-\alpha_{B_0}$,

$$
\begin{aligned}
& e^{T(\widetilde{-\alpha_{B_0}})} P^{m_1,\ldots,m_n}\left(\tilde{A}_1(\cdot),\ldots,\tilde{A}_n(\cdot)\right) \\
&= e^{T(\widetilde{-\alpha_{B_0}})}\left(\int_{[0,T]} \tilde{A}_1(s)\,\mu_1(ds)\right)^{m_1}\cdots\left(\int_{[0,T]} \tilde{A}_n(s)\,\mu_n(ds)\right)^{m_n} \\
&= e^{T(\widetilde{-\alpha_{B_0}})}\left(\int_{[0,T]\setminus U_1} \tilde{A}_1(s)\,\mu_1(ds)\right)^{m_1}\cdots\left(\int_{[0,T]\setminus U_n} \tilde{A}_n(s)\,\mu_n(ds)\right)^{m_n} \\
&= e^{T(\widetilde{-\alpha_{B_0}})}\left\{\sum_{\pi\in S_m}\int_{\Delta'_m(\pi)} \tilde{C}_{\pi(m)}\left(s_{\pi(m)}\right)\cdots\tilde{C}_{\pi(1)}\left(s_{\pi(1)}\right)\cdot\right. \\
&\qquad\qquad\qquad\qquad\left.\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\ldots,ds_m)\right\} \\
&= \sum_{\pi\in S_m} e^{T(\widetilde{-\alpha_{B_0}})}\int_{\Delta'_m(\pi)} \tilde{C}_{\pi(m)}\left(s_{\pi(m)}\right)\cdots\tilde{C}_{\pi(1)}\left(s_{\pi(1)}\right) \\
&\qquad\qquad\qquad\qquad\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\ldots,ds_m) \\
&= \sum_{\pi\in S_m}\exp\left\{\int_{[s_{\pi(m)},T]}\widetilde{-\alpha_{B_0}}(s)\,ds+\int_{[s_{\pi(m-1)},s_{\pi(m)}]}\widetilde{-\alpha_{B_0}}(s)\,ds+\cdots\right. \\
&\qquad\left.+\int_{[0,s_{\pi(1)}]}\widetilde{-\alpha_{B_0}}(s)\,ds\right\}\int_{\Delta'_m(\pi)}\tilde{C}_{\pi(m)}\left(s_{\pi(m)}\right)\cdots \\
&\quad\tilde{C}_{\pi(1)}\left(s_{\pi(1)}\right)\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\ldots,ds_m) \\
&\quad= \sum_{\pi\in S_m}\int_{\Delta'_m(\pi)} e^{\left(T-s_{\pi(m)}\right)(\widetilde{-\alpha_{B_0}})}\tilde{C}_{\pi(m)}\left(s_{\pi(m)}\right)e^{\left(s_{\pi(m)}-s_{\pi(m-1)}\right)(\widetilde{-\alpha_{B_0}})}\ldots \\
&\quad\quad e^{\left(s_{\pi(2)}-s_{\pi(1)}\right)(\widetilde{-\alpha_{B_0}})}\tilde{C}_{\pi(1)}\left(s_{\pi(1)}\right)e^{s_{\pi(1)}(\widetilde{-\alpha_{B_0}})}. \\
&\quad\quad\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\ldots,ds_m).
\end{aligned}
$$

The computation above leads to the following definition.

Definition 6.1. On the Banach space $\left(E_{B_0},|\cdot|_{B_0}\right)$ let $\{S(t)\}_{t\geq 0}$ be a (C_0)-semigroup of linear contractions and write

$$S(t)=e^{-t\alpha_{B_0}}$$

where $-\alpha_{B_0}$ is the infinitesimal generator of the semigroup. We associate Lebesgue measure ℓ on $[0,T]$ to $-\alpha_{B_0}$. Define

$$\mathcal{T}_{\ell;\mu_1,\ldots,\mu_n}:\mathscr{F}_1\left(\widetilde{-\alpha_{B_0}}\right)\hat{\otimes}_\pi\mathbb{D}_{B_0}\to\mathcal{L}\left(E_{B_0}\right)$$

by first defining

$$
\begin{aligned}
&\mathcal{T}_{\ell;\mu_1,\dots,\mu_n}\left[e^{T\left(\widetilde{-\alpha_{B_0}}\right)} \otimes P^{m_1,\dots,m_n}\left(\tilde{A}_1(\cdot),\dots,\tilde{A}_n(\cdot)\right)\right] \\
&= \sum_{\pi\in S_m}\int_{\Delta_m'(\pi)} e^{-\left(T-s_{\pi(m)}\right)\alpha_{B_0}} C_{\pi(m)}\left(s_{\pi(m)}\right) e^{-\left(s_{\pi(m)}-s_{\pi(m-1)}\right)\alpha_{B_0}} \dots \\
&\qquad e^{-\left(s_{\pi(2)}-s_{\pi(1)}\right)\alpha_{B_0}} C_{\pi(1)}\left(s_{\pi(1)}\right) e^{-s_{\pi(1)}\alpha_{B_0}}\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\dots,ds_m).
\end{aligned} \tag{6.2}
$$

With $f \in \mathbb{D}_{B_0}$ written as in (5.4), we define

$$
\begin{aligned}
&\mathcal{T}_{\ell;\mu_1,\dots,\mu_n}\left[e^{T\left(\widetilde{-\alpha_{B_0}}\right)} \otimes f\left(\tilde{A}_1(\cdot),\dots,\tilde{A}_n(\cdot)\right)\right] \\
&= \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n}\mathcal{T}_{\ell;\mu_1,\dots,\mu_n}\left[e^{T\left(\widetilde{-\alpha_{B_0}}\right)} \otimes P^{m_1,\dots,m_n}\left(\tilde{A}_1(\cdot),\dots,\tilde{A}_n(\cdot)\right)\right] \\
&= \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n}\sum_{\pi\in S_m}\int_{\Delta_m'(\pi)} e^{-\left(T-s_{\pi(m)}\right)\alpha_{B_0}} C_{\pi(m)}\left(s_{\pi(m)}\right) \\
&\qquad e^{-\left(s_{\pi(m)}-s_{\pi(m-1)}\right)\alpha_{B_0}} \dots e^{-\left(s_{\pi(2)}-s_{\pi(1)}\right)\alpha_{B_0}} C_{\pi(1)}\left(s_{\pi(1)}\right) e^{-s_{\pi(1)}\alpha_{B_0}} \\
&\qquad \left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\dots,ds_m).
\end{aligned} \tag{6.3}
$$

It is important to note that, in this setting (i.e. the operational calculus with a strongly continuous semigroup), the disentangling map gives rise to a linear map in $\mathcal{L}\left(E_{B_0}\right)$ (see the following theorem).

Remark 6.2. The reader will note that we are using only the exponential functions e^{az} ($a \in \mathbb{R}$) from $\mathcal{F}_1$ as we're focussed on inserting a semigroup of operators into the operational calculus. We could then restrict our attention to a subalgebra of $\mathcal{F}_1$ consisting of these exponentials functions (see [26] for similar comments and a slightly different presentation).

Theorem 6.3. *Using the notation introduced above and the content of Definition 6.1, the disentangling map as defined in this definition is continuous and linear in $f \in \mathbb{D}_{B_0}$ and its image is a bounded linear map in $\mathcal{L}\left(E_{B_0}\right)$.*

Proof. It is clear that $\mathcal{T}_{\ell;\mu_1,\dots,\mu_n}$ is linear in $f \in \mathbb{D}_{B_0}$. Let $\phi \in E_{B_0}$ and fix $m_1,\dots,m_n \in \mathbb{N}$ and $\pi \in S_m$ (recall that $m := m_1 + \cdots + m_n$). We have, because

$\{e^{-t\alpha_{B_0}}\}_{t\geq 0}$ is a (C_0)-semigroup of contractions,

$$\begin{aligned}
&\left|\int_{\Delta'_m(\pi)} e^{-(T-s_{\pi(m)})\alpha_{B_0}} C_{\pi(m)}\left(s_{\pi(m)}\right)\cdots e^{-(s_{\pi(2)}-s_{\pi(1)})\alpha_{B_0}} C\left(s_{\pi(1)}\right)\right.\\
&\qquad\left. e^{-s_{\pi(1)}\alpha_{B_0}}\phi\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\dots,ds_m)\right|_{B_0}\\
&\leq \int_{\Delta'_m(\pi)}\left|e^{-(T-s_{\pi(m)})\alpha_{B_0}} C_{\pi(m)}\left(s_{\pi(m)}\right)\right|_{B_0}\cdot\\
&\left|e^{-(s_{\pi(m)}-s_{\pi(m-1)})\alpha_{B_0}} C_{\pi(m-1)}\left(s_{\pi(m-1)}\right)\right|_{B_0}\\
&\cdots\left|e^{-(s_{\pi(2)}-s_{\pi(1)})\alpha_{B_0}} C_{\pi(1)}\left(s_{\pi(1)}\right)\right|_{B_0}\left|e^{-s_{\pi(1)}\alpha_{B_0}}\phi\right|_{B_0}\cdot\\
&\qquad\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\dots,ds_m)\\
&\leq \int_{\Delta'_m(\pi)}\left|C_{\pi(m)}\left(s_{\pi(m)}\right)\right|_{B_0}\cdots\left|C_{\pi(1)}\left(s_{\pi(1)}\right)\right|_{B_0}|\phi|_{B_0}\cdot\\
&\qquad\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\dots,ds_m)\\
&= \int_{\Delta'_m(\pi)}\left|C_{\pi(m)}\left(s_{\pi(m)}\right)\right|_{B_0}\cdots\left|C_{\pi(1)}\left(s_{\pi(1)}\right)\right|_{B_0}\cdot\\
&\qquad\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\dots,ds_m)|\phi|_{B_0}
\end{aligned}\tag{6.4}$$

and therefore

$$\begin{aligned}
&\left|P^{m_1,\dots,m_n}_{\ell;\mu_1,\dots,\mu_n}(A_1(\cdot),\dots,A_n(\cdot))\phi\right|_{B_0}\\
&\leq\sum_{\pi\in S_m}\int_{\Delta'_m(\pi)}\left|C_{\pi(m)}\left(s_{\pi(m)}\right)\right|_{B_0}\cdots\left|C_{\pi(1)}\left(s_{\pi(1)}\right)\right|_{B_0}\\
&\qquad\qquad\left(\mu_1^{m_1}\times\cdots\times\mu_n^{m_n}\right)(ds_1,\dots,ds_m)|\phi|_{B_0}\\
&=\left(\int_{[0,T]\setminus U_1}|A_1(s)|_{B_0}\mu_1(ds)\right)^{m_1}\cdots\left(\int_{[0,T]\setminus U_n}|A_n(s)|_{B_0}\mu_n(ds)\right)^{m_n}|\phi|_{B_0}\\
&=\left(\int_{[0,T]}|A_1(s)|_{B_0}\mu_1(ds)\right)^{m_1}\cdots\left(\int_{[0,T]}|A_n(s)|_{B_0}\mu_n(ds)\right)^{m_n}|\phi|_{B_0}\\
&=\left(R_1^{m_1}\cdots R_n^{m_n}\right)|\phi|_{B_0}.
\end{aligned}\tag{6.5}$$

Given $f \in \mathbb{D}_{B_0}$ written as (5.4), we have

$$\begin{aligned}
&\left|\mathcal{T}_{\ell;\mu_1,\dots,\mu_n}\left[e^{T(\widetilde{-\alpha_{B_0}})} \otimes f\left(\tilde{A}_1(\cdot),\dots,\tilde{A}_n(\cdot)\right)\right]\phi\right|_{B_0} \\
&\le \sum_{m_1,\dots,m_n=0}^{\infty} |a_{m_1,\dots,m_n}| \left|P^{m_1,\dots,m_n}_{\ell;\mu_1,\dots,\mu_n}\left(A_1(\cdot),\dots,A_n(\cdot)\right)\phi\right|_{B_0} \\
&\le \left\{\sum_{m_1,\dots,m_n=0}^{\infty} |a_{m_1,\dots,m_n}| R_1^{m_1}\cdots R_n^{m_n}\right\} |\phi|_{B_0} \\
= {}& \|f\|_{\mathbb{D}_{B_0}} |\phi|_{B_0} \\
\le {}& \left\|e^{Tz}\right\|_{\mathscr{F}_1} \|f\|_{\mathbb{D}_{B_0}} |\phi|_{B_0},
\end{aligned}$$

where we've used (6.1). □

The result above made use of time-ordering measures which are continuous (as was the case in the previous section). When we allow time-ordering measures with, for instance, nonzero finitely supported discrete parts, then the disentangling proceeds much as seen in [18], [17, Chapter 8] though the remarks above at the end of Section 5 come into play along the way. Also, if we use arbitrary time-ordering measures similar comments hold but the combinatorial details are more formidable (see [27]).

7. Stability of the operational calculus in E with respect to the time-ordering measures

We begin this section with some brief remarks concerning stability of the operational calculus with respect to time time-ordering measures. (For a more detailed discussion, see [17].) Note that, for a given n-tuple $(\mu_1,\dots,\mu_n)$ of time ordering measures, we have a particular operational calculus determined by the disentangling map $\mathcal{T}_{\mu_1,\dots,\mu_n}$. If we have sequences $\{\mu_{1k}\}_{k=1}^{\infty},\dots,\{\mu_{nk}\}_{k=1}^{\infty}$ measures, then for each $k \in \mathbb{N}$, we have a particular operational calculus induced by the n-tuple $(\mu_{1k},\dots,\mu_{nk})$ via the disentangling map $\mathcal{T}_{\mu_{1k},\dots,\mu_{nk}}$. Therefore, we have a countably infinite family $\left\{\mathcal{T}_{\mu_{1k},\dots,\mu_{nk}}\right\}_{k=1}^{\infty}$ of operational calculi given by the action of the disentangling map for each $k \in \mathbb{N}$. If the sequences $\left\{\mu_{jk}\right\}_{k=1}^{\infty}$, $j = 1,\dots,n$ converge to a measure μ_j in the appropriate fashion, we ask if the sequence of operational calculi $\left\{\mathcal{T}_{\mu_{1k},\dots,\mu_{nk}}\right\}_{k=1}^{\infty}$ converges to the operational calculus $\mathcal{T}_{\mu_1,\dots,\mu_n}$. This question (and other, related, questions) are investigated in, among others, [23], [24], [25], [17] (and references therein), [27].

7.1. An aside on continuity. To begin our discussion, we will assume that $A_i : [0,T] \to E$, $i = 1,\dots,n$, is continuous. To each $A_i(\cdot)$ we will, as above, associate a continuous Borel probability measure μ_i on $[0,T]$. Choose positive integers $q_1,\dots,q_n$ and put $q := q_1 + \cdots + q_n$. Also choose $\phi_1,\dots,\phi_r \in E$ and nonnegative integers $\theta_1, \dots, \theta_{r+1}$ which satisfy $\theta_1 + \cdots + \theta_{r+1} = q$. For $\pi \in S_q$ define

$$F_q : [0,T]^q \to E$$

by

$$\begin{aligned} & F_q\left(t_1,\dots,t_q\right) := C_{\pi(q)}\left(t_{\pi(q)}\right) \cdots C_{\pi(\theta_1+\cdots+\theta_r+1)}\left(t_{\pi(\theta_1+\cdots+\theta_r+1)}\right) \phi_r \\ & C_{\pi(\theta_1+\cdots+\theta_r)}\left(t_{\pi(\theta_1+\cdots+\theta_r)}\right) \cdots C_{\pi(\theta_1+1)}\left(t_{\pi(\theta_1+1)}\right) \phi_1 C_{\pi(\theta_1)}\left(t_{\pi(\theta_1)}\right) \\ & \cdots C_{\pi(1)}\left(t_{\pi(1)}\right) \end{aligned} \tag{7.1}$$

where $C_j(t)$ has been defined above in (5.10). We will first verify that F_q is continuous.

Theorem 7.1. *Let $B_i : [0,T] \to E$, $i = 1,2$, be continuous. Define $F_2 : [0,T]^2 \to E$ by*

$$F_2(t_1,t_2) := B_1(t_1)B_2(t_2).$$

We take the norm on $[0,T]^2$ to be $\|(a,b)\| := \max\{|a|,|b|\}$. The map F_2 is continuous.

Proof. Let $t_0 \in [0,T]$ and choose $\epsilon > 0$ and $\rho_1, \dots, \rho_k \in \Gamma_E$. By assumption, for each $i = 1,2$, there is a $\delta_i > 0$ such that if $|t - t_0| < \delta_i$, then

$$\left|\rho_j\left(B_i(t)\right) - \rho_j\left(B_i(t_0)\right)\right| < \epsilon$$

for each $j = 1, \dots, k$. Define $\delta := \min\{\delta_1, \delta_2\}$. If $(a,b) \in [0,T]^2$ and if

$$\|(t_1,t_2) - (a,b)\| < \delta,$$

then $|t_1 - a| < \delta$ and $|t_2 - b| < \delta$. Since each of our seminorms $\rho_1, \dots, \rho_k$ are m-seminorms we can then write, for each j,

$$\begin{aligned} & \left|\rho_j\left(B_1(t_1)B_2(t_2)\right) - \rho_j\left(B_1(a)B_2(b)\right)\right| \\ & \le \rho_j\left(B_1(t_1)\left(B_2(t_2) - B_2(b)\right)\right) + \rho_j\left(\left(B_1(t_1) - B_1(a)\right)B_2(b)\right) \\ & \le \rho_j\left(B_1(t_1)\right)\rho_j\left(B_2(t_2) - B_2(b)\right) + \rho_j\left(B_1(t_1) - B_1(a)\right)\rho_j\left(B_2(b)\right) \\ & \le 2M\epsilon \end{aligned}$$

where M is defined as follows. For each $j = 1, \dots, n$, $\rho_j \circ B_i$ is continuous from $[0,T]$ into $[0,\infty)$ for $i = 1,2$. Hence there are constants $M_{j,1}, M_{j,2}$ for which

$$\rho_j\left(B_i(t)\right) \le M_{j,i},$$

$i = 1,2$ and $t \in [0,T]$. We then define

$$M := \max\left\{M_{j,i} : j = 1, \dots, n; i = 1,2\right\}.$$

□

With Theorem 7.1 in hand, the general result follows via an easy induction argument (considering the elements $\phi_1, \dots, \phi_r$ as constant functions $g_p(s) = \phi_p$, $p = 1, \dots, r$). Thus F_q is continuous and so $F_q\left([0,T]^q\right) \subseteq E$ is compact and, using Proposition 1 on page 45 of [28], $F_q\left([0,T]^q\right)$ is bounded so that it follows that F_q is $\mu_1^{q_1} \times \cdots \times \mu_n^{q_n}$-measurable (see Section 4 above or [31]). Because our measures are finite, F_q is totally $\mu_1^{q_1} \times \cdots \times \mu_n^{q_n}$-summable. Therefore there is a closed, bounded and absolutely convex $B_q \subseteq E$ for which

$$\int_{[0,T]^q} |F_q(s_1, \dots, s_q)|_{B_q} \left(\mu_1^{q_1} \times \cdots \times \mu_n^{q_n}\right)(ds_1, \dots, ds_q) < \infty. \tag{7.2}$$

Now, by part (4) of Lemma 5.50 in [3], $|\cdot|_{B_q}$ is a continuous seminorm since B_q is closed and absolutely convex. (Recall that "absolutely convex," "convex and balanced," and "convex and circled" mean the same thing.) It then follows from Proposition 1.4.11 of [2] that

$$\sup_{[0,T]^q} \left|F_q(t_1, \dots, t_q)\right|_{B_q} < \infty. \tag{7.3}$$

When considering the operational calculus with time-ordering measures which have finitely supported or arbitrary discrete parts, it is functions like F_q which appear as integrands in our integrals of time-ordered products. Of course, when we use continuous time-ordering measures, we no longer need to consider F_q as above in (7.1) but need only the version of F_q which does not have the $\phi_1, \dots, \phi_r$.

7.2. Stability in the time-ordering measures. We are now ready to address the question of stability in the time-ordering measures. We remind the reader that we will take each $A_j : [0,T] \to E$, $j = 1, \dots, n$, to be continuous and we will associate to each $A_j(\cdot)$ a continuous Borel probability measure μ_j on $[0,T]$. As we have seen, it follows that each $A_j(\cdot)$ is μ_j-measurable and because $A_j([0,T])$ is compact, $A_j([0,T]) \subseteq E$ is bounded.

Remark 7.2. We recall that if $\{\nu_k\}_{k=1}^{\infty}$, ν are Borel probability measures on a metric space S, we say that $\{\nu_k\}_{k=1}^{\infty}$ *converges weakly* to ν as $k \to \infty$ if

$$\lim_{k\to\infty} \int_S f \, d\nu_k = \int_S f \, d\nu$$

for every bounded continuous real-valued function f on S. We write $\nu_k \rightharpoonup \nu$.

We also remark that if we have a weakly convergent sequence $\{\nu_k\}_{k=1}^{\infty}$ of Borel probability measures on a metric space S with weak limit ν, then given a continuous norm-bounded function $f : S \to X$, X a Banach space, we have

$$\lim_{k\to\infty} \int_S f \, d\nu_k = \int_S f \, d\nu$$

in norm on X where the integrals are Bochner integrals. See [23].

For each $j = 1, \dots, n$ we select a sequence $\{\mu_{jk}\}_{k=1}^{\infty}$ of continuous Borel probability measures on $[0,T]$ for which $\mu_{jk} \rightharpoonup \mu_j$ as $k \to \infty$. It is clear that, for each $j = 1, \dots, n$, $A_j(\cdot)$ is μ_{jk}-measurable for every $k \in \mathbb{N}$. Also, as seen in Subsection 7.1, each of our $A_j(\cdot)$, $j = 1, \dots, n$, is totally μ_{jk}-summable for every $k \in \mathbb{N}$ and is also totally μ_j-summable.

Fix $j \in \{1, \dots, n\}$. There is a closed, bounded and absolutely convex $B_{j0} \subseteq E$ for which

$$\int_{[0,T]} |A_j(s)|_{B_{j0}} \mu_j(ds) < \infty. \tag{7.4}$$

Similarly, for each $k \in \mathbb{N}$, there is a closed, bounded and absolutely convex $B_{jk} \subseteq E$ for which

$$\int_{[0,T]} |A_j(s)|_{B_{jk}} \mu_{jk}(ds) < \infty. \tag{7.5}$$

Define $B_j \subseteq E$ to be the closed absolutely convex hull of the union $\cup_{k=0}^{\infty} B_{jk}$ and assume that this union is a bounded set in E. It then follows from Proposition 1.4.12 of [2] that B_j is also bounded. Also, because $|\cdot|_{B_j} \leq |\cdot|_{B_{jk}}$ for all $k \in \mathbb{N} \cup \{0\}$, it follows that

$$\int_{[0,T]} |A_j(s)|_{B_j} \mu_{jk}(ds) < \infty \tag{7.6}$$

and

$$\int_{[0,T]} |A_j(s)|_{B_j} \mu_j(ds) < \infty. \tag{7.7}$$

We carry out the process above for each $j \in \{1, \dots, n\}$, obtaining the closed, bounded and absolutely convex subsets $B_1, \dots, B_n$ of E. We now define $B \subseteq E$ to be the closed absolutely convex hull of the union $B_1 \cup \cdots \cup B_n$. Again using Proposition 1.4.12 of [2], B is also bounded. Since $B_j \subseteq B$ for each $j = 1, \dots, n$, $|\cdot|_B \leq |\cdot|_{B_j}$. It therefore follows that for each $j = 1, \dots, n$ and $k \in \mathbb{N}$,

$$\int_{[0,T]} |A_j(s)|_B \mu_{jk}(ds) < \infty \text{ and } \int_{[0,T]} |A_j|_B \mu_j(ds) < \infty.$$

We also note that, because $A_j([0, T])$ is compact and so bounded,

$$\sup_{s \in [0,T]} |A_j(s)|_B < \infty \tag{7.8}$$

for each $j = 1, \dots, n$. (See [2, Proposition 1.4.11].)

Now, given $m_1, \dots, m_n \in \mathbb{N}$, we have

$$\begin{aligned}
&P^{m_1,\dots,m_n}_{\mu_1,\dots,\mu_n}(A_1(\cdot), \dots, A_n(\cdot)) \\
&= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}\left(s_{\pi(m)}\right) \cdots C_{\pi(1)}\left(s_{\pi(1)}\right) \left(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}\right)(ds_1, \dots, ds_m)
\end{aligned}$$

and

$$\begin{aligned}
&P^{m_1,\dots,m_n}_{\mu_{1k},\dots,\mu_{nk}}(A_1(\cdot), \dots, A_n(\cdot)) \\
&= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}\left(s_{\pi(m)}\right) \cdots C_{\pi(1)}\left(s_{\pi(1)}\right) \left(\mu_{1k}^{m_1} \times \cdots \times \mu_{nk}^{m_n}\right)(ds_1, \dots, ds_m).
\end{aligned}$$

Note that we do not need to use $\Delta'_m(\pi)$ here due to the fact that each $A_j(\cdot)$ is continuous and so we have (7.8). We may write, then,

$$\begin{aligned}
&\left| P^{m_1,\dots,m_n}_{\mu_{1k},\dots,\mu_{nk}}\left(A_1(\cdot),\dots,A_n(\cdot)\right) - P^{m_1,\dots,m_n}_{\mu_1,\dots,\mu_n}\left(A_1(\cdot),\dots,A_n(\cdot)\right)\right|_B \\
&\le \sum_{\pi\in S_m}\left| \int_{\Delta_m(\pi)} C_{\pi(m)}\left(s_{\pi(m)}\right)\cdots C_{\pi(1)}\left(s_{\pi(1)}\right)\left(\mu_{1k}^{m_1}\times\cdots\times\mu_{nk}^{m_n}\right)(ds_1,\dots,ds_m)\right. \\
&\qquad \left. - \int_{\Delta_m(\pi)} C_{\pi(m)}\left(s_{\pi(m)}\right)\cdots C_{\pi(1)}\left(s_{\pi(1)}\right)\left(\mu_{1}^{m_1}\times\cdots\times\mu_{n}^{m_n}\right)(ds_1,\dots,ds_m)\right|_B .
\end{aligned}$$

Now, in view of (7.8) and from the definition of E_B (see (4.2)) we see that the continuous maps $A_j(\cdot)$, $j = 1,\dots,n$, take values in E_B; i.e. they are Banach space-valued maps. Using the main result of [23], we obtain

$$\begin{aligned}
&\lim_{k\to\infty}\left| \int_{\Delta_m(\pi)} C_{\pi(m)}\left(s_{\pi(m)}\right)\cdots C_{\pi(1)}\left(s_{\pi(1)}\right)\left(\mu_{1k}^{m_1}\times\cdots\times\mu_{nk}^{m_n}\right)(ds_1,\dots,ds_m)\right. \\
&\qquad \left. - \int_{\Delta_m(\pi)} C_{\pi(m)}\left(s_{\pi(m)}\right)\cdots C_{\pi(1)}\left(s_{\pi(1)}\right)\left(\mu_{1}^{m_1}\times\cdots\times\mu_{n}^{m_n}\right)(ds_1,\dots,ds_m)\right|_B \\
&= 0. \qquad (7.9)
\end{aligned}$$

Consequently,

$$\lim_{k\to\infty}\left| P^{m_1,\dots,m_n}_{\mu_{1k},\dots,\mu_{nk}}\left(A_1(\cdot),\dots,A_n(\cdot)\right) - P^{m_1,\dots,m_n}_{\mu_1,\dots,\mu_n}\left(A_1(\cdot),\dots,A_n(\cdot)\right)\right|_B = 0. \qquad (7.10)$$

To proceed further, we need some preparatory work. For each $k \in \mathbb{N}$, define

$$R_{j,k} := \int_{[0,T]} |A_j(s)|_B \mu_{jk}(ds) \qquad (7.11)$$

for each $j = 1,\dots,n$. Construct the commutative Banach algebras

$$\mathbb{A}_B\left(R_{1,k},\dots,R_{n,k}\right) =: \mathbb{A}_{B,k}, \text{ and}$$
$$\mathbb{D}_B\left((A_1(\cdot),\mu_{1k})^{\sim},\dots,(A_n(\cdot),\mu_{nk})^{\sim}\right) =: \mathbb{D}_{B,k}$$

as well as $\mathbb{A}_B(R_1,\dots,R_n) =: \mathbb{A}_{B,0}$ and $\mathbb{D}_B\left((A_1(\cdot),\mu_1)^{\sim},\ \dots,(A_n(\cdot),\mu_n)^{\sim}\right) =: \mathbb{D}_{B,0}$. Define

$$\mathbb{A}_{B,\infty} := \bigoplus_{k=0}^{\infty}\mathbb{A}_{B,k} \text{ and } \mathbb{D}_{B,\infty} := \bigoplus_{k=0}^{\infty}\mathbb{D}_{B,k}. \qquad (7.12)$$

As is well-known, these direct sums are also Banach algebras via the standard coordinatewise operations and the norms

$$\left\|\{g_k\}_{k=0}^{\infty}\right\|_{\mathbb{A}_{B,\infty}} := \sup_{k\in\mathbb{N}\cup\{0\}}\|g_k\|_k \text{ and } \left\|\{f_k\}_{k=0}^{\infty}\right\|_{\mathbb{D}_{B,\infty}} := \sup_{k\in\mathbb{N}\cup\{0\}}\|f_k\|_k. \qquad (7.13)$$

(Note that we are using the same notation $\|\cdot\|_k$ on the right-hand sides of each norm definition. This is because the norms on $\mathbb{A}$ and $\mathbb{D}$ are the same; see Section 2.)

Remark 7.3. The reason why a direct sum Banach algebra is used is that, in the time-dependent setting of the operational calculus, the weights used for the Banach algebra $\mathbb{A}$ (and so also $\mathbb{D}$) depend explicitly on the time-ordering measures (see, for instance, Section 6.1 of [17]). If an E-valued function $A(\cdot)$ is associated with a time-ordering measure μ on $[0,T]$, the weight associated with this function is

$$\int_{[0,T]} |A(s)|_\beta \mu(ds)$$

where $\beta \subset E$ is closed and absolutely convex. Now, if $\{\mu_k\}_{k=1}^\infty$ is a sequence of measures on $[0,T]$ and if $\mu_k \rightharpoonup \mu$ as $k \to \infty$, then for each $k \in \mathbb{N}$ we have a weight

$$\int_{[0,T]} |A(s)|_\beta \mu_k(ds),$$

so the sequence of measures determine a sequence of weights. Consequently, in our setting, the sequences $\{\mu_{jk}\}_{k=1}^\infty$ induce a countably infinite family of $\mathbb{A}$-algebras as well as a countably infinite family of disentangling algebras. It is the presence of the countable sequence of disentangling algebras that gives us a sequence $\{\mathcal{T}_{\mu_{1k},\dots,\mu_{nk}}\}_{k=1}^\infty$ of disentangling maps. It is here that the need for the direct sum Banach algebra arises. For, if we make no additional assumptions about the sequences of measures, there is no particular relation between the polydisks on which the $\mathbb{A}$-algebras are defined and so no particular relation between the corresponding Banach algebras.

For each $k \in \mathbb{N} \cup \{0\}$ we have the disentangling map $\mathcal{T}_{\mu_{1k},\dots,\mu_{nk}} : \mathbb{D}_{B,k} \to E$. Our stability theorem can now be stated as follows.

Theorem 7.4. *We use the notation and definitions from above. Let* $\pi_k : \mathbb{D}_{B,\infty} \to \mathbb{D}_{B,k}$ *be the canonical projection for each* $k = 0,1,2,\dots$ *. For* $\theta_f := (f,f,f,\dots) \in \mathbb{D}_{B,\infty}$ *we have*

$$\lim_{k\to\infty} \left|\mathcal{T}_{\mu_{1k},\dots,\mu_{nk}}\left(\pi_k\left(\theta_f\right)\right) - \mathcal{T}_{\mu_1,\dots,\mu_n}\left(\pi_0\left(\theta_f\right)\right)\right|_B = 0. \tag{7.14}$$

Proof. We have already established

$$\lim_{k\to\infty} \left|P^{m_1,\dots,m_n}_{\mu_{1k},\dots,\mu_{nk}}\left(A_1(\cdot),\dots,A_n(\cdot)\right) - P^{m_1,\dots,m_n}_{\mu_1,\dots,\mu_n}\left(A_1(\cdot),\dots,A_n(\cdot)\right)\right|_B = 0.$$

Note that we do not need to use the projections π_k here, since the monomial $P^{m_1,\dots,m_n}$ is an entire function and so is an element of all of our disentangling algebras.

To continue, we select a vector $\theta_f = (f, f, f, \ldots) \in \mathbb{D}_{B,\infty}$ where f has the power series expansion given in (5.4). We obtain, successively,

$$\begin{aligned}
&\left|\mathcal{T}_{\mu_{1k},\ldots,\mu_{nk}}\pi_k\left(\theta_f\right) - \mathcal{T}_{\mu_1,\ldots,\mu_n}\pi_0\left(\theta_f\right)\right|_B \\
&\le \sum_{m_1,\ldots,m_n=0}^{\infty} |a_{m_1,\ldots,m_n}| \left|P^{m_1,\ldots,m_n}_{\mu_{1k},\ldots,\mu_{nk}}\left(A_1(\cdot),\ldots,A_n(\cdot)\right)\right. \\
&\left.-P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}\left(A_1(\cdot),\ldots,A_n(\cdot)\right)\right|_B \\
\le &\sum_{m_1,\ldots,m_n=0}^{\infty} |a_{m_1,\ldots,m_n}| \left\{\left|P^{m_1,\ldots,m_n}_{\mu_{1k},\ldots,\mu_{nk}}\left(A_1(\cdot),\ldots,A_n(\cdot)\right)\right|_B\right. \\
+ &\left.\left|P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}\left(A_1(\cdot),\ldots,A_n(\cdot)\right)\right|_B\right\} \\
\le &\sum_{m_1,\ldots,m_n=0}^{\infty} |a_{m_1,\ldots,m_n}| \left\{P^{m_1,\ldots,m_n}_{\mu_{1k},\ldots,\mu_{nk}}\left(|A_1(\cdot)|_B,\ldots,|A_n(\cdot)|_B\right)\right. \\
+ &\left.P^{m_1,\ldots,m_n}_{\mu_1,\ldots,\mu_n}\left(|A_1(\cdot)|_B,\ldots,|A_n(\cdot)|_B\right)\right\} \\
= &\|f\|_k + \|f\|_0 \\
\le &\|\theta_f\|_{\mathbb{D},\infty} + \|f\|_0.
\end{aligned} \tag{7.15}$$

Let $\epsilon > 0$ be given. From the definition of $\|\cdot\|_{\mathbb{D},\infty}$ in (7.13) there is a $k_0 \in \mathbb{N}$ such that $\|\theta_f\|_{\mathbb{D},\infty} \le \|f\|_{k_0} + \epsilon$. Therefore

$$\|\theta_f\|_{\mathbb{D},\infty} + \|f\|_0 \le \|f\|_{k_0} + \|f\|_0 + \epsilon.$$

It follows from the definition of the norms $\|\cdot\|_k$ for $k \in \mathbb{N} \cup \{0\}$ that the map

$$(m_1, \ldots, m_n) \mapsto |a_{m_1,\ldots,m_n}| \left[R_1^{m_1} \cdots R_n^{m_n} + R_{1,k_0}^{m_1} \cdots R_{n,k_0}^{m_n}\right] + \frac{\epsilon}{2^{m_1+\cdots+m_n}}$$

is a summable dominating function for the norm difference after the second inequality in (7.15) above. Furthermore, we recall that, as our functions $A_j(\cdot)$ are continuous and because $[0, T]$ is compact, we are working entirely in the Banach space $(E_B, |\cdot|_B)$. We are therefore working with Bochner integrals in this setting and we can apply the dominated convergence theorem for Bochner integrals to pass the limit $k \to \infty$ through the sum over $m_1, \ldots, m_n$ in the expression after the first inequality in (7.15). In view of (7.10), we have now established our theorem. □

We have, of course, assumed above that our time-ordering measures are continuous. However, when our measures have nonzero, finitely supported discrete parts, we can appeal to [25] (and, for a somewhat different presentation, [17]). At this time, there is no stability theory for Feynman's operational calculus with arbitrary time-ordering measures and so, while a stability theorem such as Theorem 7.4 likely exists, it is not currently known.

7.3. Stability with respect to the E-valued functions. We now turn to consideration of the stability of the E-valued version of Feynman's operational calculus with respect to the E-valued functions. As has been the case, we will continue to use continuous time-ordering measures. For stability of the operational calculus with respect to $\mathcal{L}(X)$-valued functions see, for example, [17], [24], [25], [23]. For Theorem 7.9, we need the following definition,

Definition 7.5. Let $A : [0,T] \to E$ and let μ be a measure on $[0,T]$. Assume that $A(\cdot)$ is totally μ-summable; i.e. $A \in L^1(\mu; E_B; E)$ for a closed, bounded and absolutely convex $B \subseteq E$. Define

$$\|A\|_\infty := \text{ess sup}|A(\cdot)|_B. \tag{7.16}$$

We also take the time here to outline some ideas which will arise in the statement and proof of Theorem 7.9 below. (We will make use of the notation $a_k \nearrow a$ to mean that the sequence $\{a_k\}_{k=1}^\infty$ is nondecreasing and $\lim_{k\to\infty} a_k = a$.) Let F be a Banach algebra. Given any continuous F-valued functions $A_1(\cdot), \ldots, A_n(\cdot)$ on $[0,T]$ with associated Borel measures $\mu_1, \ldots, \mu_n$ on $[0,T]$ we select sequences $\{A_{1,k}(\cdot)\}_{k=1}^\infty, \ldots, \{A_{n,k}(\cdot)\}_{k=1}^\infty$ of continuous F-valued functions for which

$$\int_{[0,T]} \|A_{j,k}\|_F \, \mu_j(ds) \nearrow \int_{[0,T]} \|A_j(s)\|_F \, \mu_j(ds) \tag{7.17}$$

as $k \to \infty$ for $j = 1, \ldots, n$. Let

$$r_{j,k} := \int_{[0,T]} \|A_{j,k}(s)\|_F \, \mu_j(ds)$$

and let

$$r_j := \int_{[0,T]} \|A_j(s)\|_F \, \mu_j(ds).$$

In view of our assumption in (7.17), $r_{j,k} \nearrow r_j$ as $k \to \infty$ for each $j = 1, \ldots, n$. We construct the families

$$\mathbb{A}_k := \mathbb{A}\left(r_{1,k}, \ldots, r_{n,k}\right) \text{ and } \mathbb{D}_k := \mathbb{D}\left(\tilde{A}_1(\cdot), \ldots, \tilde{A}_n(\cdot)\right)$$

for each $k \in \mathbb{N}$. Define the polydisk P_k, $k \in \mathbb{N}$, by

$$P_k := \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1| < r_{1,k}, \ldots, |z_n| < r_{n,k}\right\}. \tag{7.18}$$

Since each sequence $\{r_{j,k}\}_{k=1}^\infty$, $j = 1, \ldots, n$, is nondecreasing, we have

$$P_k \subseteq P_l \tag{7.19}$$

whenever $k \leq l$, and so

$$\mathbb{A}_l \subseteq A_k \tag{7.20}$$

whenever $k \leq l$.

Next, for $k, l \in \mathbb{N}$ with $k \leq l$, we define

$$g_{kl} : \mathbb{A}_l \to \mathbb{A}_k \tag{7.21}$$

by

$$g_{kl}(f) := f\Big|_{P_k}. \tag{7.22}$$

Because our sequences $\{r_{j,k}\}_{k=1}^{\infty}$, $j = 1, \dots, n$, are nondecreasing, it follows that

$$\|g_{kl}(f)\|_{\mathbb{A}_k} \le \|f\|_{\mathbb{A}_l}. \tag{7.23}$$

As g_{kl} is obviously linear and since (7.23) shows that g_{kl} is bounded, the maps g_{kl} are continuous for every k, l with $k \le l$. It follows that the spaces $\mathbb{A}_k$ and the maps g_{kl} form a projective system (see [30, Chapter 2, Section 5] for a discussion of projective topologies). We define

$$\mathbb{A}_\infty := \varprojlim_k \mathbb{A}_k = \left\{ \{f_k\}_{k=1}^{\infty} \in \prod_{k=1}^{\infty} \mathbb{A}_k \,:\, g_{kl}(f_l) = f_k \text{ when } k \le l \right\}, \tag{7.24}$$

the projective limit of the Banach algebras $\mathbb{A}_k$. Since the construction of a projective system is categorical, the projective limit $\mathbb{A}_\infty$ is itself a commutative Banach algebra in the norm

$$\|f\|_{\mathbb{A}_\infty} := \lim_{k\to\infty} \|f_k\|_{\mathbb{A}_k} \tag{7.25}$$

where $f := \{f_k\}_{k=1}^{\infty} \in \mathbb{A}_\infty$ and the norm on $\mathbb{A}_k$ is defined in Section 2. It is clear that the family $\{\mathbb{A}_k\}_{k=1}^{\infty}$ of commutative Banach algebras forms a nonincreasing sequence

$$\mathbb{A}_1 \supseteq \mathbb{A}_2 \supseteq \mathbb{A}_3 \supseteq \cdots \supseteq \mathbb{A}_k \supseteq \cdots \supseteq \mathbb{A}(r_1, \dots, r_n). \tag{7.26}$$

Using an identical construction with the disentangling algebras $\{\mathbb{D}_k\}_{k=1}^{\infty}$ gives us the projective limit

$$\mathbb{D}_\infty := \varprojlim_k \mathbb{D}_k = \left\{ \{f_k\}_{k=1}^{\infty} \in \prod_{k=1}^{\infty} \mathbb{D}_k \,:\, g_{kl}(f_l) = f_k \text{ when } k \le l \right\} \tag{7.27}$$

of the disentangling algebras $\mathbb{D}_k$, $k \in \mathbb{N}$. Moreover, the corresponding version of (7.26) holds for the family $\{\mathbb{D}_k\}_{k=1}^{\infty}$.

For exactly the same reason as above, the "disentangling algebra" $\mathbb{D}_\infty$ is a commutative Banach algebra when equipped with the norm

$$\|f\|_{\mathbb{D}_\infty} := \lim_{k\to\infty} \|f_k\|_{\mathbb{D}_k}$$

where $f := \{f_k\}_{k=1}^{\infty} \in \mathbb{D}_\infty$ and $\|\cdot\|_{\mathbb{D}_k}$ is defined in Section 2.

Remark 7.6. The commutative Banach algebras $\mathbb{A}$ and $\mathbb{D}$ are isometrically isomorphic (see Chapters 2 and 6, [17]). Using very similar arguments as those seen in Chapters 2 and 6 of [17], one can also show that $\mathbb{A}_\infty$ and $\mathbb{D}_\infty$ are also isometrically isomorphic.

We next define an analog of the disentangling map on the algebra $\mathbb{D}_\infty$. We begin by defining the Banach algebra

$$\mathcal{L}_F := \left\{ \{y_k\}_{k=1}^\infty \in \prod_{j=1}^\infty F : \left\| \{y_k\}_{k=1}^\infty \right\|_{\mathcal{L}_F} := \sup_k \|y_k\|_F < \infty \right\}. \tag{7.28}$$

We now define the analogue of the disentangling map on $\mathcal{L}_F$.

Definition 7.7. We define the disentangling map $\mathcal{T} : \mathbb{D}_\infty \to \prod_{j=1}^\infty F$ by

$$\mathcal{T}\left(\{f_k\}_{k=1}^\infty\right) := \left\{\mathcal{T}_{\mathbb{D}_k;\vec{\mu}} f_k\right\}_{k=1}^\infty, \tag{7.29}$$

where, for each $k \geq 1$, $\mathcal{T}_{\mathbb{D}_k;\vec{\mu}}$ denotes the disentangling map on $\mathbb{D}_k$, and where $\vec{\mu} := (\mu_1, \dots, \mu_n)$. Hence

$$\mathcal{T}_{\mathbb{D}_k;\vec{\mu}} f_k = \mathcal{T}_{\mu_1,\dots,\mu_n} f_k\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right).$$

The following proposition gives the essential properties of $\mathcal{T}$. Its proof is the same (with some notational changes) as that of Proposition 7.1.8 of [17].

Proposition 7.8. *The disentangling map $\mathcal{T}$ satisfies the following properties*:
(*1*) $\mathcal{T} : \mathbb{D}_\infty \to \prod_{j=1}^\infty F$ is linear.
(*2*) $\mathcal{T}\left(\{f_k\}_{k=1}^\infty\right) \in \mathcal{L}_F$ *for every* $\{f_k\}_{k=1}^\infty \in \mathbb{D}_\infty$.
(*3*) $\mathcal{T}$ *is continuous from* $\mathbb{D}_\infty$ *into* $\mathcal{L}_F$*; hence it is a bounded linear operator.*

The discussion above, of course, considered F-valued functions where F is a Banach algebra. In the stability theorem below, we will use E-valued functions $A : [0,T] \to E$ which will be associated with a continuous Borel probability measure μ on $[0,T]$. We will assume that these functions are continuous and totally μ-summable. The assumption of continuity tells us that $A([0,T])$ is compact in E and so is bounded in the topological algebra E. Hence, for the closed, bounded and absolutely convex $B \subseteq E$ such that

$$\int_{[0,T]} |A(s)|_B \,\mu(ds) < \infty,$$

we also have

$$\sup_{[0,T]} |A(s)|_B < \infty.$$

(See the discussion following Remark 5.2.) Hence $A(s) \in E_B$ for every $s \in [0,T]$; i.e. we do not have to deal with any null sets (as we did above Section 5, for instance). Finally, $E_B \subseteq E$ is a Banach space with the norm $|\cdot|_B$ and, because E is an algebra, E_B is a Banach algebra with norm $|\cdot|_B$.

The following theorem concerning the stability of the operational calculus with respect to the E-valued functions is stated using the notation and definitions introduced above.

Theorem 7.9. *For $j = 1, \ldots, n$ let $A_j : [0,T] \to E$, be associated with the continuous Borel probability measures μ_j on $[0,T]$. Assume that each $A_j(\cdot)$, $j = 1, \ldots, n$, is continuous and totally μ_j-summable. For each $j = 1, \ldots, n$ select a sequence $\{A_{j,k}(\cdot)\}_{k=1}^{\infty}$ of continuous and totally μ_j-summable functions. Fix $j \in \{1, \ldots, n\}$. There is a closed, bounded and absolutely convex $B_{j,0} \subseteq E$ for which*

$$\int_{[0,T]} |A_j(s)|_{B_{j,0}} \mu_j(ds) < \infty. \tag{7.30}$$

Next, for each $k \in \mathbb{N}$, there is a closed, bounded and absolutely convex $B_{j,k} \subseteq E$ for which

$$\int_{[0,T]} |A_{j,k}(s)|_{B_{j,k}} \mu_j(ds) < \infty. \tag{7.31}$$

Define U_j to be the closed absolutely convex hull of the union

$$\bigcup_{k=0}^{\infty} B_{j,k} \tag{7.32}$$

and assume that this union is a bounded subset of E. We obtain a collection $U_1, \ldots, U_n$ of closed, bounded and absolutely convex subsets of E. Define U to be the closed absolutely convex hull of the union

$$\bigcup_{j=1}^{n} U_j. \tag{7.33}$$

Then $U$is closed, bounded (see Proposition 1.4.12 of [2]*) and absolutely convex and we have*

$$\int_{[0,T]} |A_j(s)|_U \mu_j(ds) \le \int_{[0,T]} |A_j|_{B_{j,0}} \mu_j(ds) < \infty$$

as well as

$$\int_{[0,T]} |A_{j,k}(s)|_U \mu_j(ds) \le \int_{[0,T]} |A_{j,k}(s)|_{B_{j,k}} \mu_j(ds) < \infty.$$

We will consider each $A_j(\cdot), A_{j,k}(\cdot)$ as elements of $L^1(\mu_j; E_U; E)$, $j = 1, \ldots, n$, $k \in \mathbb{N}$. Assume that

$$\int_{[0,T]} |A_{j,k}(s)|_U \mu_j(ds) \nearrow \int_{[0,T]} |A_j(s)|_U \mu_j(ds) \tag{7.34}$$

as $k \to \infty$. Assume as well that

$$\sup\{\|A_{j,k}(s)\|_\infty : k \in \mathbb{N}\} < \infty \tag{7.35}$$

and that

$$\|A_j\|_\infty < \infty \tag{7.36}$$

for each $j \in \{1, \dots, n\}$*. For each* $j \in \{1, \dots, n\}$ *and for* $k \in \mathbb{N}$ *define*

$$R_{j,k} := \int_{[0,T]} |A_{j,k}(s)|_U \mu_j(ds) \tag{7.37}$$

and

$$R_j := \int_{[0,T]} |A_j(s)|_U \mu_j(ds). \tag{7.38}$$

Construct the algebras $\mathbb{A}_U, \mathbb{D}_U, \mathbb{A}_{U,k}, \mathbb{D}_{U,k}, \mathbb{A}_\infty$ *and* $\mathbb{D}_\infty$ *as above (see the discussion following Definition 7.5). Also define* $\mathcal{L}_{E_U}$ *as in (7.28) and define the disentangling map* $\mathcal{T} : \mathbb{D}_\infty \to \prod_{j=1}^\infty E_U$ *as in (7.29). Let* $f \in \mathbb{D}_U\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right)$ *and let* $\{f_k\}_{k=1}^\infty \in \mathbb{D}_\infty$ *be the sequence determined by* f*. Then*

$$\lim_{k\to\infty} \left|\mathcal{T}_{\mathbb{D}_{U,k};\vec{\mu}} f_k - \mathcal{T}_{\mathbb{D}_U;\vec{\mu}} f\right|_U = 0.$$

Proof. Let $\{f_k\}_{k=1}^\infty \in \mathbb{D}_\infty$ and let $f \in \mathbb{D}_U$ be the function which determines this sequence. (For the existence of such an f, see Remark 7.1.9 on page 243 of [17].) Fix $k \in \mathbb{N}$. We have

$$\begin{aligned}
&\left|\mathcal{T}_{\mathbb{D}_{U,k};\vec{\mu}} f_k - \mathcal{T}_{\mathbb{D}_U;\vec{\mu}} f\right|_U \\
&\le \sum_{m_1,\dots,m_n=0}^\infty |a_{m_1,\dots,m_n}| \sum_{\pi\in S_m} \int_{\Delta_m(\pi)} \Big| C_{\pi(m),k}(s_{\pi(m)}) \cdots C_{\pi(1),k}(s_{\pi(1)}) \\
&- C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \Big|_U \left(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}\right)(ds_1, \dots, ds_m),
\end{aligned} \tag{7.39}$$

where k is the sequence index. Fix $m_1, \dots, m_n \in \mathbb{N} \cup \{0\}$ and $\pi \in S_m$, with $m := m_1 + \cdots + m_n$. Using (7.34), (7.35), and (7.36), we have

$$\begin{aligned}
&\lim_{k\to\infty} \int_{\Delta_m(\pi)} \left| C_{\pi(m),k}\left(s_{\pi(m)}\right) \cdots C_{\pi(1),k}\left(s_{\pi(1)}\right) - C_{\pi(m)}\left(s_{\pi(m)}\right) \cdots C_{\pi(1)}\left(s_{\pi(1)}\right) \right|_U \\
&\left(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}\right)(ds_1, \dots, ds_m) \\
&= 0,
\end{aligned}$$

and so

$$\begin{aligned}
&\lim_{k\to\infty} \sum_{\pi\in S_m} \int_{\Delta_m(\pi)} \Big| C_{\pi(m),k}\left(s_{\pi(m)}\right) \cdots C_{\pi(1),k}\left(s_{\pi(1)}\right) - \\
&C_{\pi(m)}\left(s_{\pi(m)}\right) \cdots C_{\pi(1)}\left(s_{\pi(1)}\right) \Big|_U \left(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}\right)(ds_1, \dots, ds_m) \\
&= 0.
\end{aligned}$$

It remains to show that the limit on k can be interchanged with the sum over $m_1, \dots, m_n$. To do this, note that a scalar-valued dominating function for the right-hand side of the inequality (7.39) is

$$(m_1, \dots, m_n) \mapsto 2|a_{m_1,\dots,m_n}| \sum_{\pi\in S_m} \int_{\Delta_m(\pi)} \left|C_{\pi(m)}\left(s_{\pi(m)}\right)\right|_U \cdots$$

$$\left|C_{\pi(1)}\left(s_{\pi(1)}\right)\right|_U \left(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}\right)(ds_1, \ldots, ds_n), \tag{7.40}$$

where we've made use of the hypothesis (7.34). Since

$$\sum_{m_1,\ldots,m_n=0}^{\infty} 2|a_{m_1,\ldots,m_n}| \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} \left|C_{\pi(m)}\left(s_{\pi(m)}\right)\right|_U \cdots$$
$$\left|C_{\pi(1)}\left(s_{\pi(1)}\right)\right|_U \left(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}\right)(ds_1, \ldots, ds_m)$$
$$= 2 \sum_{m_1,\ldots,m_n=0}^{\infty} |a_{m_1,\ldots,m_n}| \left(\int_{[0,T]} |A_1(s)|_U \ \mu_1(ds)\right)^{m_1} \cdots$$
$$\left(\int_{[0,T]} |A_n(s)|_U \ \mu_n(ds)\right)^{m_n}$$
$$= 2\|f\|_{\mathbb{D}} < \infty,$$

we see that (7.40) defines a summable dominating function for the right-side of the inequality (7.39). The limit interchange can therefore be carried out and the proof is finished. □

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