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Brunnian planar braids and simplicial groups

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Abstract. Twin groups are planar analogues of Artin braid groups and play a crucial role in the Alexander-Markov correspondence for the isotopy classes of immersed circles on the 2-sphere without triple and higher intersections. These groups admit diagrammatic representations, leading to maps obtained by the addition and deletion of strands. This paper explores Brunnian twin groups, which are subgroups of twin groups composed of twins that become trivial when any of their strands are deleted. We establish that Brunnian twin groups consisting of more than two strands are free groups. Furthermore, we provide a necessary and sufficient condition for a Brunnian doodle on the 2-sphere to be the closure of a Brunnian twin. Additionally, we delve into two generalizations of Brunnian twins, namely, k-decomposable twins and Cohen twins, and prove some structural results about these groups. We also investigate a simplicial structure on pure twin groups that admits a simplicial homomorphism from Milnor's construction of the simplicial 2-sphere. This gives a possibility to provide a combinatorial description of homotopy groups of the 2-sphere in terms of pure twins.

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1. Introduction

The twin group, or the planar braid group T_n on $n \geq 2$ strands, is a right angled Coxeter group generated by $n-1$ involutions that admit only far commutativity relations. These groups appeared in the work of Khovanov [19] on real $K(\pi, 1)$ subspace arrangements and were further investigated in [17]. Twin groups have a geometrical interpretation similar to the one for Artin braid groups [17, 19]. We fix parallel lines $y = 0$ and $y = 1$ on the plane $\mathbb{R}^2$ with n marked points on each line. Consider the set of configurations of n strands in the strip $\mathbb{R} \times [0, 1]$ connecting the n marked points on the line $y = 1$ to those on the line $y = 0$ such that each strand is monotonic and no three strands have a point in common. Two such configurations are equivalent if one can be deformed into the other by a homotopy of strands, keeping the end points fixed throughout the homotopy. Such an equivalence class is called a *twin*. Placing one twin on top of another and rescaling the interval turns the set of all twins on n strands into a group isomorphic to T_n. The generators t_i of T_n can be geometrically represented by configurations as shown in Figure 1.

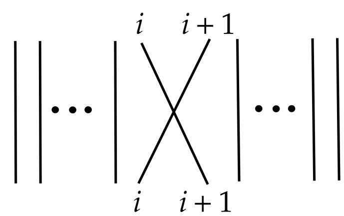

FIGURE 1. The generator t_i of T_n.

Analogous to classical knot theory, it is evident that the closure of a twin gives a *doodle* on the 2-sphere. In general, a doodle on a closed surface is a collection of finitely many piecewise-linear closed curves without triple intersections. These objects first appeared in the work of Fenn and Taylor [13]. In [17], Khovanov proved that every oriented doodle on the 2-sphere is the closure of a twin. A Markov theorem for doodles on the 2-sphere has been established by Gotin [16], although the idea has been implicit in [18]. These constructions have been generalised by Bartholomew-Fenn-Kamada-Kamada [5], where they consider a collection of immersed circles in closed oriented surfaces of arbitrary genus.

The pure twin group, denoted as PT_n, is defined as the kernel of the natural homomorphism from the twin group T_n to the symmetric group S_n, which maps the twin t_i to the transposition $(i, i+1)$. A nice topological interpretation of PT_n is known due to Khovanov. Consider the space

$$X_n = \mathbb{R}^n \setminus \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = x_j = x_k \text{ for } i \neq j \neq k \neq i\},$$

which is the complement of triple diagonals $x_i = x_j = x_k$. In [19], Khovanov proved that the fundamental group $\pi_1(X_n)$ is isomorphic to PT_n. Prior to this, Björner and Welker [9] had investigated the cohomology of these spaces, establishing that each $H^i(X_n, \mathbb{Z})$ is free.

Simplicial structures on braid groups are connected with homotopy groups of some manifolds [6, 8, 20, 23, 27]. Notably, they provide a description of elements in homotopy groups of the 2-sphere in terms of Brunnian braids [6], with a generalization to higher dimensional spheres [27]. A set of generators for the Brunnian braid group of a surface other than the 2-sphere and the projective plane has been provided in [2]. Furthermore, Brunnian subgroups of mapping class groups have been considered in [7]. In this paper, we explore simplicial structures on pure twin groups. The geometrical interpretation of elements in twin groups allows us to define face and degeneracy maps obtained by the deletion and addition of strands, thereby transforming the family of pure twin groups into a simplicial group. We adopt the approach introduced by Cohen and Wu in [11] for Artin pure braid groups.

The paper is organised as follows. In Section 2, we prove that the natural maps of deletion and addition of strands turn the sequence $\{T_n\}_{n\geq 1}$ into a bi-Δ-set, whereas the sequence $\{PT_n\}_{n\geq 1}$ is turned into a bi-Δ-group (Proposition 2.4). In Section 3, we investigate Brunnian twins, which are twins that become trivial when any one of their strands is removed. We prove that the group $\mathrm{Brun}(T_n)$ of Brunnian twins on n strands is free for $n \geq 3$ (Proposition 3.9), and give an infinite free generating set for $\mathrm{Brun}(T_4)$ (Theorem 3.5). In Section 4, we consider two generalisations of Brunnian twins, namely, k-decomposable twins and Cohen twins. A twin is k-decomposable if it becomes trivial after removing any k of its strands. We give a complete description of k-decomposable twins on $n \geq 4$ strands (Proposition 4.4). A twin on n strands is said to be Cohen if the twins obtained by removing any one of its strands are all the same. We give a characterisation for a twin to be Cohen (Theorem 4.10). In Section 5, we consider Brunnian doodles on the 2-sphere, and prove that an m-component Brunnian doodle on the 2-sphere is the closure of a Brunnian twin if and only if its twin index is m (Theorem 5.4). In Section 6, we observe that pure twin groups admit the structure of a simplicial group SPT_*. We relate it with the well-known Milnor's construction for simplicial spheres by establishing a homomorphism $\Theta : F[S^2]_* \longrightarrow SPT_*$ of simplicial groups. We also identify some low degree terms of the image of Θ as free groups (Theorem 6.6). A complete description of the image of Θ gives a possibility to provide a combinatorial description of homotopy groups of the 2-sphere in terms of pure twins.

2. Bi-Δ-set structure on twin and pure twin groups

For $n \geq 2$, the *twin group* T_n on n strands is generated by $\{t_1, \dots, t_{n-1}\}$ and it is defined by the following relations:

$$t_i^2 = 1 \quad \text{for} \quad 1 \leq i \leq n-1$$

and

$$t_i t_j \quad = \quad t_j t_i \quad \text{for} \quad |i-j| \geq 2.$$

Clearly, each T_n is a right angled Coxeter group. Further, there is a surjective homomorphism $\nu : T_n \to S_n$, that sends the generator t_i to the transposition $\tau_i = (i, i+1)$ in the symmetric group S_n. It's kernel, denoted as PT_n, is called the pure twin group. It is not difficult to see that PT_2 is trivial and PT_3 is the infinite cyclic group generated by the pure twin $(t_1t_2)^3$ [1]. Figure 2 represents the pure twin $(t_1t_2)^3$.

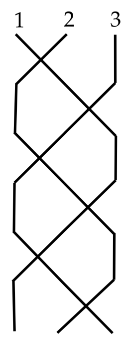

FIGURE 2. The pure twin $(t_1t_2)^3$.

Let us consider the following definitions [33, p.16].

Definition 2.1. *A sequence of sets $\{G_n\}_{n\geq 0}$ is called a Δ-set if there are maps $d_i : G_n \to G_{n-1}$ for each $0 \leq i \leq n$ such that*

$$d_j d_i = d_i d_{j+1} \tag{2.1}$$

for all $j \geq i$. The maps d_i are called face maps. If each G_n is a group and each face map is a group homomorphism, then $\{G_n\}_{n\geq 0}$ is called a Δ-group.

Definition 2.2. *A sequence of sets $\{G_n\}_{n\geq 0}$ is called a bi-Δ-set if there are face maps $d_i : G_n \to G_{n-1}$ and coface maps $d^i : G_{n-1} \to G_n$ for each $0 \leq i \leq n$ such that the following identities hold:*

1. $d_j d_i = d_i d_{j+1}$ *for $j \geq i$,*
2. $d^j d^i = d^{i+1} d^j$ *for $j \leq i$,*
3. $d_j d^i = d^{i-1} d_j$ *for $j < i$,*
4. $d_j d^i = \mathrm{id}$ *for $j = i$,*
5. $d_j d^i = d^i d_{j-1}$ *for $j > i$.*

Moreover, if each G_n is a group and each face and coface map is a group homomorphism, then $\{G_n\}_{n\geq 0}$ is called a bi-Δ-group.

We define a bi-Δ-set structure on twin groups that would induce a bi-Δ-group structure on pure twin groups. For geometrical reasons, we take $G_n = T_{n+1}$ or

PT_{n+1} for each $n \geq 0$. For each $0 \leq i \leq n$, define the map

$$d_i : T_{n+1} \to T_n$$

that deletes the $(i+1)$-th strand from the diagram of a twin on $n+1$ strands. Note that, d_i is not a group homomorphism, but it satisfies

$$d_i(uw) = d_i(u)d_{\nu(u)(i+1)-1}(w) \tag{2.2}$$

for all $u, w \in T_{n+1}$, where $\nu : T_{n+1} \to S_{n+1}$ is the natural surjection. On the other hand, we have $d_i(PT_{n+1}) \subseteq PT_n$ for each $0 \leq i \leq n$. Further, it follows from (2.2) that $d_i : PT_{n+1} \to PT_n$ is a surjective group homomorphism for each $0 \leq i \leq n$.

Remark 2.3. The homomorphism $d_i : PT_{n+1} \to PT_n$ has an alternative interpretation. Consider the space

$$X_n = \mathbb{R}^n \setminus \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = x_j = x_k, \quad i \neq j \neq k \neq i\},$$

which is the complement of triple diagonals $x_i = x_j = x_k$ in $\mathbb{R}^n$. For each $1 \leq i \leq n+1$, let

$$(p_i)_\# : \pi_1(X_{n+1}) \to \pi_1(X_n)$$

be the group homomorphism induced by the coordinate projection $p_i : X_{n+1} \to X_n$, where

$$p_i(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}).$$

By [19, Proposition 3.1], we identify $\pi_1(X_n)$ with PT_n, and observe that $(p_i)_\# = d_{i-1}$ for each $1 \leq i \leq n+1$.

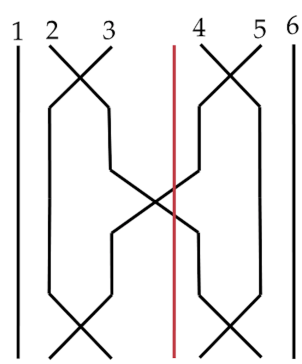

FIGURE 3. The geometrical interpretation of a coface map.

In analogy with [33, Example 1.2.8], for each $0 \leq i \leq n$, we define $d^i : T_n \to T_{n+1}$ on generators by

$$d^i(t_j) = \begin{cases} t_j & \text{for} \quad j < i, \\ t_{i+1}t_it_{i+1} & \text{for} \quad j = i, \\ t_{j+1} & \text{for} \quad j > i. \end{cases} \tag{2.3}$$

A direct calculation shows that each d^i satisfy the defining relations in T_n, and hence is a group homomorphism. Geometrically, the coface maps d^0 and d^n simply insert a trivial strand on the left and on the right of the diagram of a twin, respectively. Further, for $1 \leq i \leq n-1$, the map d^i inserts a trivial strand between the i-th and the $(i+1)$-th strands of the diagram of the twin generator t_i such this new strand passes from the right of the crossing corresponding to t_i. For other twin generators t_j with $j \neq i$, the map d^i simply inserts a trivial strand between the i-th and the $(i+1)$-th strands of t_j. See Figure 3 for an illustration.

Proposition 2.4. *Consider the sequence of groups $\{T_n\}_{n\geq 1}$. For each $0 \leq i \leq n$, let $d_i : T_{n+1} \to T_n$ be the map satisfying* (2.2) *and $d^i : T_n \to T_{n+1}$ be the homomorphism defined by* (2.3)*. Then $\{T_n, d_i, d^i\}_{n\geq 1}$ is a bi-Δ-set and $\{PT_n, d_i, d^i\}_{n\geq 1}$ is a bi-Δ-group.*

Proof. For each $0 \leq i \leq n$, the map $d_i : T_{n+1} \to T_n$ clearly satisfies (2.1). Let $d^i : T_n \to T_{n+1}$ be the homomorphism defined by (2.3). A direct computation yields

$$d^j d^i(t_k) = d^{i+1} d^j(t_k) = \begin{cases} t_k & \text{for} \quad k < j \leq i, \\ t_{k+2}t_{k+1}t_k t_{k+1}t_{k+2} & \text{for} \quad j = k = i, \\ t_{k+1}t_k t_{k+1} & \text{for} \quad j = k < i, \\ t_{k+2}t_{k+1}t_{k+2} & \text{for} \quad j < k = i, \\ t_{k+1} & \text{for} \quad j < k < i, \\ t_{k+2} & \text{for} \quad j \leq i < k, \end{cases}$$

for all $j \leq i$. This proves the identity (2). The identities (3)-(5) follows from the geometrical interpretation of d_i and d^i. Hence, $\{T_n, d_i, d^i\}_{n\geq 1}$ is a bi-Δ-set.

We already noticed that, for each $0 \leq i \leq n$, $d_i(PT_{n+1}) \subseteq PT_n$ and $d_i : PT_{n+1} \to PT_n$ is a group homomorphism. The inclusion $d^i(PT_n) \subseteq PT_{n+1}$ follows from the geometrical interpretation of the map d^i. Alternatively, for each $n \geq 1$, let $\eta^i : S_n \to S_{n+1}$ be the map defined by

$$\eta^i(\tau_j) = \begin{cases} \tau_j & \text{for} \quad j < i, \\ \tau_{i+1}\tau_i\tau_{i+1} & \text{for} \quad j = i, \\ \tau_{j+1} & \text{for} \quad j > i. \end{cases}$$

As with the case of d^i, each η^i satisfies the far commutativity and involutory relations of generators of S_n. For braid relations, we see that

$$\begin{aligned} \eta^i(\tau_k)\eta^i(\tau_{k+1})\eta^i(\tau_k) &= \begin{cases} \tau_k\tau_{k+1}\tau_k & \text{for} \quad k+1 < i, \\ \tau_k\tau_{k+1}\tau_{k+2}\tau_{k+1}\tau_k & \text{for } i = k, k+1, \\ \tau_{k+1}\tau_{k+2}\tau_{k+1} & \text{for} \quad k > i, \end{cases} \\ &= \eta^i(\tau_{k+1})\eta^i(\tau_k)\eta^i(\tau_{k+1}), \end{aligned}$$

and hence each η^i is a group homomorphism. Then the inclusion $d^i(PT_n) \subseteq PT_{n+1}$ also follows from the commutativity of the following diagram

$$\begin{array}{ccc} T_n & \xrightarrow{d^i} & T_{n+1} \\ \nu\big\downarrow & & \big\downarrow\nu \\ S_n & \xrightarrow{\eta^i} & S_{n+1}. \end{array}$$

Finally, we prove that each d^i is group homomorphism at the level of twin groups itself. Clearly, $(d^i(t_k))^2 = 1$ for all i and k. Further, for $k < \ell$ with $|k-\ell| \geq 2$, we have

$$\begin{aligned} d^i(t_k)d^i(t_\ell) &= \begin{cases} t_k t_\ell & \text{for} \quad \ell < i, \\ t_k t_{\ell+1} t_\ell t_{\ell+1} & \text{for} \quad \ell = i, \\ t_k t_{\ell+1} & \text{for} \quad k < i < \ell, \\ t_{k+1} t_k t_{k+1} t_{\ell+1} & \text{for} \quad k = i, \\ t_{k+1} t_{\ell+1} & \text{for} \quad i < k, \end{cases} \\ &= d^i(t_\ell)d^i(t_k). \end{aligned}$$

This proves that $\{PT_n, d_i, d^i\}_{n\geq 1}$ is a bi-Δ-group. □

Remark 2.5. For each $0 \leq i \leq n$, we can also define the coface maps $d^i : T_n \to T_{n+1}$ by

$$d^i(t_j) = \begin{cases} t_j & \text{for} \quad j < i, \\ t_i t_{i+1} t_i & \text{for} \quad j = i, \\ t_{j+1} & \text{for} \quad j > i. \end{cases}$$

It can be verified that the analogue of Proposition 2.4 holds with these coface maps.

We now use the bi-Δ-set structure on $\{T_n\}_{n\geq 1}$ to give a new presentation for T_{n+1}. We use the coface maps d^i as defined in Proposition 2.4.

Proposition 2.6. *Let $q_k := d^n d^{n-1} \cdots d^k(t_k)$ for $1 \leq k \leq n-1$ and $q_n := d^{n-2}(t_{n-1})$. Then T_{n+1} admits a presentation with generating set $\{q_1, \dots, q_n\}$ and the following defining relations:*

(1) $q_i^2 = 1$ *for all i,*
(2) $[q_{i+1}q_iq_{i+1}, q_n] = 1$ *for $i < n-1$,*
(3) $[q_{i+1}q_iq_{i+1}, q_{j+1}q_jq_{j+1}]$ *for $|i-j| > 2$ and $i, j \leq n-1$.*

Proof. Using the simplicial identity $d^j d^i = d^{i+1} d^j$ for $j \leq i$, we can assume that $i_{n-1} > i_{n-2} > \cdots > i_1$ in the composite map $d^{i_{n-1}} d^{i_{n-2}} \cdots d^{i_1}$. We see that

$$\begin{aligned} q_k &= d^{n-1} d^{n-2} \cdots d^k(t_k) \\ &= t_n t_{n-1} \cdots t_{k+1} t_k t_{k+1} \cdots t_{n-1} t_n \end{aligned}$$

for $1 \leq k \leq n-1$ and

$$q_n = d^{n-2}(t_{n-1}) = t_n.$$

A direct check gives $t_k = q_{k+1}q_kq_{k+1}$ for each $1 \leq k \leq n-1$, and hence $\{q_1, \dots, q_n\}$ generates T_{n+1}. Further, the defining relations of T_{n+1} in terms of the Coxeter generating set $\{t_1, \dots, t_n\}$ gives the defining relations for the new generating set as follows:

(1) $q_i^2 = 1$ for all i,
(2) $[q_{i+1}q_iq_{i+1}, q_n] = 1$ for $i < n-1$,
(3) $[q_{i+1}q_iq_{i+1}, q_{j+1}q_jq_{j+1}]$ for $|i-j| > 2$ and $i, j \leq n-1$.

□

3. Brunnian twins

In the influential work [6], a connection has been established between certain quotients of the Brunnian braid groups of the 2-sphere and its higher homotopy groups.

Definition 3.1. *A pure twin is said to be Brunnian if it becomes trivial after removing any one of its strands.*

Let $\mathrm{Brun}(T_n)$ denote the set of all Brunnian twins on n strands.

Proposition 3.2. $\mathrm{Brun}(T_n)$ *is a normal subgroup of* PT_n.

Proof. For each $0 \leq i \leq n-1$, let $d_i : PT_n \to PT_{n-1}$ be the face map of Proposition 2.4. Since each d_i is a group homomorphism and

$$\mathrm{Brun}(T_n) = \bigcap_{i=0}^{n-1} \ker(d_i),$$

it follows that $\mathrm{Brun}(T_n)$ is a normal subgroup of T_n. □

Next, we attempt to understand the groups of Brunnian twins.

Proposition 3.3. *For* $n \geq 4$, $\mathrm{Brun}(T_n)$ *does not contain any element from* $\{(t_it_{i+1})^3 \mid 1 \leq i \leq n-1\}$.

Proof. For $n \geq 4$, removing a trivial strand from $(t_it_{i+1})^3$ gives a non-trivial twin, and hence the assertion follows. □

Proposition 3.4. $\mathrm{Brun}(T_3) \cong PT_3 \cong \mathbb{Z}$.

Proof. We already have $\mathrm{Brun}(T_3) \subseteq PT_3$. By [1, Theorem 2], we know that PT_3 is the infinite cyclic group generated by $(t_1t_2)^3$ (see Figure 2), and clearly $(t_1t_2)^3 \in \mathrm{Brun}(T_3)$. □

In contrast, it is proved in [22] that $\mathrm{Brun}(B_3)$ is the commutator subgroup of the Artin pure braid group P_3.

Theorem 3.5. $\mathrm{Brun}(T_4)$ *is a free group of infinite rank.*

Proof. By [1, Theorem 2], PT_4 is a free group of rank 7 generated by

$$x_1 = (t_1t_2)^3, \quad x_2 = ((t_1t_2)^3)^{t_3}, \quad x_3 = ((t_1t_2)^3)^{t_3t_2}, \quad x_4 = ((t_1t_2)^3)^{t_3t_2t_1},$$

$$x_5 = (t_2t_3)^3, \quad x_6 = ((t_2t_3)^3)^{t_1}, \quad x_7 = ((t_2t_3)^3)^{t_1t_2}.$$

Denote the generator $(t_1t_2)^3$ of PT_3 by y. Direct computations show that images of x_i's under the face maps d_i's are as follows:

$$\begin{aligned}
d_0(x_1) = d_0(x_2) = d_0(x_3) = d_0(x_6) = d_0(x_7) = 1,\ d_0(x_4) = d_0(x_5) = y,\\
d_1(x_1) = d_1(x_2) = d_1(x_4) = d_1(x_5) = d_1(x_7) = 1,\ d_1(x_3) = d_1(x_6) = y,\\
d_2(x_1) = d_2(x_3) = d_2(x_4) = d_2(x_5) = d_2(x_6) = 1,\ d_2(x_2) = d_2(x_7) = y,\\
d_3(x_2) = d_3(x_3) = d_3(x_4) = d_3(x_5) = d_3(x_6) = d_3(x_7) = 1,\ d_3(x_1) = y.
\end{aligned}$$

For each generator x_i, let $\log_i(w)$ denote the sum of the powers of x_i in the word w. Then, it follows that

$$\begin{aligned}
\ker(d_0) &= \{w \in PT_4 \mid \log_4(w) + \log_5(w) = 0\},\\
\ker(d_1) &= \{w \in PT_4 \mid \log_3(w) + \log_6(w) = 0\},\\
\ker(d_2) &= \{w \in PT_4 \mid \log_2(w) + \log_7(w) = 0\},\\
\ker(d_3) &= \{w \in PT_4 \mid \log_1(w) = 0\},
\end{aligned}$$

and hence

$$\begin{aligned}
&\mathrm{Brun}(T_4)\\
= &\bigcap_{i=0}^{3} \ker(d_i)\\
= &\Big\{w \in PT_4 \mid \log_4(w) + \log_5(w) = 0, \log_3(w) + \log_6(w) = 0,\\
&\log_2(w) + \log_7(w) = 0, \log_1(w) = 0\Big\}.
\end{aligned}$$

Clearly, $\mathrm{Brun}(T_4)$ is free being a subgroup of the free group PT_4. We now find an infinite free basis for $\mathrm{Brun}(T_4)$. It follows from the preceding description of $\mathrm{Brun}(T_4)$ that the commutator subgroup of PT_4 is contained in $\mathrm{Brun}(T_4)$. In fact, the containment is strict since $x_4x_5^{-1} \in \mathrm{Brun}(T_4)$, but $x_4x_5^{-1} \notin PT_n'$. Thus, $PT_4/\mathrm{Brun}(T_4)$ is a non-trivial abelian group. Let $q : PT_4 \to PT_4/\mathrm{Brun}(T_4)$ be the quotient map with $q(x_i) = y_i$ for $1 \le i \le 7$. Since $x_2x_7^{-1}, x_3x_6^{-1}, x_4x_5^{-1} \in \mathrm{Brun}(T_4)$, the group $PT_4/\mathrm{Brun}(T_4)$ is generated by the set $\{y_1, y_2, y_3, y_4\}$. Note that $x_ix_j^{-1} \notin \mathrm{Brun}(T_4)$ for all $i \neq j \in \{1, 2, 3, 4\}$ and $x_i^k \notin \mathrm{Brun}(T_4)$ for $k > 0$. Thus, by the fundamental theorem for finitely generated abelian groups, $PT_4/\mathrm{Brun}(T_4)$ is a free abelian group of rank 4.

Consider the short exact sequence

$$1 \to \mathrm{Brun}(T_4) \to PT_4 \to \mathbb{Z}^4 \to 1.$$

We fix a Schreier system $\{x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4} \mid k_1, k_2, k_3, k_4 \in \mathbb{Z}\}$ of coset representatives of $\mathrm{Brun}(T_4)$ in PT_4. This gives a free basis for $\mathrm{Brun}(T_4)$ consisting of elements of the form

$$x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4}x_1(x_1^{k_1+1}x_2^{k_2}x_3^{k_3}x_4^{k_4})^{-1}, \qquad x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4}x_2(x_1^{k_1}x_2^{k_2+1}x_3^{k_3}x_4^{k_4})^{-1},$$
$$x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4}x_3(x_1^{k_1}x_2^{k_2}x_3^{k_3+1}x_4^{k_4})^{-1}, \qquad x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4}x_4(x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4+1})^{-1},$$
$$x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4}x_5(x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4+1})^{-1}, \qquad x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4}x_6(x_1^{k_1}x_2^{k_2}x_3^{k_3+1}x_4^{k_4})^{-1},$$
$$x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4}x_7(x_1^{k_1}x_2^{k_2+1}x_3^{k_3}x_4^{k_4})^{-1},$$

for $k_1, k_2, k_3, k_4 \in \mathbb{Z}$. This completes the proof. □

Remark 3.6. $PT_n/\mathrm{Brun}(T_n)$ is non-abelian for $n \geq 5$ since $PT_n' \not\subseteq \mathrm{Brun}(T_n)$ for $n \geq 5$. For example, if $w = [(t_1t_2)^3, (t_2t_3)^3]$, then $d_4(w) \neq 1$.

Proposition 3.7. *$PT_n/\mathrm{Brun}(T_n)$ is torsion free for each $n \geq 4$ and*

$$PT_4/\mathrm{Brun}(T_4) \cong \mathbb{Z}^4.$$

Proof. The homomorphisms $d_i : PT_n \to PT_{n-1}$ induce an injective homomorphism

$$PT_n/\mathrm{Brun}(T_n) \hookrightarrow \underbrace{PT_{n-1} \times \cdots \times PT_{n-1}}_{n \text{ times}}.$$

By [1, Theorem 3], PT_n is torsion-free for $n \geq 3$. Hence, $PT_{n-1} \times \cdots \times PT_{n-1}$ is torsion free, and therefore $PT_n/\mathrm{Brun}(T_n)$ is so. The second assertion is proven in the proof of Theorem 3.5. □

Problem 3.8. *Describe the structure of the group $PT_n/\mathrm{Brun}(T_n)$ for $n \geq 5$.*

Recall from [1, 30] that the *virtual twin group* VT_n on $n \geq 2$ strands is the group generated by $\{t_1, \dots, t_{n-1}, \rho_1, \dots, \rho_{n-1}\}$ and having the following defining relations:

$$\begin{aligned}
t_i^2 &= 1 \quad \text{for} \quad 1 \leq i \leq n-1,\\
t_it_j &= t_jt_i \quad \text{for} \quad |i-j| \geq 2,\\
\rho_i^2 &= 1 \quad \text{for} \quad 1 \leq i \leq n-1,\\
\rho_i\rho_j &= \rho_j\rho_i \quad \text{for} \quad |i-j| \geq 2,\\
\rho_i\rho_{i+1}\rho_i &= \rho_{i+1}\rho_i\rho_{i+1} \quad \text{for} \quad 1 \leq i \leq n-2,\\
\rho_it_j &= t_j\rho_i \quad \text{for} \quad |i-j| \geq 2,\\
\rho_i\rho_{i+1}t_i &= t_{i+1}\rho_i\rho_{i+1} \quad \text{for} \quad 1 \leq i \leq n-2.
\end{aligned}$$

The group VT_n plays the role of virtual braid groups in the Alexander-Markov correspondence for the planar analogue of virtual knot theory. There is a surjective homomorphism $\mu : VT_n \to S_n$ given by

$$\mu(t_i) = \mu(\rho_i) = (i, i+1)$$

for all $1 \leq i \leq n-1$. The kernel PVT_n of this surjection is called the *pure virtual twin group* on n strands.

For each $n \geq 2$, we have surjective homomorphisms $d_{n-1} : PT_n \to PT_{n-1}$ and $\overline{d}_{n-1} : PVT_n \to PVT_{n-1}$ that delete the n-th strand from the diagram of a pure twin and pure virtual twin. In the reverse directions, we have homomorphisms $d^{n-1} : PT_{n-1} \to PT_n$ and $\overline{d}^{n-1} : PVT_{n-1} \to PVT_n$ that add a trivial strand to the right side of the diagram. Further, we have $d_{n-1}\, d^{n-1} = \mathrm{id}_{PT_{n-1}}$ and $\overline{d}_{n-1}\, \overline{d}^{n-1} = \mathrm{id}_{PVT_{n-1}}$. Setting $U_n = \ker(d_{n-1})$ and $V_n = \ker(\overline{d}_{n-1})$, we have split short exact sequences

$$1 \to U_n \to PT_n \to PT_{n-1} \to 1$$

and

$$1 \to V_n \to PVT_n \to PVT_{n-1} \to 1.$$

In other words, $PT_n \cong U_n \rtimes PT_{n-1}$ and $PVT_n \cong V_n \rtimes PVT_{n-1}$.

Proposition 3.9. $\mathrm{Brun}(T_n)$ *is free for all* $n \geq 3$.

Proof. The map $t_i \mapsto t_i$ gives an embedding of T_n into PVT_n [31, Corollary 3.5]. Restricting to PT_n, this gives an inclusion $\psi_n : PT_n \to PVT_n$ such that the following diagram commutes

$$\begin{array}{ccc} PT_n & \xrightarrow{d_{n-1}} & PT_{n-1} \\ \downarrow \psi_n & & \downarrow \psi_{n-1} \\ PVT_n & \xrightarrow{\overline{d}_{n-1}} & PVT_{n-1}. \end{array}$$

This gives $U_n \cong \psi_n(U_n) = \psi_n(\ker(d_{n-1})) \leq \ker(\overline{d}_{n-1}) = V_n$. Since V_n is free for $n \geq 2$ [30, Theorem 4.1], it follows that U_n is also free. Note that the subgroup $U_i = \ker(d_i)$ is conjugate to U_n by the element $t_{n-1}t_{n-2}\cdots t_{i+1}$. Thus, U_i is free group for each $1 \leq i \leq n$, and hence $\mathrm{Brun}(T_n) = \cap_{i=1}^{n} U_i$ is a free group. □

At this juncture, the ensuing problem naturally arises.

Problem 3.10. *Determine a free generating set for* $\mathrm{Brun}(T_n)$ *for* $n \geq 5$.

We conclude the section with a consequence of the Decomposition Theorem for bi-Δ-groups in our setting [33, Proposition 1.2.9].

Proposition 3.11. *The pure twin group* PT_{n+1} *is the iterated semi-direct product of subgroups*

$$\left\{ d^{i_k} d^{i_{k-1}} \cdots d^{i_1}(\mathrm{Brun}(T_{n-k+1})) \mid 0 \leq i_1 < i_2 < \cdots < i_k \leq n \text{ and } 0 \leq k \leq n \right\}$$

with the lexicographic order on the indexing set

$$\left\{ (i_k, i_{k-1}, \ldots, i_1, \underbrace{i_0, i_0, \ldots, i_0}_{n-k \text{ times}}) \mid 0 \leq i_1 < i_2 < \cdots < i_k \leq n \text{ and } 0 \leq k \leq n \right\}$$

from the left, where i_0 *is the blank symbol considered smaller than all other indices.*

Example 3.12. For $n = 3$, we have

$$d^0(PT_3) = \langle (t_2t_3)^3 \rangle, \qquad d^1(PT_3) = \langle (t_2t_1t_2t_3)^3 \rangle,$$
$$d^2(PT_3) = \langle (t_1t_3t_2t_3)^3 \rangle, \qquad d^3(PT_3) = \langle (t_1t_2)^3 \rangle.$$

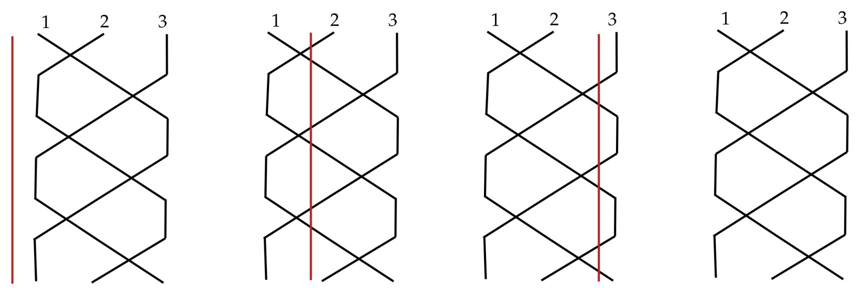

FIGURE 4. Images of $(t_1t_2)^3$ under the coface maps d^0, d^1, d^2 and d^3.

See Figure 4. Observing the proof of [33, Proposition 1.2.9], we get

$$PT_4 = \ker(d_3) \rtimes \langle (t_1t_2)^3 \rangle,$$

where $\ker(d_3)$ is normal in PT_4 and $\langle (t_1t_2)^3$ acts on $\ker(d_3)$ via conjugation. At the second stage, we obtain

$$\ker(d_3) = (\ker(d_2) \cap \ker(d_3)) \rtimes \langle (t_1t_3t_2t_3)^3 \rangle,$$

where $\ker(d_2) \cap \ker(d_3)$ is normal in $\ker(d_3)$ and the subgroup

$$\langle (t_1t_3t_2t_3)^3 \rangle = \langle d^2((t_1t_2)^3) \rangle \leq \ker(d_3)$$

acts on $\ker(d_2) \cap \ker(d_3)$ via conjugation. At the third stage, we get

$$\ker(d_2) \cap \ker(d_3) = (\ker(d_1) \cap \ker(d_2) \cap \ker(d_3)) \rtimes \langle (t_2t_1t_2t_3)^3 \rangle,$$

where $\ker(d_1) \cap \ker(d_2) \cap \ker(d_3)$ is normal in $\ker(d_2) \cap \ker(d_3)$ and the subgroup $\langle (t_2t_1t_2t_3)^3 \rangle = \langle d^1((t_1t_2)^3) \rangle \leq \ker(d_2) \cap \ker(d_3)$ acts on $\ker(d_1) \cap \ker(d_2) \cap \ker(d_3)$ via conjugation. Finally, we have

$$\ker(d_1) \cap \ker(d_2) \cap \ker(d_3) = \mathrm{Brun}(T_4) \rtimes \langle (t_2t_3)^3 \rangle,$$

where $\mathrm{Brun}(T_4)$ is normal and the subgroup $\langle (t_2t_3)^3 \rangle = \langle d^0((t_1t_2)^3) \rangle$ acts on $\mathrm{Brun}(T_4)$ via conjugation. Thus, we obtain the following decomposition of PT_4 as an iterated semi-direct product

$$PT_4 = \Big(\big(\big(\mathrm{Brun}(T_4) \rtimes \langle (t_2t_3)^3 \rangle \big) \rtimes \langle (t_2t_1t_2t_3)^3 \rangle \big) \rtimes \langle (t_1t_3t_2t_3)^3 \rangle \Big) \rtimes \langle (t_1t_2)^3 \rangle.$$

Similarly, there are 16 non-trivial terms in the decomposition of PT_5 with the leftmost term being the Brunnian subgroup $\mathrm{Brun}(T_5)$.

4. *k*-decomposable twins and Cohen twins

In this section, we consider two generalisations of Brunnian twins.

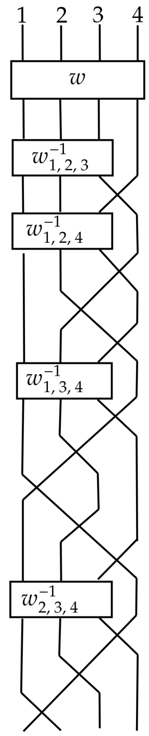

FIGURE 5. Converting a pure twin into a Brunnian twin.

4.1. *k*-decomposable twins.

We begin with the following definition.

Definition 4.1. *A pure twin on n strands is said to be k-decomposable if it becomes trivial after removing any k of its strands.*

Clearly, a 1-decomposable twin is simply a Brunnian twin. Further, the set of all k-decomposable twins on n strands forms a normal subgroup of PT_n and we denote this subgroup by $D_{k,n}$. For $w \in PT_n$ and $1 \leq i < j < k \leq n$, let $w_{i,j,k}$ be the pure twin obtained from w by deleting all the strands except those indexed i, j, k. We can still view each $w_{i,j,k}$ as an element of PT_n by adding trivial $(n-3)$ strands on its right. See Figure 5 for an example for $n = 4$. Using ideas from [21], we prove the following result.

Proposition 4.2. *For $n \geq 4$,*

$$D_{n-3,n} = \{w \prod_{1 \leq i<j<k \leq n} (w^{-1}_{i,j,k})^{c_{i,j,k}} \mid w \in PT_n\},$$

where $c_{i,j,k} \in T_n$ is a coset representative of the permutation in $T_n/PT_n \cong S_n$ which takes i, j, k to $1, 2, 3$, respectively, and fix everything else.

Proof. In view of Proposition 3.4, we have $w_{i,j,k} \in \mathrm{Brun}(T_3)$. A direct check shows that for any $w \in PT_n$, the pure twin

$$w \prod_{1 \le i < j < k \le n} (w_{i,j,k}^{-1})^{c_{i,j,k}}$$

is a $(n-3)$-decomposable twin on n strands. Note that the map $\phi : PT_n \to D_{n-3,n}$ given by

$$\phi(w) = w \prod_{1 \le i < j < k \le n} (w_{i,j,k}^{-1})^{c_{i,j,k}}$$

is a retraction, that is, the restriction of ϕ on $D_{n-3,n}$ is the identity map. Hence, it follows that each element of $D_{n-3,n}$ arises in this fashion. □

Corollary 4.3. $\mathrm{Brun}(T_4) = \{ww_{1,2,3}^{-1}(w_{1,2,4}^{-1})^{t_3}(w_{1,3,4}^{-1})^{t_2t_3}(w_{2,3,4}^{-1})^{t_1t_2t_3} \mid w \in PT_4\}$.

Next, we describe a process of constructing $D_{k-1,n}$ from $D_{k,n}$. Let $w \in D_{k,n}$ and $1 \le i_1 < i_2 \cdots < i_{n-k+1} \le n$. Let $w_{i_1,i_2,\dots,i_{n-k+1}}$ be the pure twin obtained from w by removing the $k-1$ strands except those indexed $i_1, i_2, \dots, i_{n-k+1}$. Since $w \in D_{k,n}$, we have $w_{i_1,i_2,\dots,i_{n-k+1}} \in \mathrm{Brun}(T_{n-k+1})$. The following result can be proved along the lines of Proposition 4.2.

Proposition 4.4. *For $n \ge 4$,*

$$D_{k-1,n} = \{w \prod_{1 \le i_1 < i_2 \cdots < i_{n-k+1} \le n} (w_{i_1,i_2,\dots,i_{n-k+1}}^{-1})^{c_{i_1,i_2,\dots,i_{n-k+1}}} \mid w \in D_{k,n}\},$$

where $c_{i_1,i_2,\dots,i_{n-k+1}} \in T_n$ is a coset representative of the permutation in $T_n/PT_n \cong S_n$ which takes $i_1, i_2, \dots, i_{n-k+1}$ to $1, 2, \dots, n-k+1$, respectively, and fix everything else.

Beginning with $PT_n = D_{n-2,n} = D_{n-1,n}$ and iterating the procedure of constructing $D_{k-1,n}$ from $D_{k,n}$, we can construct all Brunnian twins on n strands.

4.2. Cohen twins.

Next, we consider another generalisation of Brunnian twins motivated by an idea due to Fred Cohen [10], and developed further for surface braid groups in [4]. Recall that, for $0 \le i \le n-1$, the face map $d_i : T_n \to T_{n-1}$ deletes the $(i+1)$-st strand from the diagram of a twin. Although d_i is not a group homomorphism, it satisfies

$$d_i(uw) = d_i(u)d_{\nu(u)(i+1)-1}(w), \tag{4.1}$$

where $\nu : T_{n+1} \to S_{n+1}$ is the natural surjection. For an arbitrary $u \in T_{n-1}$, we ask whether there exists $w \in T_n$ which is a solution of the system of equations

$$\begin{cases} d_0(w) = u, \\ d_1(w) = u, \\ \vdots \\ d_{n-1}(w) = u. \end{cases} \tag{4.2}$$

Taking $u = 1$ amounts to $w \in T_n$ being a Brunnian twin.

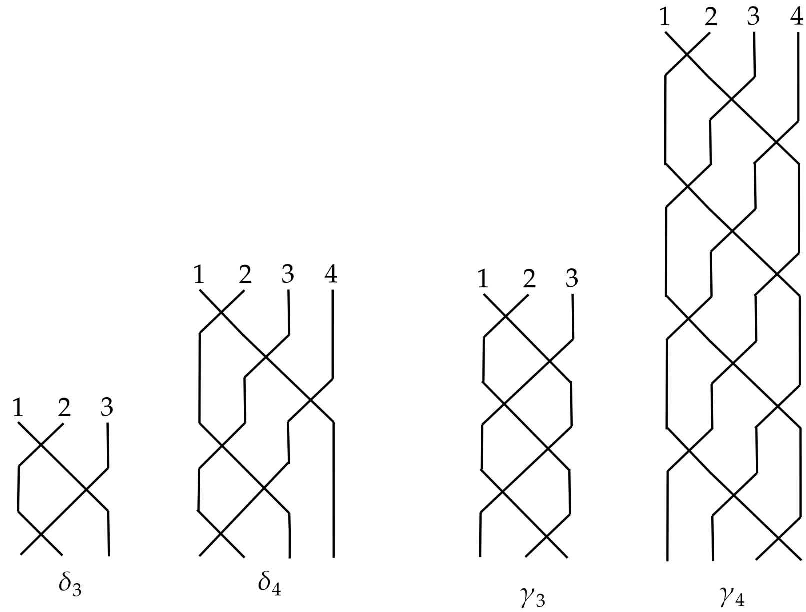

FIGURE 6. Elements δ_3, δ_4 and γ_3, γ_4.

Definition 4.5. *A twin $w \in T_n$ is called a Cohen twin if $d_0(w) = d_1(w) = \cdots = d_{n-1}(w)$.*

For $n \geq 2$, let us set

$$CT_n = \{w \in T_n \mid d_0(w) = d_1(w) = \cdots = d_{n-1}(w)\}.$$

In other words, a twin on n strands lie in CT_n if it gives the same twin on $(n-1)$ strands after removing any one of its strands. For example, the twin

$$\delta_n := (t_1 t_2 \cdots t_{n-1})(t_1 t_2 \cdots t_{n-2}) \cdots (t_1 t_2) t_1$$

lies in CT_n for all $n \geq 2$ and $d_0(\delta_n) = \delta_{n-1}$ (see Figure 6). Similarly, we define

$$CPT_n = CT_n \cap PT_n = \{w \in PT_n \mid d_0(w) = d_1(w) = \cdots = d_{n-1}(w)\}.$$

We refer to elements of CPT_n as *pure Cohen twins*. For instance, the pure twin

$$\gamma_n := (t_1 t_2 \cdots t_{n-1})^n$$

lies in CPT_n for all $n \geq 2$ and $d_0(\gamma_n) = \gamma_{n-1}$ (see Figure 6).

If $\phi, \psi : G \to H$ are group homomorphisms, then their *equalizer* is the subgroup of G given by

$$\{g \in G \mid \phi(g) = \psi(g)\}.$$

Hence, CPT_n is a subgroup of PT_n being the equalizer of group homomorphisms $d_0, d_1, \ldots, d_{n-1} : PT_n \to PT_{n-1}$.

Proposition 4.6. *The following assertions hold:*

(1) *For each* $0 \leq i \leq n-1$, $d_i(CPT_n) \subseteq CPT_{n-1}$ *and the map* $d_0 = d_1 = \cdots = d_{n-1} : CPT_n \to CPT_{n-1}$ *is a group homomorphism.*
(2) *The set* CT_n *is a subgroup of* T_n. *Moreover, for each* $0 \leq i \leq n-1$, $d_i(CT_n) \subseteq CT_{n-1}$ *and the map* $d_0 = d_1 = \cdots = d_{n-1} : CT_n \to CT_{n-1}$ *is a group homomorphism.*

Proof. Let $w \in CPT_n$ and $0 \leq i \leq n-1$. Then, using (2.1), we obtain

$$d_j(d_i(w)) = d_j(d_0(w)) = d_0(d_{j+1}(w)) = d_0(d_i(w)) \tag{4.3}$$

for each $0 \leq j \leq n-2$, and hence $d_i(CPT_n) \subseteq CPT_{n-1}$. That $d_0 = d_1 = \cdots = d_{n-1} : CPT_n \to CPT_{n-1}$ is a group homomorphism follows from Proposition 2.4.

For the second assertion, let $u, w \in CT_n$. By (4.1), we have

$$d_i(uw) = d_i(u) d_{\nu(u)(i+1)-1}(w) = d_0(u) d_{\nu(u)(1)-1}(w) = d_0(uw) \tag{4.4}$$

for each $0 \leq i \leq n-1$, and hence $uw \in CT_n$. Further, the equation

$$1 = d_i\left(u^{-1}u\right) = d_i\left(u^{-1}\right) d_{\nu(u^{-1})(i+1)-1}(u) = d_i\left(u^{-1}\right) d_0(u),$$

gives

$$d_i\left(u^{-1}\right) = \left(d_0(u)\right)^{-1}$$

for each $0 \leq i \leq n-1$, and hence CT_n is a subgroup of T_n. The proof of $d_i(CT_n) \subseteq CT_{n-1}$ follows from 4.3. Finally, (4.4) also shows that $d_0 = d_1 = \cdots = d_{n-1} : CT_n \to CT_{n-1}$ is a group homomorphism. □

Proposition 4.7. *CPT_n is an index two subgroup of CT_n for $n \geq 3$.*

Proof. The topological interpretation of elements of T_n can be applied to elements of S_n as well by allowing triple intersection points. Thus, for each $0 \leq i \leq n-1$, there is a map $\bar{d}_i : S_n \to S_{n-1}$ (thought of as deleting the $(i+1)$-st strand) such the following diagram commutes

$$\begin{array}{ccccc} PT_n & \hookrightarrow & T_n & \xrightarrow{\nu_n} & S_n \\ \downarrow d_i & & \downarrow d_i & & \downarrow \bar{d}_i \\ PT_{n-1} & \hookrightarrow & T_{n-1} & \xrightarrow{\nu_{n-1}} & S_{n-1}. \end{array}$$

Set $CS_n := \nu_n(CT_n)$ for each $n \geq 2$. Note that $CS_2 = \nu_2(T_2) = S_2 \cong \mathbb{Z}_2$. The commutativity of the preceding diagram shows that every $\tau \in CS_n$ satisfy $\bar{d}_0(\tau) = \bar{d}_1(\tau) = \cdots = \bar{d}_{n-1}(\tau)$. By Proposition 4.6(2), we have $d_0(CT_n) \subseteq CT_{n-1}$. The commutativity of the preceding diagram implies that $\bar{d}_0(CS_n) = \bar{d}_0 \nu_n(CT_n) = \nu_{n-1} d_0(CT_n) \subseteq \nu_{n-1}(CT_{n-1}) = CS_{n-1}$. Thus, for $n \geq 3$, the restriction of the map $\bar{d}_0 : S_n \to S_{n-1}$ induces a map $\bar{d}_0 : CS_n \to CS_{n-1}$ such that $\ker(\bar{d}_0) = \cap_{i=0}^{n-1} \ker(\bar{d}_i)$. Direct computation gives $\ker(\bar{d}_0) = 1$, and hence the map $\bar{d}_0 \cdots \bar{d}_0 : CS_n \to CS_2$ is injective. Since $\nu_n(\delta_n) \neq 1$, we have $CT_n / CPT_n \cong CS_n \cong \mathbb{Z}_2$, and the proof is complete. □

The following result follows along the lines of [32, Lemma 2.10].

Proposition 4.8. *For each* $1 \leq k \leq n-1$*, the map*

$$\underbrace{d_0 \cdots d_0}_{(n-k)\ times} : CPT_n \to CPT_k$$

is surjective. In particular, $d_0 : CPT_n \to CPT_{n-1}$ *is surjective for* $n \geq 2$*.*

Proof. Let us set $d_{n-k,n} = \underbrace{d_0 \cdots d_0}_{(n-k)\text{ times}}$. We use induction on k. Clearly, for $k = 1$, the map $d_{n-1,n} : CPT_n \to CPT_1$ is surjective. Assume that $d_{n-k+1,n}$ is surjective with $k > 1$, and let $w \in CPT_k$.

Case 1: Suppose that $w \in \ker(d_0 : CPT_k \to CPT_{k-1})$. Then consider the element

$$w_{k,n} = \prod_{0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} d^{i_{n-k}} d^{i_{n-k-1}} \cdots d^{i_1}(w)$$

of PT_n with lexicographic order on the indices from the right. Since $w \in \ker(d_0 : CPT_k \to CPT_{k-1})$, a straightforward computation shows that $w_{k,n} \in CPT_n$ and $d_{n-k,n}(w_{n,k}) = w$. For instance, taking $n = 4$ and $k = 1$, we have

$$w_{1,4} = \prod_{0 \leq i_1 < i_2 < i_3 \leq 3} d^{i_3} d^{i_2} d^{i_1}(w)$$

with lexicographic order from the right. Note that (i_1, i_2, i_3) all lie in the set $\{(0,1,2), (0,1,3), (0,2,3), (1,2,3)\}$ and

$$w_{1,4} = d^2 d^1 d^0(w)\, d^3 d^1 d^0(w)\, d^3 d^2 d^0(w)\, d^3 d^2 d^1(w).$$

Direct computations give

$$\begin{aligned}
d_0(w_{1,4}) &= d^1 d^0(w)\, d^2 d^0(w)\, d^2 d^1(w)\, d^2 d^1 d^0(d_0(w)),\\
d_1(w_{1,4}) &= d^1 d^0(w)\, d^2 d^0(w)\, d^2 d^1 d^0(d_0(w))\, d^2 d^1(w),\\
d_2(w_{1,4}) &= d^1 d^0(w)\, d^2 d^1 d^0(d_0(w))\, d^2 d^0(w)\, d^2 d^1(w),\\
d_3(w_{1,4}) &= d^2 d^1 d^0(d_0(w))\, d^1 d^0(w)\, d^2 d^0(w)\, d^2 d^1(w).
\end{aligned}$$

Since $w \in \ker(d_0 : CPT_k \to CPT_{k-1})$, $d^2 d^1 d^0(d_0(w)) = 1$, and hence $w_{1,4} \in CPT_4$.

Case 2: Now, suppose that $1 \neq \delta = d_0(w) \in CPT_{k-1}$. By induction hypothesis, there exists $\gamma \in CPT_n$ such that $d_{n-k+1,n}(\gamma) = d_0(d_{n-k,n}(\gamma)) = \delta$. Note that

$$w\, d_{n-k,n}(\gamma)^{-1} \in \ker(d_0 : CPT_k \to CPT_{k-1}).$$

Thus, by Case 1, there exists $\lambda \in CPT_n$ such that

$$d_{n-k,n}(\lambda) = w\, d_{n-k,n}(\gamma)^{-1},$$

and hence $d_{n-k,n}(\lambda\gamma) = w$. This proves that the map $d_{n-k,n}$ is surjective. □

Proposition 4.9. *The map* $d_0 : CT_n \to CT_{n-1}$ *is surjective for each* $n \geq 2$*.*

Proof. In view of Proposition 4.7, we can write $CT_{n-1} = CPT_{n-1} \cup \delta_{n-1} CPT_{n-1}$. Let us take $w \in CT_{n-1}$. If $w \in CPT_{n-1}$, then by Proposition 4.8, there exists an $u \in CPT_n$ such that $d_0(u) = w$. If $w \in \delta_{n-1} CPT_{n-1}$, then again by Proposition 4.8, there exists $v \in CPT_n$, such that $d_0(v) = \delta_{n-1}^{-1} w$, and hence $d_0(\delta_n v) = w$. This complete the proof. □

Thus, we obtain the following short exact sequences

$$1 \to \mathrm{Brun}(T_n) \to CT_n \to CT_{n-1} \to 1$$

and

$$1 \to \mathrm{Brun}(T_n) \to CPT_n \to CPT_{n-1} \to 1.$$

Observe that $CPT_2 = \mathrm{Brun}(T_2) = PT_2 = 1$ and $CPT_3 = \mathrm{Brun}(T_3) = PT_3 = \langle (t_1 t_2)^3 \rangle \cong \mathbb{Z}$. Thus, the preceding exact sequence gives $CPT_4 = \mathrm{Brun}(T_4) \rtimes \langle (t_1 t_2)^3 \rangle$.

Theorem 4.10. *For each $u \in PT_{n-1}$ or $u \in T_{n-1}$, the system of equations*

$$\begin{cases} d_0(w) = u, \\ d_1(w) = u, \\ \vdots \\ d_{n-1}(w) = u, \end{cases} \tag{4.5}$$

has a solution if and only if u satisfies the condition

$$d_0(u) = d_1(u) = \cdots = d_{n-2}(u).$$

Proof. Let $u \in PT_{n-1}$ such that the system of equations (4.2) has a solution. Then there exists $w \in PT_n$ such that $d_0(w) = \cdots = d_{n-1}(w) = u$. It follows from Proposition 4.6 that $u \in CPT_{n-1}$, and hence $d_0(u) = \cdots = d_{n-2}(u)$. Conversely, suppose that $d_0(u) = \cdots = d_{n-2}(u)$, that is, $u \in CPT_{n-1}$. By Proposition 4.8, $d_0 : CPT_n \to CPT_{n-1}$ is surjective, and hence there exists $w \in CPT_n$ which is a solution to (4.2). The proof for the case when $u \in T_{n-1}$ is similar. □

5. Brunnian doodles on the 2-sphere

Note that the closure of a Brunnian braid is a Brunnian link. The converse is not true and there exist Brunnian links that cannot be obtained as the closure of Brunnian braids (see [12]). The same scenario occurs with doodles on the 2-sphere. Consider the Brunnian doodle on the 2-sphere as shown in Figure 7. We will justify in Remark 5.6 that this Brunnian doodle cannot be realised as the closure of a Brunnian twin.

Definition 5.1. *A doodle diagram on the 2-sphere is called minimal if it has no monogons and bigons.*

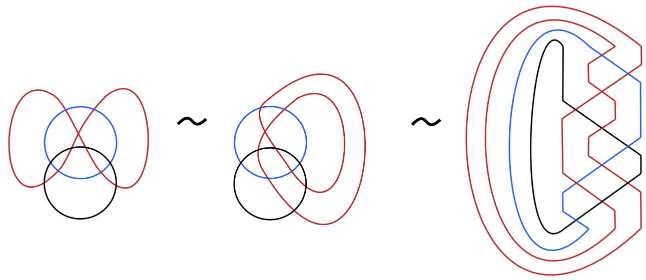

FIGURE 7. A Brunnian doodle which is not the closure of a Brunnian twin.

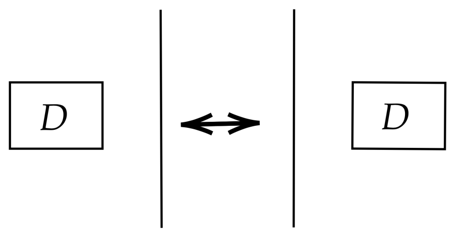

FIGURE 8. Transformation of doodle diagrams.

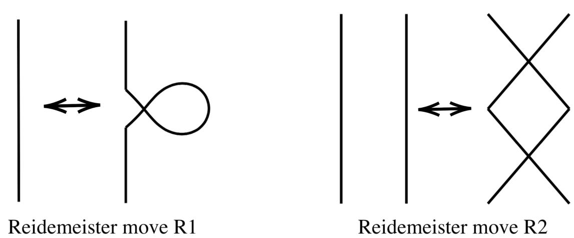

FIGURE 9. Reidemeister moves.

Theorem 5.2. [17, Theorem 2.2] *Any doodle has a unique (up to the transformation shown in Figure* 8*) minimal doodle diagram with a minimal number of intersection points. Further, this minimal doodle diagram can be constructed from any other doodle diagram by applying Reidemeister moves R*1 *and R*2 *(see Figure* 9*) that reduce the number of intersection points.*

For a given reduced word $w = t_{i_1} \dots t_{i_k} \in T_n$, let $\ell(w) = k$ be the *length* of w. For each $1 \leq i \leq n-1$, if $\log_i(w)$ denote the number of t_i's present in the expression w, then

$$\ell(w) = \sum_{i=1}^{n-1} \log_i(w).$$

A *cyclic permutation* of a word $w = t_{i_1} \dots t_{i_k} \in T_n$ (not necessarily reduced) is a word $w' = t_{i_r} t_{i_{r+1}} \dots t_{i_k} t_{i_1} t_{i_2} \cdots t_{i_{r-1}}$ for some $1 \leq r \leq k$. It is easy to see that w and w' are conjugate to each other in T_n, in fact,

$$w' = (t_{i_1} t_{i_2} \dots t_{i_{r-1}})^{-1} w (t_{i_1} t_{i_2} \dots t_{i_{r-1}}).$$

A word w is called *cyclically reduced* if each cyclic permutation of w is reduced. Clearly, a cyclically reduced word is reduced.

Lemma 5.3. *Let $w \in PT_n$ be a pure twin. Then the following assertions hold:*

(1) *If $\ell(w)$ is minimal among all the elements in the conjugacy class of w, then the closure of w is a minimal doodle diagram.*
(2) *The closure of w is an n-component trivial doodle if and only if w is a trivial twin.*

Proof. It follows from [29, Corollary 2.4] that each word in T_n is conjugate to some cyclically reduced word. Since $\ell(w)$ is minimal among all the elements in the conjugacy class of w, it follows that w is a cyclically reduced word. Hence, the closure of w has no bigons. Since w is pure twin, its closure has no monogons, and hence the diagram is minimal.

By Markov Theorem for doodles on the 2-sphere [16, Theorem 4.1], conjugate twins have the same closure. Thus, we can assume that $\ell(w)$ is minimal among all the elements in the conjugacy class of w. It follows from assertion (1) that the closure of w is a minimal doodle diagram. Note that the number of double points in the closure of the twin w equals $\ell(w)$, and hence $\ell(w) = 0$. But, this implies that w is trivial twin. The converse implication in assertion (2) is obvious. □

Let $\widehat{w}$ denote the closure of a twin w on the 2-sphere. By [17, Theorem 2.1], every oriented doodle on the 2-sphere is the closure of a twin. The *twin index* $I(D)$ of a doodle D on the 2-sphere is the minimal n such that there is a twin $w \in T_n$ whose closure is equivalent to D.

Theorem 5.4. *An m-component Brunnian doodle D on the 2-sphere is the closure of a Brunnian twin if and only if $I(D) = m$.*

Proof. If u is a Brunnian twin on m strands, then its closure on the 2-sphere is a Brunnian doodle on m components with $I(\widehat{u}) = m$. Conversely, if D is a Brunnian doodle on m components and $I(D) = m$, then there exist $w \in PT_m$ such that $\widehat{w} = D$. Removing any strand from w corresponds to removing a component from D. Thus, $\widehat{d_i(w)}$ is a trivial doodle for each i. By Lemma 5.3, $d_i(w) = 1$ for each i, and hence w is a Brunnian twin. □

Remark 5.5. An analogue of Theorem 5.4 for Brunnian links in S^3 is proved in [24, Theorem 2.2].

Remark 5.6. The Brunnian doodle in Figure 7 cannot have twin index 3, since the closure of a pure twin on 3 strands is a minimal doodle diagram with the number of crossings being a multiple of 6. Hence, this Brunnian doodle cannot be realised as the closure of a Brunnian twin.

6. Simplicial structure on pure twin groups

In this section, we discuss simplicial structures on twin and pure twin groups and relate them with Milnor's construction for simplicial spheres.

6.1. Simplicial sets and simplicial groups.

We recall some basic definitions and constructions [25, 26].

Definition 6.1. *A sequence of sets $X_* = \{X_n\}_{n\geq 0}$ is called a simplicial set if there are face maps*

$$d_i : X_n \longrightarrow X_{n-1} \text{ for } 0 \leq i \leq n$$

and degeneracy maps

$$s_i : X_n \longrightarrow X_{n+1} \text{ for } 0 \leq i \leq n,$$

which satisfy the following simplicial identities:

(1) $d_i d_j = d_{j-1} d_i$ *if* $i < j$,
(2) $s_i s_j = s_{j+1} s_i$ *if* $i \leq j$,
(3) $d_i s_j = s_{j-1} d_i$ *if* $i < j$,
(4) $d_j s_j = \mathrm{id} = d_{j+1} s_j$,
(5) $d_i s_j = s_j d_{i-1}$ *if* $i > j + 1$.

We view X_n geometrically as the set of n-simplices including all possible degenerate simplices. Here, a simplex x is *degenerate* if $x = s_i(y)$ for some simplex y and degeneracy operator s_i, otherwise x is *non-degenerate*. A simplicial set X_* is *pointed* if we fix a basepoint $\star \in X_0$ that creates one and only one degenerate n-simplex in each X_n by applying iterated degeneracy operations on it. A *simplicial group* is a simplicial set X_* such that each X_n is a group and all face and degeneracy maps are group homomorphisms.

Remark 6.2. In the context of braid-type groups (for example, braid group B_n, virtual braid group VB_n, welded braid group WB_n, etc.), the maps d_i usually represents deleting of the $(i + 1)$-th strand and s_i represents doubling of the $(i + 1)$-th strand.

Remark 6.3. Note that the defining identities of a bi-Δ-set and that of a simplicial set are similar. The only differences are that we don't have $d_{j+1}s_j = \mathrm{id}$ for bi-Δ-sets, and when viewed as maps from $X_{n-1} \to X_n$, the number of degeneracy maps is one less than the number of coface maps. We have used the bi-Δ-set structure at three instances in the preceding sections. The first instance of usage of a bi-Δ-set is Proposition 2.4, though its arguments can be modified to adapt to a simplicial set structure. The second instance is the proof of Proposition 4.8, where we defined the element $w_{k,n}$ and showed that $w_{k,n} \in CPT_n$. In the latter case, a simplicial structure would not be helpful. Finally, using the Decomposition Theorem for bi-Δ-groups, we have given a decomposition of pure twin groups in Proposition 3.11 with Brunnian subgroups as constituents.

Let $G_* = \{G_n\}_{n\geq 0}$ be a simplicial group. The group of *Moore n-cycles* $Z_n(G_*) \leq G_n$ is defined by

$$Z_n(G_*) = \bigcap_{i=0}^{n} \mathrm{Ker}(d_i : G_n \to G_{n-1})$$

and the group of *Moore n-boundaries* $B_n(G_*) \leq G_n$ is defined by

$$B_n(G_*) = d_0\left(\bigcap_{i=1}^{n+1} \mathrm{Ker}(d_i : G_{n+1} \to G_n)\right).$$

Simplicial identities guarantees that $B_n(G_*)$ is a (normal) subgroup of $Z_n(G_*)$ (see [6, Proposition 4.1.3] or [14, Example 7.7]). The n-th *Moore homotopy group* $\pi_n(G_*)$ of G_* is defined by

$$\pi_n(G_*) = Z_n(G_*)/B_n(G_*).$$

It is a classical result due to Moore [28] that $\pi_n(G_*) \cong \pi_n(|G_*|)$, where $|G_*|$ is the geometric realisation of G_*. A simplicial group G_* is called *contractible* if $\pi_n(G_*) = 1$ for all $n > 0$.

Milnor's $F[K]$ construction is the adjoint functor to the forgetful functor from the category of pointed simplicial groups to the category of pointed simplicial sets. For a given pointed simplicial set $K_* = \{K_n, \star\}_{n\geq 0}$, Milnor's $F[K]$ construction is the simplicial group with $F[K]_n = F(K_n \setminus \star)$, the free group on $K_n \setminus \star$, with the face and the degeneracy maps induced from the face and degeneracy maps of K_*. It is well-known from [26] that there is weak homotopy equivalence

$$|F[K]_*| \simeq \Omega\Sigma|K_*|, \tag{6.1}$$

where $|X_*|$ denotes the geometric realisation of a simplicial set X_*. Here, ΩZ is the loop space of all based loops in a pointed topological space Z and ΣZ is the reduced suspension of Z.

Consider the pointed simplicial 2-sphere $S^2 = \Delta[2]/\partial\Delta[2]$ with

$$S^2_0 = \{\star\},\ S^2_1 = \{\star\},\ S^2_2 = \{\star, \sigma\},\ S^2_3 = \{\star, s_0(\sigma), s_1(\sigma), s_2(\sigma)\}, \dots,$$

$$S^2_n = \{\star, x_{ij} \mid 0 \leq i < j \leq n-1\}, \dots$$

where $\sigma = (0, 1, 2)$ is the non-degenerate 2-simplex and

$$x_{ij} = s_{n-1} \dots s_{j+1}\widehat{s_j}s_{j-1} \dots s_{i+1}\widehat{s_i}s_{i-1} \dots s_0(\sigma)$$

with $\widehat{s_k}$ meaning that the degeneracy map s_k is omitted. Then $F[S^2]$ construction has the following terms:

$$
\begin{aligned}
F[S^2]_0 &= 1,\\
F[S^2]_1 &= 1,\\
F[S^2]_2 &= F(\sigma),\\
F[S^2]_3 &= F(s_0(\sigma), s_1(\sigma), s_2(\sigma)),\\
F[S^2]_4 &= F(s_1s_0(\sigma), s_2s_0(\sigma), s_3s_0(\sigma), s_2s_1(\sigma), s_3s_1(\sigma), s_3s_2(\sigma)),\\
&\vdots\\
F[S^2]_n &= F(x_{ij};\ 0 \le i < j \le n-1),\\
&\vdots
\end{aligned}
$$

For each $n \ge 2$, the group $F[S^2]_n$ is a free group of rank $n(n-1)/2$. In this construction of the simplicial 2-sphere, it is convenient to present the degeneracy map s_i as a doubling of the $(i+1)$-th component and the face map d_i as deletion of the $(i+1)$-th component. For example,

$$
s_0(\sigma) = (0,0,1,2), \quad s_1(\sigma) = (0,1,1,2), \quad s_2(\sigma) = (0,1,2,2),
$$
$$
s_1s_0(\sigma) = (0,0,0,1,2), \quad s_2s_0(\sigma) = (0,0,1,1,2), \quad s_3s_0(\sigma) = (0,0,1,2,2),
$$
$$
s_2s_1(\sigma) = (0,1,1,1,2), \quad s_3s_1(\sigma) = (0,1,1,2,2), \quad s_3s_2(\sigma) = (0,1,2,2,2).
$$

The face and degeneracy maps are determined with respect to the standard simplicial identities for simplicial groups. For example, the first non-trivial face maps $d_i : F[S^2]_3 \to F[S^2]_2$ are given by

$$
\begin{aligned}
d_0 &: s_0(\sigma) \mapsto \sigma, \quad s_1(\sigma) \mapsto \star, \quad s_2(\sigma) \mapsto \star,\\
d_1 &: s_0(\sigma) \mapsto \sigma, \quad s_1(\sigma) \mapsto \sigma, \quad s_2(\sigma) \mapsto \star,\\
d_2 &: s_0(\sigma) \mapsto \star, \quad s_1(\sigma) \mapsto \sigma, \quad s_2(\sigma) \mapsto \sigma,\\
d_3 &: s_0(\sigma) \mapsto \star, \quad s_1(\sigma) \mapsto \star, \quad s_2(\sigma) \mapsto \sigma.
\end{aligned}
$$

Milnor's construction gives a possibility to define the homotopy groups $\pi_n(S^3)$ combinatorially, in terms of free groups. By (6.1), the geometric realisation of $F[S^2]_*$ is weakly homotopically equivalent to the loop space ΩS^3. Thus, the homotopy groups of S^3 are isomorphic to the Moore homotopy groups of $F[S^2]$, that is,

$$
\pi_{n+1}(S^3) \cong Z_n(F[S^2]_*)/B_n(F[S^2]_*). \tag{6.2}
$$

6.2. Simplicial pure twin group. By [1, Theorem 2], we have

$$
PT_3 = \langle (t_1t_2)^3 \rangle \cong \mathbb{Z}, \qquad PT_4 \cong F_7,
$$

where F_7 is the free group on the elements

$$
x_1 = (t_1t_2)^3, \quad x_2 = \left((t_1t_2)^3\right)^{t_3}, \quad x_3 = \left((t_1t_2)^3\right)^{t_3t_2}, \quad x_4 = \left((t_1t_2)^3\right)^{t_3t_2t_1},
$$

$$x_5 = (t_2t_3)^3, \quad x_6 = \left((t_2t_3)^3\right)^{t_1}, \quad x_7 = \left((t_2t_3)^3\right)^{t_1t_2}.$$

Let $SPT_* = \{SPT_n\}_{n\geq 0}$, where $SPT_n = PT_{n+1}$ for each $n \geq 0$. Following the methodology of [11], consider the sequence of groups

$$\cdots \; PT_4 \; \substack{\longrightarrow\\ \longleftarrow} \; PT_3 \; \substack{\longrightarrow\\ \longleftarrow} \; PT_2 \; \substack{\longrightarrow\\ \longleftarrow} \; PT_1$$

with face and degeneracy homomorphisms

$$\begin{aligned} d_i &: SPT_n = PT_{n+1} \to SPT_{n-1} = PT_n, \\ s_i &: SPT_n = PT_{n+1} \to SPT_{n+1} = PT_{n+2}, \end{aligned}$$

where the face map d_i is the deleting of the $(i+1)$-th strand and the degeneracy map s_i is the doubling of the $(i+1)$-th strand for each $0 \leq i \leq n$. For example, we prove in the proof of Proposition 3.5 that $d_3 : PT_4 \to PT_3$ is given by $d_3(x_1) = y$ and

$$d_3(x_2) = d_3(x_3) = d_3(x_4) = d_3(x_5) = d_3(x_6) = d_3(x_7) = 1,$$

where $y = (t_1t_2)^3 \in PT_3$. As in the classical case it is not difficult to prove the following result, whose proof is adapted from [3, Proposition 3.1].

Proposition 6.4. *SPT_* is a contractible simplicial group.*

Proof. Let $x \in Z_n(SPT_*)$ be a Moore n-cycle, that is, $x \in SPT_n$ and $d_i(x) = 1$ for all $0 \leq i \leq n$. Note that SPT_* admits an additional degeneracy map $\iota_{n+1} : SPT_n \to SPT_{n+1}$, which adds a trivial strand on the left of the diagram of the twin. If we set $y = \iota_{n+1}(x) \in SPT_{n+1}$, then we see that $d_j(y) = 1$ for all $1 \leq j \leq n+1$ and $d_0(y) = x$. Thus, $x \in B_n(SPT_*)$ is a Moore n-boundary, and hence $\pi_n(SPT_*) = 1$ for all n. □

We write $U_{n,i} := \mathrm{Ker}(d_i : PT_n \to PT_{n-1})$ for each $0 \leq i \leq n-1$. Then, we have the following short exact sequence

$$1 \longrightarrow U_{n,i} \longrightarrow PT_n \xrightarrow{d_i} PT_{n-1} \longrightarrow 1$$

with the splitting given by $d^i : PT_{n-1} \to PT_n$ as defined in Proposition 2.4. This gives a semi-direct product decomposition $PT_n = U_{n,i} \rtimes PT_{n-1}$. Clearly, $U_{3,0} = U_{3,1} = U_{3,2} = PT_3$. The following problem seems interesting.

Problem 6.5. *Find presentations of $U_{n,i}$ for $n \geq 4$.*

We construct a simplicial subgroup K_* of SPT_*, which would be the image of the simplicial sphere S^2 under a simplicial map. We set $K_0 = K_1 = 1$, $K_2 = SPT_2 = \langle c_{0,1;2} \rangle$, the infinite cyclic group generated by $c_{0,1;2} = (t_1t_2)^3$, and

$$K_3 = \langle c_{1,2;3} = s_0(c_{0,1;2}),\ c_{0,2;3} = s_1(c_{0,1;2}),\ c_{0,1;3} = s_2(c_{0,1;2}) \rangle.$$

In general, we define

$$K_n = \langle c_{i,j;n} = s_{n-1} \dots s_{j+1}\widehat{s_j}s_{j-1} \dots s_{i+1}\widehat{s_i}s_{i-1} \dots s_0(c_{0,1;2}) \mid 0 \leq i < j \leq n-1 \rangle,$$

the subgroup of SPT_n generated by $n(n-1)/2$ elements. It follows from the simplicial identities that $d_i(c_{i,j;n}) \in K_{n-1}$ and $s_j(c_{i,j;n}) \in K_{n+1}$ for each generator $c_{i,j;n}$ of K_n and all d_i, s_j. Thus, for each $n \geq 0$, restriction of face maps $d_i : SPT_n \to SPT_{n-1}$ gives face maps $d_i : K_n \to K_{n-1}$. Similarly, restriction of degeneracy maps $s_i : SPT_n \to SPT_{n+1}$ induce degeneracy maps $s_i : K_n \to K_{n+1}$, turning $K_* = \{K_n\}_{n\geq 0}$ into a simplicial subgroup of SPT_*.

Theorem 6.6. *$K_3 \cong F[S^2]_3$ and $K_4 \cong F[S^2]_4$.*

Proof. Using the geometrical interpretation of $c_{0,1;2}$ (see Figure 2) and degeneracy maps s_i, we write the generators of K_3 in terms of the generators of PT_4 as follows:

$$c_{1,2;3} = (t_2t_1t_3t_2)(t_1t_2t_3)(t_1t_2t_3) = \left((t_1t_2)^3\right)^{t_3t_2}(t_2t_3)^3 = x_3x_5,$$
$$c_{0,2;3} = (t_1t_2t_3)(t_2t_1t_3t_2)(t_1t_2t_3) = \left((t_2t_3)^3\right)^{t_1}\left((t_1t_2)^3\right)^{t_3} = x_6x_2,$$
$$c_{0,1;3} = (t_1t_2t_3)(t_1t_2t_3)(t_2t_1t_3t_2) = (t_1t_2)^3\left((t_2t_3)^3\right)^{t_1t_2} = x_1x_7.$$

Since PT_4 is a free group of rank 7, it follows that K_3 is a free group of rank 3, and hence $K_3 \cong F[S^2]_3$. It is known from [15, Section 5] that $SPT_4 = PT_5$ is free group of rank 31, but [15] does not give any free generating set for PT_5. However, using [1, Theorem 4], we obtain a generating set for PT_5 of cardinality 43. By removing the redundant generators, we obtain the following minimial generating set for PT_5:

$$a_1 = (t_1t_2)^3 \qquad a_{17} = \left((t_2t_3)^3\right)^{t_4}$$
$$a_2 = \left((t_1t_2)^3\right)^{t_3} \qquad a_{18} = \left((t_2t_3)^3\right)^{t_1t_2t_4t_3t_2t_1}$$
$$a_3 = \left((t_1t_2)^3\right)^{t_3t_2} \qquad a_{19} = \left((t_2t_3)^3\right)^{t_1t_2t_4t_3t_2}$$
$$a_4 = \left((t_1t_2)^3\right)^{t_3t_2t_1} \qquad a_{20} = \left((t_2t_3)^3\right)^{t_1t_2t_4t_3}$$
$$a_5 = \left((t_1t_2)^3\right)^{t_3t_2t_1t_4t_3t_2} \qquad a_{21} = \left((t_2t_3)^3\right)^{t_1t_2t_4}$$
$$a_6 = \left((t_1t_2)^3\right)^{t_3t_2t_1t_4t_3} \qquad a_{22} = \left((t_2t_3)^3\right)^{t_1t_4t_3t_2t_1}$$
$$a_7 = \left((t_1t_2)^3\right)^{t_3t_2t_1t_4} \qquad a_{23} = \left((t_2t_3)^3\right)^{t_1t_4t_3t_2}$$
$$a_8 = \left((t_1t_2)^3\right)^{t_3t_2t_4t_3} \qquad a_{24} = \left((t_2t_3)^3\right)^{t_1t_4t_3}$$
$$a_9 = \left((t_1t_2)^3\right)^{t_3t_4t_3t_2} \qquad a_{25} = \left((t_2t_3)^3\right)^{t_1t_4}$$
$$a_{10} = \left((t_1t_2)^3\right)^{t_3t_4} \qquad a_{26} = \left((t_3t_4)^3\right)$$
$$a_{11} = (t_2t_3)^3 \qquad a_{27} = \left((t_3t_4)^3\right)^{t_2t_1t_3t_2}$$
$$a_{12} = \left((t_2t_3)^3\right)^{t_1} \qquad a_{28} = \left((t_3t_4)^3\right)^{t_2t_1t_3}$$
$$a_{13} = \left((t_2t_3)^3\right)^{t_1t_2} \qquad a_{29} = \left((t_3t_4)^3\right)^{t_2t_1}$$
$$a_{14} = \left((t_2t_3)^3\right)^{t_4t_3t_2t_1} \qquad a_{30} = \left((t_3t_4)^3\right)^{t_2t_3}$$
$$a_{15} = \left((t_2t_3)^3\right)^{t_4t_3t_2} \qquad a_{31} = \left((t_3t_4)^3\right)^{t_2}$$
$$a_{16} = \left((t_2t_3)^3\right)^{t_4t_3}$$

By definition, we have

$$K_4 = \langle s_1 s_0(c_{0,1;2}), s_2 s_0(c_{0,1;2}), s_3 s_0(c_{0,1;2}), s_2 s_1(c_{0,1;2}), s_3 s_1(c_{0,1;2}), s_3 s_2(c_{0,1;2})\rangle.$$

Direct calculation gives

$$\begin{aligned} s_1(x_3x_5) &= a_8a_{16}a_{26}, & s_2(x_3x_5) &= a_{23}a_9a_{31}a_{17}, \\ s_3(x_3x_5) &= a_3a_{19}a_{11}a_{30}, & s_2(x_6x_2) &= a_{29}a_{25}a_{10}, \\ s_3(x_6x_2) &= a_{12}a_{28}a_2a_{20}, & s_3(x_1x_7) &= a_1a_{13}a_{27}. \end{aligned}$$

Thus, K_4 is free of rank 6, and hence $K_4 \cong F[S^2]_4$. □

Problem 6.7. *Determine a presentation of K_n for $n \geq 4$.*

We consider $c_{0,1;2}$ as a 2-simplex in the simplicial group SPT_*. Since

$$d_0(c_{0,1;2}) = d_1(c_{0,1;2}) = d_2(c_{0,1;2}) = 1,$$

there is a (unique) simplicial map

$$\theta : S^2 \to SPT_*$$

such that $\theta(\sigma) = c_{0,1;2}$, where $\sigma = (0,1,2)$ is the non-degenerate 2-simplex of the simplicial 2-sphere S^2. By Milnor's construction, the simplicial map θ extends uniquely to a simplicial homomorphism

$$\Theta : F[S^2]_* \longrightarrow SPT_*.$$

We note that $K_* = \Theta(F[S^2]_*)$ and it is the smallest simplicial subgroup of SPT_* containing $c_{0,1;2}$. Further, by Proposition 6.6,

$$\Theta_n : F[S^2]_n \longrightarrow SPT_n$$

is injective for $n \leq 4$. If each $\Theta_n : F[S^2]_n \to SPT_n$ is injective, then by (6.2), we have

$$\pi_{n+1}(S^3) \cong Z_n(F[S^2]_*)/B_n(F[S^2]_*) \cong Z_n(K_*)/B_n(K_*) \cong \pi_n(K_*).$$

Thus, if Θ is injective, then we can describe $\pi_{n+1}(S^3)$ as a quotient of a subgroup of PT_{n+1}. For instance, the generator of $\pi_3(S^3) \cong \mathbb{Z}$ can be represented by the pure twin $(t_1t_2)^3$.

It appears that the following statement holds.

Conjecture 6.8. $\Theta : F[S^2]_* \longrightarrow K_*$ *is an isomorphism.*

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7. Declaration

The authors declare that there is no data associated to this paper and that there are no conflicts of interests.

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