New York Journal of Mathematics

New York J. Math. **30** (2024) 1705–1749.

First integral cohomology group of the pure mapping class group of a non-orientable surface of infinite type

Jesús Hernández Hernández and **Cristhian E. Hidber**

ABSTRACT. In this work we compute the first integral cohomology of the pure mapping class group of a non-orientable surface of infinite topological type and genus at least 3. To this purpose, we also prove several other results already known for orientable surfaces such as the existence of an Alexander method, the fact that the mapping class group is isomorphic to the automorphism group of the curve graph along with the topological rigidity of the curve graph, and the structure of the pure mapping class group as both a Polish group and a semi-direct product.

CONTENTS

Introduction

Let N be a connected surface with empty boundary and define the mapping class group of N, denoted as Map(N), as the group of isotopy classes of self-homeomorphisms of N. If N is orientable, this is often called the extended

Received July 28, 2023.

2020 *Mathematics Subject Classification.* 57K20 (primary); 20J06(secondary); 20F65 (secondary).

Key words and phrases. Non-orientable surface; big mapping class groups; first cohomology group.

The first author was supported during the creation of this article by the research project grants UNAM-PAPIIT IA104620 and UNAM-PAPIIT IN102018. The second author received support from a CONAHCYT Posdoctoral Fellowship and from UNAM-PAPIIT-IN105318. Both authors were supported during the creation of this article by the CONAHCYT Ciencia de Frontera 2019 research project grant CF 217392.

ISSN 1076-9803/2024

mapping class group. The mapping class group has been studied for several decades now, and the most-commonly used tools for its study are the curves on the surface. This leads to the definition of the curve graph of N, denoted as $\mathcal{C}(N)$: The curve graph is defined as the simplicial graph whose vertices are isotopy classes of essential simple closed curves on N. See Section 1 for more details on these definitions. Given that $\mathrm{Map}(N)$ naturally acts on $\mathcal{C}(N)$, a lot of information and properties of $\mathrm{Map}(N)$ have been obtained by studying this action.

If N is an orientable surface of finite (topological) type, i.e. N has finitely generated fundamental group, some of the results that have been proved are the following:

(1) There exists a collection of finitely many curves that completely determine a homeomorphism of N up to isotopy. See Chapter 2 of Farb and Margalit [8].
(2) For all but finitely many surfaces, the action is rigid: every automorphism of $\mathcal{C}(N)$ is induced by an element of $\mathrm{Map}(N)$. See Ivanov [11], Korkmaz [12], Luo [13].
(3) For all but finitely many surfaces, the first integral homology groups are finite groups depending on the surface. This implies that the first integral cohomology groups are trivial. See [8] and the reference therein.

If N is a non-orientable surface of finite type, the same results are valid. However, in most of these cases either slight modifications of the proofs are needed as in (1), or whole new proofs are needed as in (2) and (3), and as such different papers have been dedicated to these results. See Paris [14], Atalan and Korkmaz [4], Stukow [19].

On the other hand, for infinite-type surfaces there is a lot of work left to be done, particularly on the non-orientable case.

If N is an orientable surface of infinite type, the same results as above have been proved recently: In Hernández Hernández, Morales and Valdez [10], it is proved that there exists a locally finite collection of curves on N that determine a homeomorphism up to isotopy. In Hernández Hernández, Morales and Valdez [9], and Bavard, Dowdall and Rafi [5] the respective authors proved independently that there is action rigidity along with topological rigidity. In Aramayona, Patel and Vlamis [2], the respective authors compute the first integral cohomology group of the pure mapping class group (the subgroup of $\mathrm{Map}(N)$ that acts trivially on the ends of the surface, denoted by $\mathrm{PMap}(N)$; see Section 1 for more details) for the case that N has genus at least two.

In this work, we prove the analogous results. As mentioned before, many of the techniques used in this work are analogous to the orientable case. That said, in several cases the non-orientable nature of the surface forces the proofs to be different (particularly for the computation of the first integral cohomology group).

As such, the main results of our work are the following.

Theorem A (Alexander method). *Let N be a (possibly non-orientable) connected surface of infinite topological type. There exists a locally finite collection of essential, simple, closed curves $\Gamma = \{\gamma_i\}_{0\leq i<\omega}$ that satisfies the following: If $h \in \mathrm{Homeo}(N)$ is such that for all $i \geq 0$, $h(\gamma_i)$ is isotopic to γ_i, then h is isotopic to the identity.*

This theorem is the analogue of Theorem 1.1 in [10]. For the proof, we delegate the case when N is orientable to the aforementioned theorem, and focus only on the non-orientable case. That said, the proof in this case is analogous, as such we only sketch the proofs of the related lemmata and theorems, while highlighting the differences.

Also, see Shapiro [17] for a more complete approach to the Alexander Method of infinite-type surfaces.

Theorem B. *Let N_1 and N_2 be two connected (possibly non-orientable) surfaces of infinite topological type, and let $\varphi : \mathcal{C}(N_1) \to \mathcal{C}(N_2)$ be an isomorphism. Then N_1 is homeomorphic to N_2, and φ is induced by a homeomorphism $N_1 \to N_2$.*

This theorem is the analogue of Theorem 1.1 in [9] and Theorem 1.3 in [5]. The proof is analogue to the proof of Theorem 1.1 in [9], and as such we only sketch the proofs of the related lemmata and theorems, while highlighting the differences both in arguments and in references needed.

Using Theorem A and Theorem B, we obtain the following classical corollary.

Corollary C. *Let N be a connected (possibly non-orientable) surface of infinite topological type. Then the natural map $\Psi : \mathrm{Map}(N) \to \mathrm{Aut}(\mathcal{C}(N))$ is an isomorphism.*

This corollary is the analogue of Theorem 1.2 in [9].

Now, as in the orientable case, $\mathrm{Map}(N)$ has a natural topology which makes it a topological group: We equip $\mathrm{Homeo}(N)$ with the compact-open topology and then $\mathrm{Map}(N)$ has the quotient of said topology.

On the other hand, Corollary C tells us that pulling the permutation topology of $\mathrm{Aut}(\mathcal{C}(N))$, we can endow $\mathrm{Map}(N)$ with a topology which makes it a Polish (separable and completely metrizable) topological group.

Using the same arguments as in the orientable case (see Aramayona, Patel and Vlamis [2], Aramayona and Vlamis [3]), we can see these two topologies coincide, and as such we have the following corollary.

Corollary D. *Let N be a connected surface of infinite topological type. Arming $\mathrm{Map}(N)$ with the quotient of the compact-open topology, and $\mathrm{Aut}(\mathcal{C}(N))$ with the permutation topology, then the natural map $\Psi : \mathrm{Map}(N) \to \mathrm{Aut}(\mathcal{C}(N))$ is an isomorphism of topological groups. In particular, $\mathrm{Map}(N)$ is a Polish group with the compact-open topology.*

Then, to compute the first cohomology group of the pure mapping class group of N, we follow the ideas of [2], and thus we need to understand more the topology and the topological generators of $\mathrm{PMap}(N)$. For this we need to recall some definitions.

A handle-shift is in essence taking an infinite strip connecting two ends of the surface with genus, and shifting said genus by one. The precise definition is given in Subsection 4.2.

The *compactly supported mapping class group*, denoted by $\mathrm{PMap}_c(N)$, is the subgroup of $\mathrm{Map}(N)$ composed of the mapping classes that have representatives with compact support.

Theorem E. *Let N be a connected (possibly non-orientable) surface of infinite topological type. If N has at most one end accumulated by genus, then* $\mathrm{PMap}(N) = \overline{\mathrm{PMap}_c(N)}$. *If N has at least two ends accumulated by genus, then there exist a constant $1 \leq r \leq \omega$ and a collection $\{h_i\}_{0\leq i<r}$ of handle-shifts, such that* $\mathrm{PMap}(N) = \overline{\langle \mathrm{PMap}_c(N), \{h_i\}_{0\leq i<r}\rangle}$ *and $\overline{\langle\{h_i\}_{0\leq i<r}\rangle}$ is isomorphic to $\mathbb{Z}^r$ as a topological group.*

The general outline of the proof of this theorem is to follow the proof of Theorem 4 in Patel and Vlamis [15], and prove that the compactly supported mapping class group and the set of all handle-shifts topologically generate the pure mapping class group. Then, we refine that result with the use of the collection $\{h_i\}_{0\leq i<r}$. This collection is obtained via a basis of $H_1^{sep}(\widehat{N};\mathbb{Z})$, and as such r is the dimension of $H_1^{sep}(\widehat{N};\mathbb{Z})$, where $\widehat{N}$ is the surface obtained from N by "forgetting" all the planar ends of N, and $H_1^{sep}(\,\cdot\,;\mathbb{Z})$ is the subgroup of $H_1(\,\cdot\,;\mathbb{Z})$ generated by the homology classes that can be represented by separating simple closed curves.

One of the main difference between $\{h_i\}_{0\leq i<r}$ and the similar collection obtained in Theorem 3 in [2] is that if N is non-orientable, there are bases of $H_1^{sep}(\widehat{N};\mathbb{Z})$ that we cannot use for our proof. For example, for some surfaces we can produce a basis for $H_1^{sep}(\widehat{N};\mathbb{Z})$ such that it is not clear how to topologically generate all the handle-shifts of N; see Subsection 4.5 for more details and an example of this pheonomenon. Thus, we construct what we call a "good basis" for $H_1^{sep}(\widehat{N};\mathbb{Z})$, which in turn produces the collection $\{h_i\}_{0\leq i<r}$ that satisfies the theorem. Also, due to the conclusion of Theorem E, we denote $\overline{\langle\{h_i\}_{0\leq i<r}\rangle}$ by $\prod_{0\leq i<r}\langle h_i\rangle$ to emphasize the fact that this group is isomorphic to $\mathbb{Z}^r$ as a topological group.

A direct consequence of Theorem E and the well-known fact that closed subgroups of Polish groups are Polish, is the following corollary.

Corollary F. *Let N be a connected (possibly non-orientable) surface of infinite topological type. Then* $\mathrm{Map}(N)$, $\mathrm{PMap}(N)$ *and* $\overline{\mathrm{PMap}_c(N)}$ *are Polish groups with their respective topologies.*

Now, using the previous results we can define a homomorphism from $\mathrm{PMap}(N)$ to $\prod_{0\leq i<r}\langle h_i\rangle$, which we use to give a semi-direct product structure to $\mathrm{PMap}(N)$.

Theorem G. *Let N be a connected (possibly non-orientable) surface of infinite topological type with at least two ends accumulated by genus. Then, we have that:*

$$\mathrm{PMap}(N) = \overline{\mathrm{PMap}_c(N)} \rtimes \prod_{0\le i<r} \langle h_i \rangle.$$

Finally, using Theorems E and G, along with the results from Stukow in [19], Dudley in [7], Specker in [18] and Blass and Göbel in [6], we obtain the following corollary.

Corollary H. *Let N be a connected (possibly non-orientable) surface of infinite topological type with genus at least* 3*. If N has at most one end accumulated by genus, then $H^1(\mathrm{PMap}(N);\mathbb{Z})$ is trivial. If N has at least two ends accumulated by genus, then $H^1(\mathrm{PMap}(N);\mathbb{Z}) = H^1(\mathbb{Z}^r;\mathbb{Z}) = \bigoplus_{0\le i<r} \mathbb{Z}$.*

This phenomenon belongs solely to the infinite-type surfaces, since if a surface N is of finite type then (except in finitely-many cases) the abelianisation of $\mathrm{PMap}(N)$ is either trivial or finite, which implies that $H^1(\mathrm{PMap}(N);\mathbb{Z})$ is trivial.

Acknowledgements

Both authors thank Ulises A. Ramos-García for his thoughtful comments on this work. Both authors also thank the referee for their comments and suggestions which helped greatly improve this work.

1. Preliminaries

A *curve* is a topological embedding of the unit circle into N. We often abuse notation and call "curve" the embedding, its image on N or its isotopy class. The context makes clear which use we mean.

A curve is *essential* if it is not isotopic to a boundary curve and if it does not bound a disk, a punctured disk or a Möbius band. Unless otherwise stated, all curves are assumed to be essential.

The *(geometric) intersection number* of two isotopy classes of essential curves α and β is defined as:

$$i(\alpha,\beta) := \min\{|a\cap b| : a\in\alpha, b\in\beta\}.$$

We say two curves α and β are in minimal position if $\alpha\cap\beta = i([\alpha],[\beta])$.

It is a well-known result (see [8]) that if N is endowed with a hyperbolic metric, then in every isotopy class of a curve, there exists a unique geodesic representative. Also (see [8]), any two geodesic representatives are in minimal position.

A set of curves $\mathcal{G}$ is *locally finite* if for every compact subset K, the set $\{\alpha\in\mathcal{G} : \alpha\cap K\neq\varnothing\}$ is finite. A set of isotopy classes of curves Γ is *locally finite* if there exists a set of representatives $\mathcal{G}$ that is locally finite.

A *multicurve* is a locally finite set of pairwise disjoint and pairwise non-isotopic curves. We often abuse notation and call "multicurve" the set of curves, their images on N or its set of isotopy classes. The context makes clear which use we mean.

In this work, unless otherwise stated, by a *subsurface* Σ *of* N we mean a closed subsurface of N such that every connected component of $\partial\Sigma$ is compact, and the natural inclusion $\Sigma \hookrightarrow N$ is π_1-injective.

A *pair of pants* is a closed subsurface whose interior is homeomorphic to a thrice-punctured sphere. With this, a *pants decomposition of* N is a maximal multicurve P, and its name comes from the fact that $N\backslash P$ is the disjoint union of pair of pants.

A curve α is *separating* if $N\backslash\alpha$ is disconnected. It is *non-separating* otherwise. Now we are ready for the following definition.

Definition 1.1. An increasing sequence of subsurfaces $\Sigma_0 \subset \Sigma_1 \subset \cdots \subset N$ is a *principal exhaustion* if it satisfies the following:

(1) For each $j \geq 0$, Σ_j is a finite-type subsurface such that each of its boundary curves are essential separating curves in N (recall that $\partial N = \varnothing$).
(2) For each $j \geq 0$, every connected component of $N \setminus \Sigma_j$ is an infinite type surface.
(3) For each $j \geq 0$ and taking $\Sigma_{-1} = \varnothing$, we have that each connected component of $\Sigma_j \setminus \Sigma_{j-1}$ satisfies one of the following conditions:
   - If it is an orientable subsurface of genus g, n punctures and b boundary components, then $3g - 3 + n + b \geq 5$.
   - If it is a non-orientable subsurface of genus g, n punctures and b boundary components, then $g + n + b \geq 8$.
(4) $\displaystyle\bigcup_{0\leq j<\omega} \Sigma_j = N$.
(5) Finally, we define B_j as the set of boundary curves of Σ_j and the set $B = \bigcup_{0\leq j<\omega} B_j$. Note that B is a multicurve of N composed of separating curves.

1.1. The genus of a surface. For a finite-type surface N, the genus can be defined as the number of projective planes needed for the connected sum to be homeomorphic to N (possibly after puncturing and deleting interiors of dsijoint discs from the connected sum). This gives a lot of freedom to how the surface can "look". A classic trick to change the look/model of the surface is to remember that the connected sum of three projective planes is homeomorphic to the connected sum of a torus and one projective plane. This allows all the surfaces in Figure 1 to be homeomorphic. That said, note that the parity of the number of projective planes (cross-caps) does not change.

For an infinite-type surface N, the genus can be defined as the supremum of the genus of its finite-type subsurfaces. Thus, it can be either finite or infinite. Moreover, if the genus is infinite, it could very well happen that there are non-orientability of the surface is "restricted" to a finite-type subsurface. If this happens, we say that N is *finitely non-orientable*, but we can further divide this case into two subcases depending on whether this (non-unique) finite-type

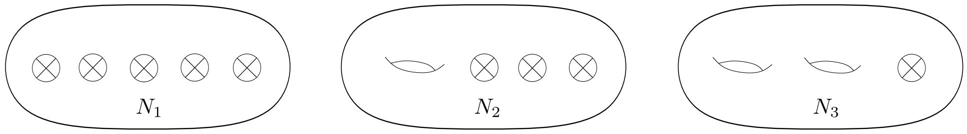

FIGURE 1. The three surfaces are homeomorphic and represents a non-orientable surface of genus 5. Each $\otimes$ represents a cross-cap.

surface has even or odd genus. This division is well-defined since if we change the subsurface, while the genus of this subsurface might be different, it will still have the same parity. See Subsection 1.2 for more information on the possible orientabilities of a surface.

That said, note that given any non-orientable surface N, we cannot truly define the "number of cross-caps" or the "number of tori" in the surface, since one could always "exchange" three cross-caps for one torus and a cross-cap.

1.2. Ends and orientability. In this subsection, we recall the definition of ends and the classification of infinite type surfaces, for details we refer to [20], [16] and [3]. First, we make a small note on the notation of the genus of a surface. By the classification of finite type surfaces, an orientable surface S of genus g is homeomorphic to the connected sum of g tori minus n points and b open disks, or equivalently it is homeomorphic to a sphere with n punctures, b holes and g handles. We say that each *handle* adds one *orientable genus* or *positive genus*. If N is a non-orientable surface of genus g then it is homeomorphic to the connected sum of g real projective planes minus n points and b open disks or equivalently homeomorphic to a sphere with g cross-caps minus n points and b open disks. We say that each cross-cap adds one *non-orinetable genus* or one *negative genus*. When we say infinite genus we think that an infinite number of handles or cross-caps have been added. Below there is a precise definition.

Let N be an infinite type surface, an *exiting sequence* is a sequence $\{U_i\}_{0\leq i<\omega}$ of connected open subsets of N such that

- $U_i \subset U_j$ whenever $j < i$;
- U_i is not relatively compact for all $0 \leq i < \omega$;
- U_i has compact boundary for all $0 \leq i < \omega$;
- any relatively compact subset of N is disjoint from all but finitely many U_i's.

See Figure 2

Let $\{U_i\}_{0\leq i<\omega}$ be an exiting sequence. We say that an element U_i is *planar* if it has genus zero.

We say that two exiting sequences are *equivalent* if every element of the first sequence is eventually contained in some element of the second, and vice versa. An equivalent class of an exiting sequence is called an *end*. We denote the set

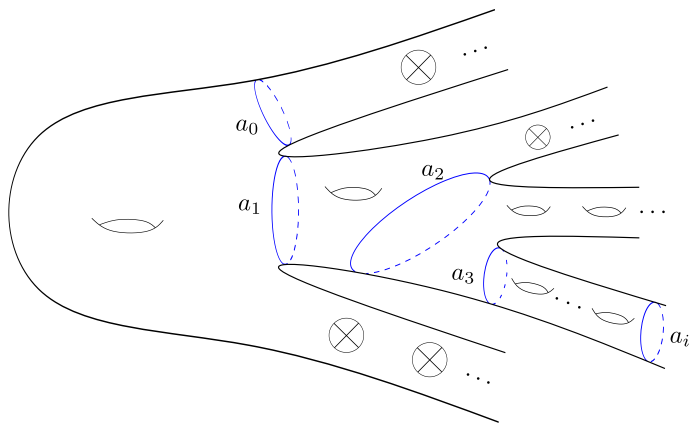

FIGURE 2. U_i is the open subsurface that contains all the curves a_j with $j > i$ and have a_i as its boundary.

of ends by $\mathcal{E}(N)$. The space of ends $\mathcal{E}(N)$ can be equipped with a topology that makes it a totally disconnected, separable and compact set, and is therefore homeomorphic to a closed subset of the Cantor set $\mathcal{C}$ (see Proposition 3 in [16]). We say that

- An end $\{U_i\}_{0\leq i<\omega}$ is *orientable accumulated by genus* (or simply orientable) if U_i is orientable and has infinite orientable genus for all i.
- An end $\{U_i\}_{0\leq i<\omega}$ is *non-orientable accumulated by genus* (or simply non-orientable) if U_i is non-orientable for all i.
- An end $\{U_i\}_{0\leq i<\omega}$ is *planar* if U_i is planar for all but finitely many i.

We denote by $\mathcal{E}_\infty(N)$ the subspace of orientable and non-orientable ends accumulated by genus, and by $\mathcal{E}_-(N)$ the subspace of non-orientable ends. The sets $\mathcal{E}_\infty(N)$ and $\mathcal{E}_-(N)$ are closed subsets of $\mathcal{E}(N)$.

Recall that one can exchange three cross-caps for a cross-cap and a torus, thus even if one exchanges infinitely many cross-caps, one cannot turn a non-orientable end accumulated by genus into an orientable end accumulated by genus.

Note that an infinitely non-orientable surface is also of infinite genus. It can happen that N is of infinite genus and non-orientable but not infinitely non-orientable, in this case we say that N is *odd* or *even* non-orientable according to whether every sufficiently large compact subsurface is non-orientable of odd or even genus, respectively (or, equivalently has an odd or an even number of cross-caps). With these definitions, we have four *orientability classes*: orientable, infinitely non-orientable, odd non-orientable and even non-orientable.

Richards (see [16]) showed that the homeomorphism type of a surface is determinated by its genus, number of boundaries, orientability class and the triple of spaces

$$(\mathcal{E}(N), \mathcal{E}_\infty(N), \mathcal{E}_-(N)).$$

2. The Alexander method

In this section, we will prove the analogue of Theorem 1.1 in [10] for non-orientable surfaces. To do this, we first recall what an application of the Alexander method is for finite-type surfaces (see [8]). Then we follow the general idea of the proof for the case of orientable infinite-type surfaces, highlighting the differences.

2.1. Finite type.

A very well-known application of the Alexander method is the following:

Theorem 2.1 (Application of the Alexander method for finite-type surfaces)**.** *Let Σ be a finite-type surface of genus g, n punctures and b boundary components. If Σ is orientable, assume that $3g-3+n+b \geq 4$. If Σ is non-orientable assume that $g+n+b \geq 5$. Then there exists a finite set of curves and arcs Γ, such that if $h \in \mathrm{Homeo}(\Sigma;\partial\Sigma)$ fixes the isotopy class of every element in Γ, then h is isotopic to the identity.*

For the sake of completeness we give a very short sketch of the proof.

Sketch of the proof. First suppose that Σ is orientable and has empty boundary. Then, let Γ be the set of curves as exemplified in Figure 3 satisfy the hypotheses for Proposition 2.8 in [8]. Thus, if $h \in \mathrm{Homeo}(\Sigma;\partial\Sigma)$ fixes the isotopy classes of every element in Γ, by construction and a quick analysis, the induced map on the graph given by the curves in Γ on Σ fixes each vertex and edge with orientation. Thus, by (2) in Proposition 2.8 in [8], h is isotopic to the identity if it preserves orientation. Since h fixes every vertex and edge with orientation of $\cup\Gamma$, a quick analysis shows that the complements of $\cup\Gamma$ cannot be permuted. This coupled with h preserving the orientation of the edges of $\cup\Gamma$, implies that h preserves orientation and thus is isotopic to the identity.

Analogously, if Σ is orientable and has non-empty boundary, and we label the boundary components by $c_1, \dots, c_b$, then we can obtain Γ as follows: For each $i = 1, \dots, b$, let α_i be an essential arc that starts and finishes at c_i, and also let Γ' be the set obtained from Theorem 2.1 for $\mathrm{int}(\Sigma)$; then Γ can be taken to be $\Gamma' \cup \{\alpha_i\}_{i=1}^{b}$. See Figure 3.

Finally, if Σ is non-orientable, let Γ be the set of curves and arcs exemplified in Figure 3 and let $h \in \mathrm{Homeo}(\Sigma;\partial\Sigma)$ fix the isotopy classes of every element in Γ. Following the same ideas as in the proof of Proposition 2.8 in [8], we can assume that h fixes $\cup\Gamma$. Then, by construction of Γ, h has to fix every vertex and edge with orientation of the induced graph given by the curves in Γ on Σ. Moreover, h has to map each cross-cap to itself. Thus, after an isotopy, the support of h has to be in the complement of the cross-caps, which is an orientable surface. By construction, the curves and arcs in Γ are precisely those given for the orientable-with-boundary case above. Thus, h is isotopic to the identity. □

Note that if Σ has empty boundary, then Γ does not contain arcs.

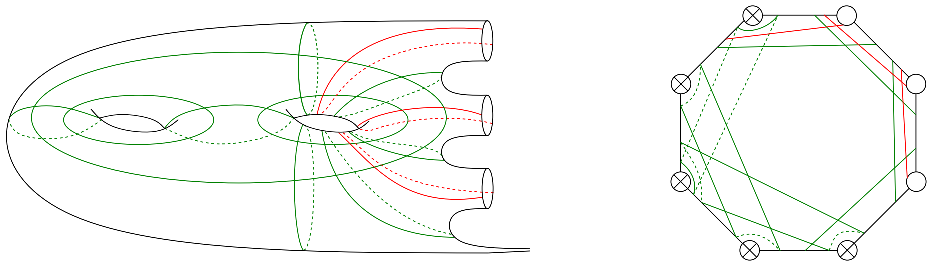

FIGURE 3. Two examples of a set of curves (in green) and arcs (in red) that determine homeomorphisms up to isotopy. On the left, an orientable surface (genus 2, 3 boundary components and 1 puncture), while on the right a non-orientable surface (genus 5 and 3 boundary components).

2.2. Infinite type. To prove Theorem A the general idea is to follow these steps:

(1) Let $h \in \mathrm{Homeo}(N)$ and a principal exhaustion of the surface.
(2) Prove that h can be isotoped to a homeomorphism that fixes the boundary curves of the subsurfaces of the principal exhaustion.
(3) Apply Theorem 2.1 to each subsurface.

So, to be able to prove step (2) above, we first need the following lemma, recalling that an *ambient isotopy of N* is a homeomorphism isotopy starting at the identity of N.

Lemma 2.2. *Let $k \geq 0$, and let $\Gamma_1 = \{\alpha_0, \ldots, \alpha_k\}$ and $\Gamma_2 = \{\beta_0, \ldots, \beta_k\}$ be two collections of curves on N that satisfy the following:*

(1) For each $i = 0, \ldots, k$, α_i is isotopic to β_i, and
(2) for each $j = 1, 2$, Γ_j is a collection of pairwise disjoint curves.

Then, there exists $f \in \mathrm{Homeo}(N)$ isotopic to the identity such that for all $i = 0, \ldots, k$, $f(\alpha_i) = \beta_i$. Moreover, f can be chosen to be the end homeomorphism of an ambient isotopy of N, i.e. there exists an isotopy $H : N \times [0,1] \to N$ such that $H|_{N\times\{0\}} = id_N$ and $H|_{N\times\{1\}} = f$.

Proof. If $|\Gamma_1| = |\Gamma_2| = 1$, let V_0 be a finite-type subsurface such that α_0 and β_0 are both contained in the interior of V_0 and they are isotopic in V_0. Then, there exists an ambient isotopy $\widetilde{H}_0 : V_0 \times I \to V_0$ that deforms α_0 into β_0, and that restricts to the identity on the boundary of V_0. We can then extend $\widetilde{H}_0$ to an ambient isotopy $H_0 : N \times I \to N$ using the identity on $N \backslash V_0$; the homeomorphism $f := H_0(\cdot, 1)$ is the desired homeomorphism.

We now proceed by induction. Suppose that there exists an ambient isotopy $H_n : N \times I \to N$ such that $f_n := H_n(\cdot, 1)$ maps each α_i to β_i for $i \leq n$. Note that the collection $f_n(\Gamma_1) = \{\beta_0, \ldots, \beta_n, f_n(\alpha_{n+1}), \cdots, f_n(\alpha_k)\}$ is again a collection of pairwise disjoint curves; in particular, $f_n(\alpha_{n+1})$ and β_{n+1} are both disjoint from all the curves in $\{\beta_0, \ldots, \beta_n\}$.

Let V_{n+1} be a finite-type subsurface such that $f_n(\alpha_{n+1})$ and β_{n+1} are both contained in the interior of V_{n+1}, they are isotopic in V_{n+1} and V_{n+1} is disjoint from all the curves in $\{\beta_0, \dots, \beta_n\}$; the existence of this subsurface is justified as follows:

(1) $f(\alpha_{n+1})$ and β_{n+1} are disjoint from and non-isotopic to all the curves in $\{\beta_0, \dots, \beta_n\}$, so we can take an isotopy between $f(\alpha_{n+1})$ and β_{n+1} with image disjoint from all the curves in $\{\beta_0, \dots, \beta_n\}$ (otherwise we can use the Bigon Criterion to argue that $f(\alpha_{n+1})$ is either non-essential or isotopic to some β_i with $1 \leq i \leq n$),
(2) then we can take V_{n+1} as a finite-type subsurface of $N\backslash\{\beta_1, \dots, \beta_n\}$ containing the image of the isotopy between these two curves (which in particular implies $f(\alpha_{n+1})$ and β_{n+1} are isotopic in V_{n+1}),
(3) since both of them are essential in N, V_{n+1} can always be taken to have them as essential curves (and in particular they are in the interior of V_{n+1}).

We then obtain an ambient isotopy $\widetilde{H}_{n+1} : V_{n+1} \times I \to V_{n+1}$ as above. Extending $\widetilde{H}_{n+1}$ by the identity on $N\backslash V_{n+1}$ and doing an isotopy composition with H_n, we obtain an ambient isotopy $H_{n+1} : N \times I \to N$ that deforms α_i into β_i for all $i \leq n+1$. We finish the proof by defining f_{n+1} as $H_{n+1}(\cdot, 1)$. □

Given a principal exhaustion of N, $\{N_i\}_{0\leq i<\omega}$, the first step for the construction of Γ is the following: For each $0 \leq i$, we define B_i as the set of boundary curves of N_i. Also, we define $B = \bigcup_{0\leq i<\omega} B_i$; we call B the *set boundaries of the principal exhaustion*.

Lemma 2.3 (cf. Lemma 3.5 in [10]). *Let $N_0 \subset N_1 \subset \cdots \subset N$ be a principal exhaustion of N, B the set of boundaries of this principal exhaustion, and $h \in \mathrm{Homeo}(N)$ be such that for all $\beta \in B$, $h(\beta)$ is isotopic to β. Then there exists $g \in \mathrm{Homeo}(N)$ isotopic to h such that $g|_B = \mathrm{id}|_B$.*

Proof. For this proof, we use Lemma 2.2 to define a sequence of homeomorphisms g_i that satisfy the conclusion of the lemma for the set B_i instead of B. Then we define the desired g from this sequence.

For B_0, by Lemma 2.2 there exists a homeomorphism $f_0 : S \to S$ such that f is isotopic to id and $f_0|_{B_0} = h|_{B_0}$. Then, we define $g_0 := f_0^{-1} \circ h$; this implies that g_0 is isotopic to h and $g_0|_{B_0} = \mathrm{id}|_{B_0}$. Note that $g_0(N_0) = N_0$ since g_0 and id coincide in the boundary of N_0.

For B_1, we define $\widetilde{g_0} := g_0|_{N\backslash \mathrm{int}(N_0)}$. Using Lemma 2.2 again for the surface $N\backslash\mathrm{int}(N_0)$, there exists $\widetilde{f_1} : N\backslash\mathrm{int}(N_0) \to N\backslash\mathrm{int}(N_0)$ such that $\widetilde{f_1}$ is isotopic to $\mathrm{id}|_{N\backslash\mathrm{int}(N_0)}$ and $\widetilde{f_1}|_{B_1} = g_0|_{B_1}$. We can then extend $\widetilde{f_1}$ to the whole surface using the identity on N_0, i.e. we define f_1 as follows:

$$f_1(s) = \begin{cases} s & s \in N_0 \\ \widetilde{f_1}(s) & \text{otherwise} \end{cases},$$

which in particular implies that f_1 is isotopic to id relative to N_0.

Afterwards, we define $g_1 := f_1^{-1} \circ g_0$. Thus, g_0 is isotopic to g_1 relative to N_0 and $g_1|_{B_0 \cup B_1} = \mathrm{id}|_{B_0 \cup B_1}$.

Inductively, following the same procedure for the definition of g_1, we define for any $i \geq 2$ a homeomorphism $g_i : N \to N$ such that $g_i|_{\cup_{0 \leq k \leq i} B_k} = \mathrm{id}|_{\cup_{0 \leq k \leq i} B_k}$, and for $i < j$ we have that g_i is isotopic to g_j relative to N_i.

Thus, the map $g : N \to N$ with $s \mapsto g_i(s)$ for $s \in N_i$ is a well-defined homeomorphism. Also, by construction, $g|_B = \mathrm{id}|_B$. Moreover, if $H_i : N \times [0,1] \to N$ is the isotopy from g_i to g_{i+1} relative to N_i, then let $\widetilde{H_i}$ be the rescaling of H_i to the interval $[\frac{i}{i+1}, \frac{i+1}{i+2}]$; thus, defining $H : N \times [0,1] \to N$ as the concatenation of the $\widetilde{H_i}$, and as $g = H|_{N \times \{1\}}$, we obtain an isotopy from g_1 to g. Therefore, by transitivity g is isotopic to h, finishing the proof. □

Proof of Theorem A. Let $\{N_i\}_{0 \leq i < \omega}$ be a principal exhaustion of N, let B be its boundaries. Let $\{\Sigma_j\}_{0 \leq j < \omega}$ be the collection of subsurfaces Σ_j of N, corresponding to the connected components of $N \backslash B$. Note that for all $j \geq 0$, Σ_j has complexity at least 5 if it is orientable, and if g is its genus and it has n punctures, then $g + n \geq 8$ if it is non-orientable. Also, if we denote by $\overline{\Sigma_j}$ the closure of Σ_j in N, then all the boundary curves of $\overline{\Sigma_j}$ are elements of B.

Now, for each $\beta \in B$, let β^* be a curve on N such that:

- $i([\beta], [\beta^*]) \geq 2$.
- For all $\gamma \in B \backslash \{\beta\}$, β^* is disjoint from γ.
- For all $\alpha, \beta, \gamma \in B$, at least one of the following is zero:

$$i(\alpha^*, \beta^*), i(\alpha^*, \gamma^*), i(\beta^*, \gamma^*).$$

Note that the choice of β^* is arbitrary, and while for every $\beta \in B$, there exist infinitely many possible choices for β^*, once the choice is made, we fix β^* for the rest of the proof. We define $B^* = \{\beta^* : \beta \in B\}$.

Then, let Γ_j be a finite set of curves in Σ_j such that it satisfies the Alexander method for Σ_j along with the arcs obtained from the elements of B^* when restricted to Σ_j (see Theorem 2.1).

We claim that the set:

$$\Gamma = B \cup B^* \cup \left(\bigcup_{0 \leq j < \omega} \Gamma_j \right),$$

satisfies Theorem A.

To prove this, let $h \in \mathrm{Homeo}(N)$ be such that $h(\gamma)$ is isotopic to γ for all $\gamma \in \Gamma$. Due to Lemma 2.3, we can suppose that $h|_B = \mathrm{id}|_B$. This implies that $h|_{\overline{\Sigma_j}} \in \mathrm{Homeo}(\overline{\Sigma_j}; \partial \overline{\Sigma_j})$ for each $0 \leq j < \omega$.

By definition of the Γ_j and Theorem 2.1, $h|_{\Sigma_j}$ is isotopic to the identity in Σ_j. Thus, doing these isotopies independently, h is isotopic to a homeomorphism f that is the identity in N except in annular neighbourhoods of the elements in B. However, by construction for each $0 \leq j < \omega$ and every boundary curve β of $\overline{\Sigma_j}$, the curves β^* plays the roll of arcs connecting the different boundary curve of the annular neighbourhood of β, namely $N(\beta)$. Then, for each β, we can do an isotopy H_β from $f|_{N(\beta)}$ to $id_{N(\beta)}$. Finally, we define an isotopy from f to the identity by defining it as the identity outside $\bigcup_{\beta\in B} N(\beta)$ and then as H_b on each $N(\beta)$. □

3. Isomorphisms between curve graphs

In this section, we prove Theorem B, which says that any isomorphism between curve graphs is induced by a homeomorphism between the underlying surfaces. Our proof of this theorem is almost the same as the proof of Theorem 1.1 in [9], with the proofs being essentially the same (simply substituting auxiliary lemmata and results in the orientable case with the corresponding lemmata and results in the possibly non-orientable case); thus, while we refer the reader to [9] for more detailed proofs, for the sake of completeness we also sketch the proofs.

Throughout this section, we let N, N_1 and N_2 be connected, possibly non-orientable surfaces of infinite type with empty boundary.

Recalling from Section 1 that in hyperbolic surfaces geodesic representatives of curves are always in minimal position, we know that for N there exists a set $\mathcal{S}$ of representatives of the isotopy classes of all essential curves on N such that any two elements of $\mathcal{S}$ are in minimal position. With this in mind, we have the following lemma.

Lemma 3.1 (cf. Lemma 2.5 in [9])**.** *Let N be an infinite-type surface and $\mathcal{S}$ be a set of representatives of the isotopy classes of all essential curves on N such that any two elements of $\mathcal{S}$ are in minimal position. Let also Γ be a set of isotopy classes of curves and $\mathcal{G}\subset\mathcal{S}$ be a set of representatives of Γ. Then the following are equivalent:*

(1) $\mathcal{G}$ is locally finite.
(2) Γ is locally finite.
(3) For every curve α the set $\{\gamma\in\Gamma : i(\alpha,\gamma)\neq 0\}$ is finite.

Since this lemma is actually more general than the analogous in [9], we give a more detailed proof.

Proof. (1) ⇒ (2): This is obvious by the definition of a set of isotopy classes being locally finite.

(2) ⇒ (3): Let $\mathcal{X}$ be a set of representatives of Γ that is locally finite, α be a curve of N and a be a representative of α. Then, we have the following:

$$\{\gamma\in\Gamma : i(\alpha,\gamma)\neq 0\}\subset\{[c]\in\Gamma : c\in\mathcal{X}, a\cap c\neq\varnothing\}.$$

Since the latter set is finite, then we obtain (3).

(3) ⇒ (1): We prove this by contrapositive. Suppose $\mathcal{G}$ is not locally finite. Then there exists a compact set K and an infinite collection $\{\gamma_i\}_{0\leq i<\omega}\subset\mathcal{G}$ such that for all i we have that $K\cap\gamma_i\neq\varnothing$. There exists a finite-type subsurface Σ that satisfies the following:

(1) Σ contains K in its interior.

(2) If $\{c_1, \dots, c_b\}$ are all the boundary curves of Σ, then $\{c_1, \dots, c_b\} \subset \mathcal{S}$.

Then we have that for all i, $\Sigma \cap \gamma_i \neq \emptyset$, and we can divide the proof into two cases:

Case 1, there exists an infinite subcollection $\{\gamma_{i_n}\}_{0 \leq n < \omega}$ contained in Σ: Let $P \subset \mathcal{S}$ be a pants decomposition of Σ. Since P is a maximal set of pairwise disjoint and pairwise non-isotopic curves of Σ, by the pigeonhole principle, there is a curve $\alpha \in P$ that intersects infinitely many elements of $\{\gamma_{i_n}\}_{0 \leq n < \omega}$. Given that all the elements in $\mathcal{S}$ are in minimal position, we have that the set $\{\gamma \in \Gamma : i(\alpha, \gamma) \neq 0\}$ is infinite.

Case 2, only finitely many elements of $\{\gamma_i\}_{0 \leq i < \omega}$ are contained in Σ: If all (but finitely many of) the elements of $\{\gamma_i\}_{0 \leq i < \omega}$ were disjoint from all the elements of $\{c_1, \dots, c_b\}$, then they would not intersect K. Thus, by the pigeonhole principle, there exists a boundary curve c_i of Σ and a subsequence $\{\gamma_{i_n}\}_{0 \leq n < \omega}$, such that they every γ_{i_n} intersects c_i. Given that the elements of $\mathcal{S}$ are in minimal position, this implies that the set $\{\gamma \in \Gamma : i([c_i], \gamma) \neq 0\}$ is infinite. □

Lemma 3.2 (cf. Corollary 2.6 in [9]). *Let N_1, N_2 be two connected (possibly non-orientable) surfaces of infinite type, M be a multicurve on N_1, and $\varphi : \mathcal{C}(N_1) \to \mathcal{C}(N_2)$ be an isomorphism. Then, $\varphi(M)$ is a multicurve. In particular, if P is a pants decomposition of N_1, then $\varphi(P)$ is a pants decomposition.*

Sketch of the proof. Note that if φ is an isomorphism and M is a multicurve, by Lemma 3.1 M is a complete subgraph satisfying (3) in Lemma 3.1, thus $\varphi(M)$ is also a complete subgraph satisfying (3), which again by Lemma 3.1 implies that $\varphi(M)$ is a multicurve.

Moreover, if P is a pants decomposition, then it is a maximal multicurve. Then, by the argument above, $\varphi(P)$ is a multicurve, and maximality is obtained by the surjectivity of φ. □

We say a set M of locally finite, pairwise disjoint curves bounds a closed subsurface $\Sigma \subset N$ if the set of boundary curves of Σ that are not boundary curves of N is exactly M.

Now, let P be a pants decomposition of N and $\alpha_1, \alpha_2 \in P$ be different. For each $i = 1, 2$, let β_i be α_i if α_i is two-sided; otherwise, let β_i be the boundary curve of the Möbius band that is the regular neighborhood of α_i (note that in this case, β_i is not essential). We say that α_1 and α_2 are *adjacent with respect to P* if there exists a set $M \supset \{\beta_1, \beta_2\}$ that bounds a pair of pants. See Figure 4.

Lemma 3.3. *Let $\alpha, \beta \in P$. Then, α and β are adjacent with respect to P if and only if there exists a curve γ such that $i(\alpha, \gamma) \neq 0 \neq i(\beta, \gamma)$ and $i(\delta, \gamma) = 0$ for all $\delta \in P \backslash \{\alpha, \beta\}$.*

Proof. If α and β are adjacent with respect to P, then we can find a curve γ satisfying the lemma as is shown in Figure 5.

If α and β are not adjacent with respect to P, then there exist finite-type closed subsurfaces Σ_1 and Σ_2 such that:

(1) They have disjoint interiors.

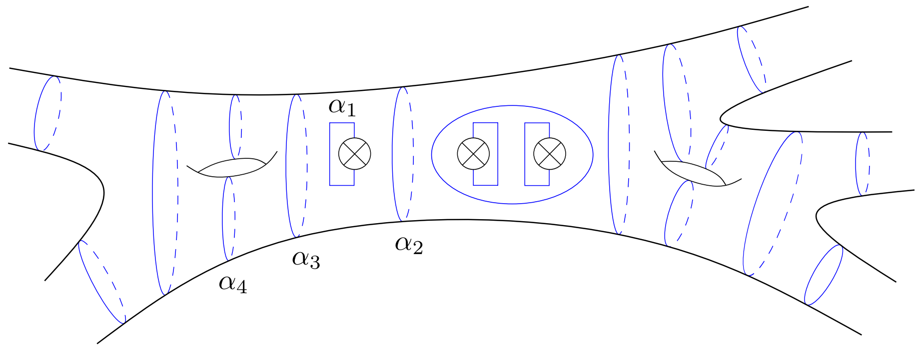

FIGURE 4. An example of a pants decomposition P. The curve α_1 is adjacent to α_2 and α_3 with respect to P. The curve α_1 is not adjacent to α_4 with respect to P.

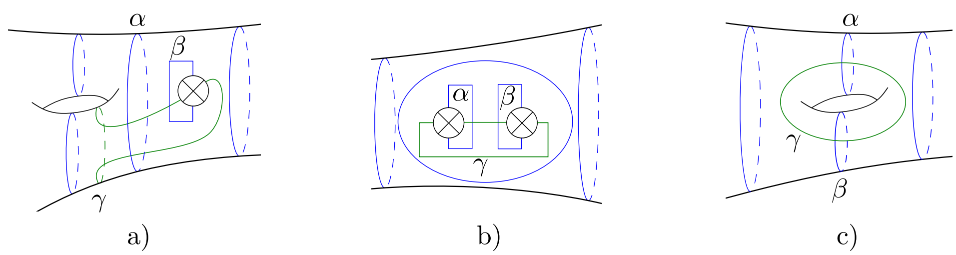

FIGURE 5. Examples for Lemma 3.3.

(2) They are bounded by elements in $P\backslash\{\alpha,\beta\}$.
(3) Σ_1 contains α and Σ_2 contains β.

Thus, any curve that intersects α and β has to intersect some curves in the boundary of Σ_1 and Σ_2. □

Note that, Lemma 3.3 implies that we can simplicially characterise adjacency with respect to P.

Lemma 3.4. *Let N_1, N_2 be two connected (possibly non-orientable) surface of infinite type, P be a pants decomposition on N_1, and $\varphi : \mathcal{C}(N_1) \to \mathcal{C}(N_2)$ be an isomorphism. Then, $\alpha, \beta \in P$ are adjacent with respect to P if and only if $\varphi(\alpha)$ and $\varphi(\beta)$ are adjacent with respect to $\varphi(P)$.*

Proof. This follows immediatly from Lemma 3.3 and the fact that φ is an isomorphism. □

Let P be a pants decomposition; we define its *adjacency graph*, denoted $\mathcal{A}(P)$, to be the simplicial graph whose vertices correspond to the curves in P, and two vertices span an edge if they are adjacent with respect to P. Note that by Lemma 3.2, if $\varphi : \mathcal{C}(N_1) \to \mathcal{C}(N_2)$ is an isomorphism, then $\varphi(P)$ is a pants decomposition; thus, we induce a map $\varphi_P : \mathcal{A}(P) \to \mathcal{A}(\varphi(P))$ defined as $\alpha \mapsto \varphi(\alpha)$. Then, the following corollary follows from Lemmata 3.2 and 3.4.

Corollary 3.5 (cf. Proposition 3.1 in [9]). *Let N_1, N_2 be two connected (possibly non-orientable) surface of infinite type, P be a pants decomposition on N_1, and $\varphi : \mathcal{C}(N_1) \to \mathcal{C}(N_2)$ be an isomorphism. Then, φ_P is an isomorphism between $\mathcal{A}(P)$ and $\mathcal{A}(\varphi(P))$.*

A separating curve is called *outer* if it bounds a twice-punctured disk, while it is called *non-outer* otherwise.

Recall that if Γ is a simplicial graph and $v \in V(\Gamma)$, we say v is a *cut vertex* if $\Gamma \backslash v$ has more connected components than Γ.

Lemma 3.6. *If P is a pants decomposition, then non-outer separating curves are exactly the cut vertices of $\mathcal{A}(P)$. Moreover, if $M \subset P$ is a finite set of non-outer separating curves, then M bounds a finite-type closed subsurface of N if and only if there is a finite subgraph of $\mathcal{A}(P)$ delimited exactly by the cut vertices corresponding to M in $\mathcal{A}(P)$.*

Proof. Let $\alpha \in P$ be a non-outer separating curve in N. Then, there $N = \Sigma_1 \cup \Sigma_2$ where Σ_1 and Σ_2 are closed subsurfaces with disjoint interior such that α is exactly their intersection. Since α is non-outer, we have that there exist curves from P in both Σ_1 and Σ_2. In particular, any curve from P in Σ_1 cannot be adjacent with respect to P to any curve from P in Σ_2. Thus, α is a cut vertex of $\mathcal{A}(P)$.

If $\alpha \in P$ is a cut vertex of $\mathcal{A}(P)$, then $\mathcal{A}(P) \backslash \alpha$ has at least two connected components Γ_1 and Γ_2. Then, by definition of adjacency with respect to P, we have that Γ_1 and Γ_2 are pants decompositions of $N \backslash \alpha$. Thus, since none of the curves in Γ_1 are adjacent to any of the curves in Γ_2, we have that $N \backslash \alpha$ has to be disconnected. Also, given that neither Γ_1 nor Γ_2 are empty graphs, we have that α is a non-outer separating curve.

The rest of the lemma follows applying the same argument as above for each element of M. □

Lemma 3.7 (cf. Lemma 3.2 in [9]). *Let N_1, N_2 be two connected (possibly non-orientable) surface of infinite type, and $\varphi : \mathcal{C}(N_1) \to \mathcal{C}(N_2)$ be an isomorphism. Then, α is a non-outer separating curve if and only if $\varphi(\alpha)$ is a non-outer separating curve.*

Proof. This follows from Lemma 3.6 and the fact that φ is an isomorphism. □

Now, to prove Theorem B we use the following theorem, which is an amalgamation of Theorems 1 and 2 in [4] (by Atalan and Korkmaz), Theorem 1 in [11] (by Ivanov), Theorem 1 in [12] (by Korkmaz), and Theorem (a) in [13] (by Luo).

Theorem 3.8. *Let N_1 and N_2 be two finite-type surfaces such that neither of them is homeomorphic to any of the following surfaces: $S_{0,4}$, $S_{1,1}$, $S_{0,5}$, $S_{1,2}$, $S_{0,6}$, $S_{2,0}$. If $\phi : \mathcal{C}(N_1) \to \mathcal{C}(N_2)$ is an isomorphism, then N_1 and N_2 are homeomorphic and ϕ is induced by a homeomorphism $N_1 \to N_2$.*

Proof. See the articles cited above. □

***Proof of Theorem* B.** Let $\Sigma_0 \subset \Sigma_1 \subset \cdots N_1$ be a principal exhaustion of N_1, B be the set of boundaries of this principal exhaustion, and for each $0 \leq i < \omega$ let B_i the boundary curves of Σ_i.

Since for each $0 \leq i < \omega$, B_i is a set of non-outer separating curves, then $\varphi(B_i)$ is composed solely of non-outer separating curves.

Then, let $P \supset B$ be a pants decomposition of N_1; for each $i \in \mathbb{Z}^+$, note the following two facts:

(1) The set of curves from $P \backslash B$ contained in Σ_i forms a pants decomposition of Σ_i. We denote this pants decomposition as P_i.
(2) By Lemma 3.6 there is a finite subgraph of $\mathcal{A}(P)$ delimited exactly by the cut vertices corresponding to B_i, and this finite subgraph corresponds to P_i.

Given that for each $0 \leq i < \omega$, the set $\varphi(B_i)$ is composed solely of non-outer separating curves, and using Lemma 3.2, Corollary 3.5 and point (2) above, there exists (for each $0 \leq i < \omega$) a finite subgraph $\varphi(P_i)$ in $\mathcal{A}(\varphi(P))$ delimited exactly by the cut vertices corresponding to $\varphi(B_i)$. Again for each $0 \leq i < \omega$, by Lemma 3.6, there exists a finite-type closed subsurface Σ_i' bounded by $\varphi(B_i)$.

Recalling that for any $i = 1, 2$ and any finite-type subsurface $\Sigma \subset N_i$, there is a natural embedding $\mathcal{C}(\Sigma) \hookrightarrow \mathcal{C}(N_i)$ induced by the inclusion, we denote the image of this inclusion in $\mathcal{C}(N_i)$ also as $\mathcal{C}(\Sigma)$. With this, for any $\alpha \in V(\mathcal{C}(\Sigma_i))$, either $\alpha \in P_i$ or there exists $\beta \in P_i$ such that $i(\alpha, \beta) \neq 0$. Since this is preserved by φ, we can induce a map $\varphi_i : \mathcal{C}(\Sigma_i) \to \mathcal{C}(\Sigma_i')$, defined as $\alpha \mapsto \varphi(\alpha)$; given that φ is an isomorphism, we have that φ_i is also an isomorphism for each $0 \leq i < \omega$.

Using Theorem 3.8, we have that (for each $0 \leq i < \omega$), Σ_i is homeomorphic to Σ_i' and there exists a homeomorphism $f_i : \mathrm{int}(\Sigma_i) \to \mathrm{int}(\Sigma_i')$ such that φ_i is induced by f_i. Given that for any $i < j$ we have that $\varphi_i = \varphi_j|_{\mathcal{C}(N_i)}$, by the Alexander method, we obtain that $f_i = f_j|_{\mathrm{int}(N_i)}$. With this, we can define a map $f : N_1 \to N_2$ as $x \mapsto f_i(x)$ if $x \in \mathrm{int}(\Sigma_i)$. By construction this map is a homeomorphism that induces φ. □

4. Topological generation of PMap(N)

The group Homeo(N) with the compact-open topology is a topological group; thus, Map(N) inherits a very natural topological group structure via the quotient of the compact-open topology. In this work, we abuse language and refer to this topology of Map(N) as the "compact-open topology" too.

In the finite-type surface case, it not hard to see that Map(N) becomes a discrete group with this topology (the Alexander Method is a simple way of seeing this). However, in the infinite-type case this does not happen; moreover, Map(N) is not even locally compact (see [3]). As such, Map(N) (and its subgroups) becomes much more interesting from a topological-group viewpoint. In this section, we obtain several results considering Map(N) with this topology; all these results have analogues in the orientable-surface case, and their proofs here are essentially the same but with subtle differences (see [15] and [2]).

4.1. Map(N) is Polish. Recall that a *Polish group* is a topological group whose underlying space is Polish (a separable and completely metrizable space). We verify that Map(N) is a Polish group.

Let Γ be a graph with a countable set of vertices. We define a topology on Aut(Γ) as follows: for any finite vertex subset A of the vertex set of Γ define

$$U_A = \{f \in \mathrm{Aut}(\Gamma) : f(a) = a \text{ for all } a \in A\}.$$

Then, the *permutation topology* on Aut(Γ) is defined as the topology with basis the translated sets $f \cdot U_A$, where A is a finite set of vertex of Γ and $f \in \mathrm{Aut}(\Gamma)$. With this topology, Aut(Γ) is separable and; moreover, it is a Polish group (see Lemma 2.2 in [2], and [3]).

The curve graph $\mathcal{C}(N)$ of an infinite type surface N (orientable or not) has a countable set of vertices. A simple way of seeing this is to consider a principal exhaustion; each curve graph of these subsurfaces has a countable set of vertices and the union of all these sets of vertices is equal to the set of vertices of $\mathcal{C}(N)$. Hence, we can equip Aut($\mathcal{C}(N)$) with the permutation topology. From Corollary C we have that $\mathrm{Map}(N) \cong \mathrm{Aut}(\mathcal{C}(N))$, and using the isomorphism Ψ we can pullback the permutation topology to Map(N). We call this topology the *permutation topology* on Map(N). Recall that this topology has as basis the Map(N)-translates of the sets

$$U_A = \{f \in \mathrm{Map}(N) : f(a) = a \text{ for all } a \in A\},$$

where A is any finite set of the vertices of $\mathcal{C}(N)$. Then, we have that Map(N) is a Polish group with the permutation topology.

Using the Alexander method for finite surfaces (see Theorem 2.1), it can be proved that the compact-open topology is the same as the permutation topology on Map(N), the proof for orientable surfaces (see Proposition 2.4 in [2]) works for the non-orientable ones too. For the sake of completeness, we include a sketch here.

First we make the following observation. Let K and U respectively be a compact set and an open set in N. Recall that in the compact-open topology in Homeo(N), the set

$$[K, U] := \{f \in \mathrm{Homeo}(N) : f(K) \subset U\}$$

is a sub-basic open set. As such, the set

$$[\![K, U]\!] := \{f \in \mathrm{Map}(N) : \exists F \in f, F(K) \subset U\}$$

is a sub-basic open set of Map(N).

For the sake of completeness, we give an example of a sub-basic open neighbourhood of the identity. Let $K, U \subset N$ be respectively a compact set and an open set in N such that K can be isotoped inside U, i.e. there exists an ambient isotopy of N such that it deforms K inside of U. Then $[\![K, U]\!]$ is an open neighbourhood of the identity, and it consists of all mapping classes that (up to isotopy) keep K inside of U. Of particular interest is when K is the image of a curve, and U is its regular neighbourhood, which leaves with an open set consisting of all mapping classes that fix the curve up to isotopy.

Lemma 4.1. *Let N be a connected (possibly non-orientable) surface of infinite topological type. Then, the compact-open topology and the permutation topology in* $\mathrm{Map}(N)$ *coincide.*

Sketch of proof. Let τ be the compact-open topology and τ' be the permutation topology. Let $A = \{\alpha_1, \ldots, \alpha_n\}$ be a finite set of vertices in $\mathcal{C}(N)$, and $U_A \in \tau'$ be a basic open neighbourhood of the identity $id_{\mathcal{C}(N)}$. Consider for each $\alpha_i \in A$ the basic open $V_i = [\![\alpha_i, N(\alpha_i)]\!]$ where $N(\alpha_i)$ is a regular neighborhood of γ_i. Let $V = V_1 \cap \cdots \cap V_n$, the set V is a basic open neighbourhood of $[id_N]$ in τ, and $V = U_A$. Then $\tau' \subset \tau$.

Conversely, let $U \in \tau$ be a sub-basic open neighbourhood of $[id_N]$, we have that $U = [\![K, V]\!]$ with K and V compact and open subsets of N respectively. Let Σ be a connected, compact subsurface of N such that $K \subset \Sigma$. Let A be a finite set of curves such that $A \cap \Sigma$ is collection of arcs and curves in Σ that satisfies the Alexander method for finite-type surfaces (see Theorem 2.1). From the Alexander method, we have that each $f \in U_A$ is isotopic to the identity in Σ. In particular, there exists $F \in f$ such that $F(K) = K \subset V$, hence $U_A \subset U$.

Therefore, $\tau = \tau'$. □

Now follows from Lemma 4.1 that the natural isomorphism $\Psi : \mathrm{Map}(N) \to \mathrm{Aut}(\mathcal{C}(N))$ is an isomorphism between topological groups, which is precisely Corollary D.To finish this subsection, we make a small note on convergence in $\mathrm{Map}(N)$.

Given Σ a compact subsurface of N and $f \in \mathrm{Map}(N)$, the set $\{g \in \mathrm{Map}(N) \mid g|_\Sigma = f|_\Sigma\}$ is open. Let $f \in \mathrm{Map}(N)$ and $(f_i)_{0\le i<\omega}$ be a sequence in $\mathrm{Map}(N)$. By definition of the topology, the sequence $(f_i)_{0\le i<\omega}$ converges to f if and only if for all open neighborhoods V of f there exists $N \ge 0$ such that $f_i \in V$ for all $i \ge N$. Then, if $\Sigma_1 \subset \Sigma_2 \subset \cdots N$ is a principal exhaustion and $(f_i)_{0\le i<\omega}$ converges to f, for all i there exists an n_i such that for all $j \ge n_i$ we have that $f_j|_{\Sigma_i} = f|_{\Sigma_i}$.

4.2. Handle-shifts. The concept of a handle-shift was introduced in [15], and intuitively it is moving/shifting the genera between two ends of N that are accumulated by genus. Here we recall their definition of a handle-shift. Let $o\Sigma$ the surface obtained by taking $\mathbb{R} \times [-1, 1]$, removing the interior of each disk of radius $\frac{1}{4}$ with center in $(k, 0)$ for each $k \in \mathbb{Z}$, and attaching a torus with one boundary component to the boundary left by removing each such a disk. If instead of a torus we attach projective planes with one boundary component, we obtain a surface denoted by $n\Sigma$. See Figure 6.

Let $\sigma : o\Sigma \to o\Sigma$ be the isotopy class of the homeomorphism determined by requiring

(1) $\sigma(x, y) = (x + 1, y)$ for $(x, y) \in \mathbb{R} \times [-1 + \epsilon, 1 - \epsilon]$ for some $\epsilon > 0$ and
(2) $\sigma(x, y) = (x, y)$ for $(x, y) \in \mathbb{R} \times \{-1, 1\}$.

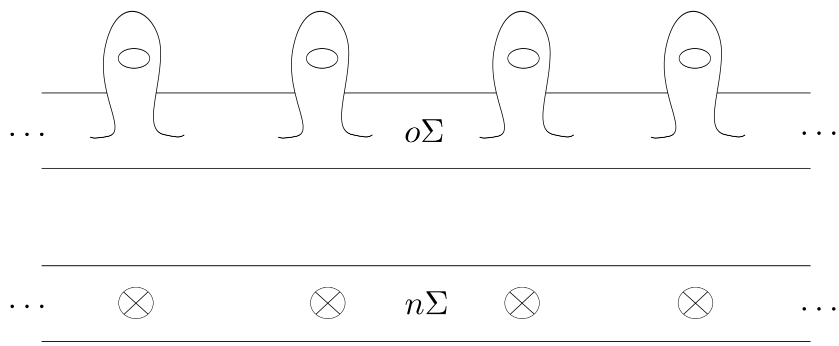

FIGURE 6. The surface $o\Sigma$ above and the surface $n\Sigma$ below.

Similarly, let $n\sigma : n\Sigma \to n\Sigma$ be the homeomorphism determined by similar conditions to that of σ.

Consider a surface N with at least two non planar ends and let $h : N \to N$ be a homemorphism. If there is an embedding $\iota : o\Sigma \hookrightarrow N$ inducing an injection $\iota_* : \mathcal{E}(o\Sigma) \hookrightarrow \mathcal{E}(N)$ such that

$$h(x) = \begin{cases} (\iota \circ \sigma \circ \iota^{-1})(x) & \text{if} \quad x \in \iota(o\Sigma), \\ x & \text{otherwise,} \end{cases}$$

we say that h is:

(1) an *orientable handle-shift* if both elements of $\iota_*(\mathcal{E}(o\Sigma))$ are orientable,
(2) a *semi-orientable handle-shift* if exactly one element of $\iota_*(\mathcal{E}(o\Sigma))$ is orientable,
(3) a *pseudo-orientable handle-shift* if both elements of $\iota_*(\mathcal{E}(o\Sigma))$ are not orientable.[1]

Finally, we say that a homeomorphism $h : N \to N$ is a *non-orientable handle-shift* if there exists an embedding $\iota : n\Sigma \to N$ inducing an injection $\iota_* : \mathcal{E}(n\Sigma) \to \mathcal{E}(N)$ and such that

$$h(x) = \begin{cases} (\iota \circ n\sigma \circ \iota^{-1})(x) & \text{if} \quad x \in \iota(n\Sigma), \\ x & \text{otherwise.} \end{cases}$$

We call the mapping class associated to a handle-shift a handle-shift as well. Notice that a power of a handle-shift is not a handle-shift; this can be proved using the Alexander method. Also notice that in all cases, the homeomorphism h has an attracting and a repelling end denoted by h^+ and h^- respectively, and they are invariant under isotopies (see [2]).

Now, let the *ends graph of* N, denoted by $EG(N)$, be the simplicial graph whose vertex set is $\mathcal{E}_\infty(N)$, and such that two vertices x, y span an edge if there

[1]The naming for this mapping class comes from the fact that even though its support is orientable, it "comes and goes" between non-orientable ends.

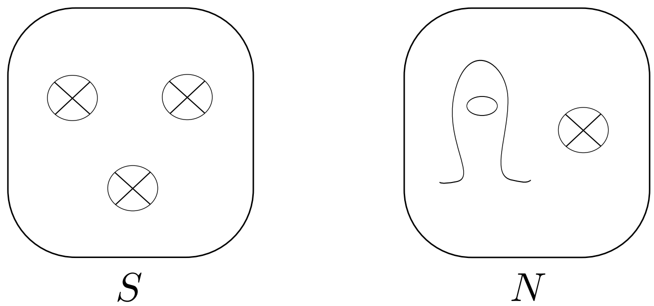

FIGURE 7. The surfaces S and N are homeomorphic.

exists a handle-shift h with $\{x, y\} = \{h^+, h^-\}$. Note that with the discourse above and Figure 7, it is clear that $EG(N)$ is a complete graph. This graph is used below in Subsection 4.5.3.

More recently, Abbott, Miller and Patel in [1] have defined "shift maps" as homeomorphisms constructed as above but with the difference that they consider $o\Sigma$ to be possibly obtained by gluing to the infinite band $\mathbb{R} \times [-1, 1]$ other surfaces besides a torus/projective plane with one boundary component.

4.3. Type of curves in N. To have a better idea of how the following proofs work, we give a small note on the different (topological) types of curves that exist in a non-orientable surface. Thus, for this section let N be a non-orientable surface with compact boundary (hence $0 \leq b < \infty$ boundary components), α be an essential curve in N, V a regular neighbourhood of α that is either an annulus (if α is two-sided) or a Möbius band (if α is one-sided), and N' be the essential subsurface defined as $N' = \overline{N \backslash V}$.

To ease the notation, we say α is *orienting* if N' is orientable, and we say it is *non-orienting* otherwise.

4.3.1. N is of finite type. Let N be a finite-type surface. Then, N can be thought as the connected sum of $g > 1$ projective planes with $n \geq 0$ punctures and $b \geq 0$ boundary components.

The first thing to notice is that α can be either separating or non-separating, and these classes of curves can be further divided into subclasses depending on N'. Using an easy Euler characteristic argument one can prove the following facts.

(1) α is separating. In this case, $N' = N_1 \sqcup N_2$, and let (g_1, g_2), (n_1, n_2) and $(b_1 + 1, b_2 + 1)$ be the respective genus, punctures and boundary components of N_1 and N_2.
Since both N_1 and N_2 have a boundary component induced by V, we have that α is always a two-sided curve.

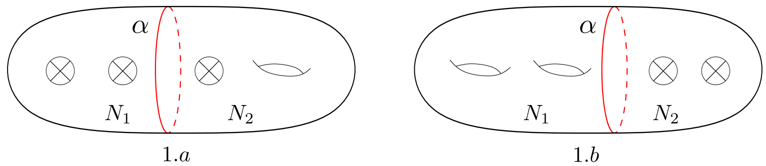

FIGURE 8. Topological types of separating curves on a non-orientable surface.

(a) If both N_1 and N_2 are non-orientable surfaces, then not only are $\{n_1, n_2\}$ and $\{b_1, b_2\}$ partitions of n and b, but also α induces a partition $\{g_1, g_2\}$ of g with $g_1, g_2 > 1$. See Figure 8 (1.a) for an example.
(b) If N_1 is orientable and N_2 is non-orientable, then not only are $\{n_1, n_2\}$ and $\{b_1, b_2\}$ partitions of n and b, but also α induces a partition $\{2g_1, g_2\}$ of g with $g_2 > 1$. See Figure 8 (1.b) for an example.

Thus, the possible topological types of α depend on the above partitions.

(2) α is non-separating. In this case, N' is connected, has genus g', n punctures and b' boundary components.
(a) If α is two-sided, then V is an annulus and $b' = b + 2$. Now, this case is subdivided analogously to case (1) depending on the orientability of N'.
(i) If α is orienting, then $2g' = g - 2$. In particular, this case is only possible if g is even and at least 2. See Figure 9 (2.a.i) for an example.
(ii) If α is non-orienting, then $0 < g' = g - 2$. In particular, this case only exists if g is at least 3. See Figure 9 (2.a.ii) for an example.

Hence, α can be two-sided only when then genus of N is at least 2.
(b) If α is one-sided, then V is a Möbius band and $b' = b + 1$. We subdivide this case depending on the orientability of N'.
(i) If α is orienting, then $2g' = g - 1$. In particular, this case only exists if g is odd. See Figure 9 (2.b.i) for an example.
(ii) If α is non-orienting, then $0 < g' = g - 1$. In particular, g is at least 2. See Figure 9 (2.b.ii) for an example.

With this we can conclude the following: In contrast with the orientable case, depending on the surface, not all non-separating curves have the same topological type.

If $g = 1$, then every non-separating is a one-sided orienting curve and there is exactly one topological type of non-separating curves.

If $g = 2$, then there are exactly two topological types of non-separating curves: a non-separating curve α can be either a two-sided orienting curve or a one-sided non-orienting curve.

If $g \geq 3$, then there exists exactly 3 topological types of non-separating curves (cases (2.a.i) and (2.b.i) cannot exist at the same time).

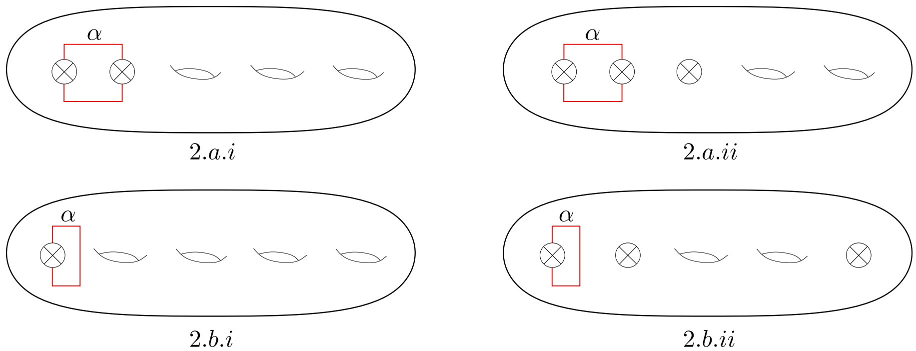

FIGURE 9. Topological types of non-separating curves on a non-orientable surface.

4.3.2. *N* is of infinite type. Let N be an infinite-type surface of genus $g \in \mathbb{N} \cup \{\infty\}$, $b \geq 0$ boundary components and triple of ends spaces[2]

$$(\mathcal{E}(N), \mathcal{E}_\infty(N), \mathcal{E}_-(N)).$$

As in the previous subsubsection, the first thing to notice is that α can be either separating or non-separating, and that these classes of curves can be further subdivided. By taking a non-orientable finite-type essential subsurface of N that contains α as an essential curve, one can verify the following facts.

(1) α is separating. In this case, as it happens in Subsubsection 4.3.1, α induces a partition of the boundary components and the genus. However, in contrast, in this case α also induces a partition of the triple of ends spaces. This partition could be trivial, i.e. one of the connected components of N' is compact. With this in mind, in the case where $b = 0$ the partition is trivial if and only if α is trivial in the integer homology of N.

(2) α is non-separating. In this case, N' is a connected surface of genus g' and b' boundary components. It is clear however that the triple of ends spaces of N and N' are homeomorphic, thus the only differences between N and N' are the boundary components ($b' > b$), possibly the genus and possibly the orientability. Having this in mind, it is clear why in the orientable case there exists exactly one topological type of non-separating curves. Now, as in the previous case, the following facts can be easily deduced from taking an essential finite-type subsurface that contains α as an essential curve.

   (a) If α is two-sided, then $b' = b + 2$.

      (i) If α is orienting, then N has to be finitely even non-orientable and in particular g has to be either finite, even and $2g' = g - 2$, or infinite.

      (ii) If α is non-orienting, then g is at least 3 and $g' = g - 2$.

[2]Recall that these are the set of all ends, the set of all ends accumulated by genus, and the set of all non-orientable ends.

(b) If α is one-sided, then $b' = b + 1$.
  (i) If α is orienting, then N has to be finitely odd non-orientable and in particular g has to be either finite, odd and $2g' = g-1$, or infinite.
  (ii) If α is non-orienting, then g is at least 2 and $g' = g - 1$.

As in the finite-type case, this means that the number of topological types can differ depending on the surface:
If $g = 1$, then every non-separating curve is one-sided and orienting.
If $g = 2$, then a non-separating curve can be either two-sided orienting or one-sided non-orienting.
If $g \geq 3$ and is finite, then a non-separating curve has three options for its topological type.
If $g = \infty$ but N is finitely odd non-orientable, then a non-separating curve can be:

- two-sided non-orienting,
- one-sided orienting, and
- one-sided non-orienting.

If $g = \infty$ but N is finitely even non-orientable, then a non-separating curve can be:

- two-sided orienting,
- two-sided non-orienting, and
- one-sided non-orienting.

If $g = \infty$ and N is infinitely non-orientable, then a non-separating curve has to be either two-sided non-orienting or one-sided non-orienting.

4.4. PMap(N) as a closed subgroup. In this section, we prove that PMap(N) is a closed subgroup, following closely the proof by Patel-Vlamis for the orientable case in [15]. However, for the proof to work on the non-orientable case, the following lemma is needed. As a reminder, recall that a curve α is called one-sided if its closed regular neighborhood is homeomorphic to a Möbius band, and it is called two-sided it if its closed regular neighborhood is an annulus.

Lemma 4.2. *If α is a non-separating curve in N and $f \in$ PMap(N), then there exists an essential finite-type subsurface $\Sigma \subset N$ such that:*

(1) α and $f(\alpha)$ are contained in Σ and they have the same topological type in Σ,
(2) each boundary curve of Σ is an essential separating curve in N,
(3) all the connected components of $N\backslash\Sigma$ share at least one end with N, and
(4) if N has exactly one end accumulated by genus, then exactly one connected component of $N\backslash\Sigma$ has positive genus.

Proof. Let Σ' be a subsurface that contains α and $f(\alpha)$, and satisfies (*2*) - (*4*). This can be obtained by taking an essential subsurface that contains α and $f(\alpha)$ as essential curves, then adding orientable subsurfaces of genus 0 so that the resulting union has every boundary curve as a separating curve in N. Call Σ'' the resulting subsurface. After that, if any of the boundary curves of Σ'' are not

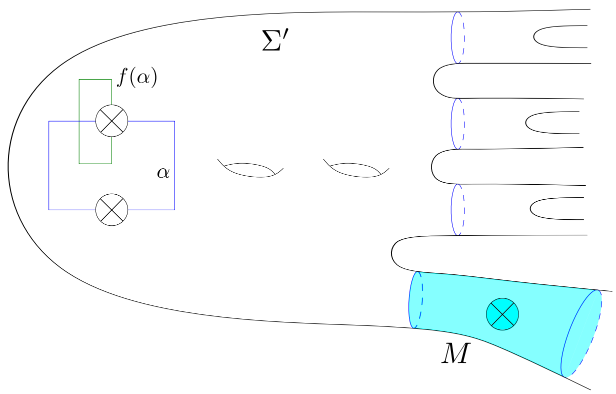

FIGURE 10. The subsurface Σ' with the curves α and $f(\alpha)$ being orienting and non-orienting in Σ', respectively.

essential, add the corresponding punctured disc or disc; this way the resulting subsurface satisfies (*2*). Call Σ''' the resulting subsurface. Then, add any finite-type connected component of $N\backslash\Sigma'''$; the resulting subsurface satisfies (*3*) and (*4*). The resulting subsurface is Σ'.

Now, Σ' is a finite-type subsurface with boundary curves being essential separating curves in N. Thus, α and $f(\alpha)$ have to be non-separating curves in Σ'. Moreover, α and $f(\alpha)$ have to be either both one-sided or both two-sided. Hence, by the discussion in Subsubsection 4.3.1, the topological types of α and $f(\alpha)$ in Σ' might differ only in whether α and $f(\alpha)$ are either orienting in Σ' or non-orienting in Σ'.

If both α and $f(\alpha)$ coincide in being orienting or non-orienting in Σ', then Σ' satisfies (*1*) and we make $\Sigma = \Sigma'$.

Otherwise, without loss of generality we can assume that α is orienting in Σ' while $f(\alpha)$ is non-orienting in Σ'. Since α and $f(\alpha)$ have the same topological type in N, then there exists at least one cross-cap in some connected component of $N\backslash\Sigma'$. Moreover, there exists a subsurface M of genus 1 and two boundary components with both boundary curves as essential separating curves in N, such that M and Σ' share one boundary component. Let Σ be the surface obtained by the union of Σ' and M; see Figure 10. Therefore, Σ satisfies (*1*) - (*4*). □

Theorem 4.3. *Let N be a connected (possibly non-orientable) surface of infinite topological type, and let H be the set of all handle-shifts of N. Then,* $\mathrm{PMap}(N) = \overline{\langle \mathrm{PMap}_c(N) \cup H \rangle}$.

Proof. The proof of this theorem is essentially the same that Patel-Vlamis have for orientable surfaces in Proposition 6.2 of [15]. We just need to take in account the following observations:

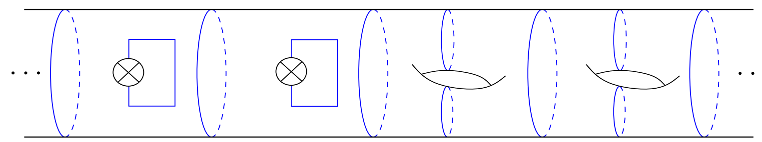

FIGURE 11. A pants decomposition of a non-orientable surface that contains one-sided curves.

(1) A pants decomposition of a non-orientable surface N may contain one-sided curves, and one-sided curves bound exactly one pair of pants (instead of two pair of pants as two-sided curves may). See Figure 11
(2) One-sided curves are not separating curves.
(3) A key step on Patel-Vlamis' proof is that in the orientable case, any two non-separating curves on a finite type surface always have the same topological type. We substitute this argument with Lemma 4.2.
(4) Some steps of Patel-Vlamis proof involve comparing two subsurfaces of N, say V and W. They do it by comparing their genus. In the general case, it is better to see whether V and W are homeomorphic or not.
(5) In the last paragraph of Patel-Vlamis proof, for the general case, it is a good idea to consider the subsurfaces V and W as connected sums of a torus plus zero, one or two projective planes. In this way, it becomes more obvious which composition of handle-shifts is the mapping class h. □

4.5. Definition of $\{h_i\}_{0\leq i<r}$. The purpose of this subsection is to construct a collection $\{h_i\}_{0\leq i<r}$ of handle-shifts that satisfies Theorem E, which we recall below.

Theorem (E). *Let N be a connected (possibly non-orientable) surface of infinite topological type. If N has at most one end accumulated by genus, then* $\mathrm{PMap}(N) = \overline{\mathrm{PMap}_c(N)}$. *If N has at least two ends accumulated by genus, then there exist a constant $1 \leq r \leq \omega$ and a collection $\{h_i\}_{0\leq i<r}$ of handle-shifts, such that* $\mathrm{PMap}(N) = \langle \overline{\mathrm{PMap}_c(N)}, \{h_i\}_{0\leq i<r}\rangle$ *and* $\overline{\langle\{h_i\}_{0\leq i<r}\rangle}$ *is isomorphic to* $\mathbb{Z}^r$ *as a topological group.*

Using Theorem 4.3, we have that if N has at most one end accumulated by genus, then $\mathrm{PMap}(N) = \overline{\mathrm{PMap}_c(N)}$, proving the first part of Theorem E. Thus, for the rest of this section, we assume that N has at least two ends accumulated by genus, unless otherwise stated.

The construction of the $\{h_i\}_{0\leq i<r}$ is inspired by the one made in the proof of Theorem 3 in [2]: Given a surface Σ with either empty boundary or compact boundary, we define $\widehat{\Sigma}$ as the surface obtained by forgetting the planar ends and capping the boundary components with disks. The idea is to find a collection of separating simple closed curves $\{\gamma_i\}_{0\leq i<r}$ such that their homology classes generate $H_1^{sep}(\widehat{N};\mathbb{Z})$ and assign to each of these curves a handle-shift.

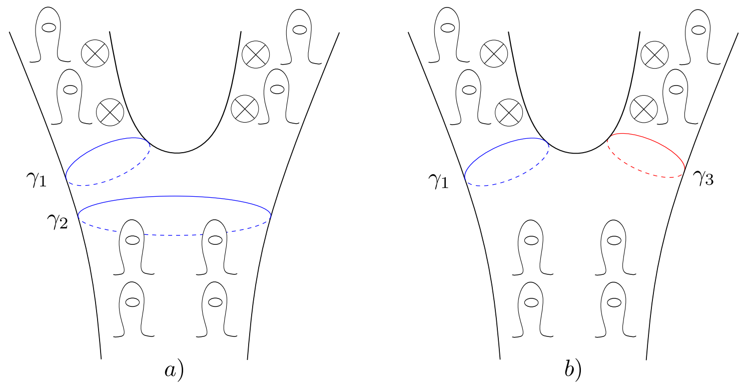

FIGURE 12. $H_1^{sep}(N;\mathbb{Z})$ can be generated by the homology classes of the curves $\{\gamma_1,\gamma_2\}$ or equivalently by the homology classes of the curves $\{\gamma_1,\gamma_3\}$. In a), we can associate to γ_1 a non-orientable handle-shift and to γ_2 a semi-orientable one. In b), we can only associate to γ_i semi-orientable handle-shifts.

In [2], the curves $\{\gamma_i\}_{0\leq i<r}$ satisfy that the components of the surface Σ' obtained by taking $\widehat{\Sigma}$ and removing disjoint regular neighborhoods of each γ_i is one-ended and has infinite genus. Then to each γ_i is associated a handle shift h_i such that the attracting end h_i^+ is in one of the components of Σ' containing a boundary component b_1 homotopic to γ_i and the repelling end h_i^- is in the other component of Σ' containing a boundary component b_2 homotopic to γ_i.

However, unlike the orientable case treated in [2], if N is a surface with at least one orientable end and two non-orientable ends, we cannot take just any such a collection of curves. The problem is that it is not clear how to build non-orientable handle-shifts using only $\overline{\mathrm{PMap}_c(N)}$ and pseudo-orientable, semi-orientable and orientable handle-shifts, see Figure 12. So we need to choose a good basis of $H_1^{sep}(\widehat{N};\mathbb{Z})$ in order to have in $\{h_i\}_{0\leq i<r}$ the minimum number of non-orientable handle-shifts needed to generate all the non-orientable handle-shifts.

4.5.1. A fixed model for N. Recalling that if N is a non-orientable surface of genus at least 3, we can exchange 3 cross-caps for one torus and one cross-cap, and thus change the model. So, we start this subsubsection by fixing a model of a given infinite-type surface N. Recall the following theorem of I. Richards.

Theorem 4.4 (cf. Theorem 2 in [16])**.** *Given a triple (X,Y,Z) of compact, separable, totally disconnected spaces $Z\subset Y\subset X$, there is a surface Σ whose ends $(\mathcal{E}(\Sigma),\mathcal{E}_\infty(\Sigma),\mathcal{E}_-(\Sigma))$ are topologically equivalent to the triple (X,Y,Z).*

In the proof of Theorem 4.4, Richards gives an explicit construction of the surface Σ. So, given an infinite-type surface N and using Richards' construction, we obtain a surface Σ homeomorphic to N. Abusing notation, we also denote it by N. We recall Richards' construction here.

The idea is to construct the surface N from a sphere S by first removing from S a set X homeomorphic to $\mathcal{E}(N)$, then removing the interiors of a finite or infinite sequence of non-overlapping closed discs in $S - X$ and finally suitably identifying the boundaries of these discs in pairs to form handles and/or cross-caps (see Theorem 3 in [16]). The non-overlapping discs "converge" to X in a sense that will be clear form the description below. See Figure 13.

Recall that $\mathcal{E}(N)$ is homeomorphic to a subset of the Cantor set $\mathcal{C}$. Embed the cantor set $\mathcal{C}$ in the one point compactification of the plane as the set of all points $(x, 0)$ such that $0 \leq x \leq 1$ and x admits a triadic expansion which does not involve the digit 1. Let $\mathcal{D}'$ be the collection of all closed disks in the plane whose diameters are the intervals in the x axis

$$\left[\frac{n-\frac{1}{3}}{3^m}, \frac{n+\frac{4}{3}}{3^m}\right],$$

with $m, n \in \mathbb{Z}$ such that $m \geq 1,\ 0 \leq n \leq 3^m$ and n admits a triadic expansion free from 1's. Let $\mathcal{D}$ be the subcollection consisting of all disks in $\mathcal{D}'$ which contain at least one point of $\mathcal{E}(N)$. For each disk $\varkappa \in \mathcal{D}$, let $\varkappa_1$ and $\varkappa_2$ the two largest disks in $\mathcal{D}'$ properly contained in $\varkappa$. Choose two circles $C^+(\varkappa)$ and $C^-(\varkappa)$ contained in the interior of $\varkappa$ such that:

(1) $C^+(\varkappa)$ is contained in the upper half-plane and $C^-(\varkappa)$ is contained in the lower half-plane.
(2) $C^+(\varkappa)$ and $C^-(\varkappa)$ do not intersect $\varkappa_1$ and $\varkappa_2$.
(3) $C^+(\varkappa)$ and $C^-(\varkappa)$ are symmetric with respect to the x axis.

Remove the interior of $C^{\pm}(\varkappa)$ for all $\varkappa \in \mathcal{D}$ such that $\varkappa \cap \mathcal{E}_\infty(N) \neq \emptyset$. If $\varkappa \cap \mathcal{E}_-(N) = \emptyset$, then identify the boundaries of $C^+(\varkappa)$ and $C^-(\varkappa)$ by reflecting $C^+(\varkappa)$ in the x axis preserving orientation, i.e. add a handle. If $\varkappa \cap \mathcal{E}_-(N) \neq \emptyset$, then identify $C^+(\varkappa)$ and $C^-(\varkappa)$ by translating $C^+(\varkappa)$ onto $C^-(\varkappa)$, i. e. add a Klein bottle.

If the surface N is of finite genus orientable or non-orientable, in the above construction, add the finite number of handles or cross-caps that are needed. Similarly, if N is of infinite genus but odd or even non-orientable, add one or two cross-caps respectively.

4.5.2. A family of curves $\mathcal{H}$. Let N be an infinite-type surface (with empty boundary). With the model of a surface described in Subsubsection 4.5.1, we construct a family of pairwise disjoint and non-homologous separating simple closed curves $\mathcal{H} = \{\gamma_i\}_{0 \leq i < r}$, with $1 \leq r \leq \omega$ and such that each connected component of the surface N' obtained from N by removing pairwise disjoint

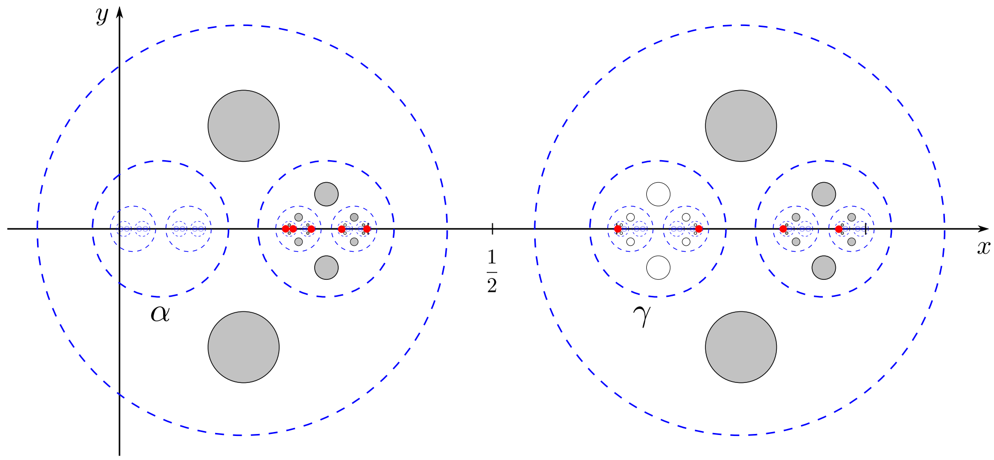

FIGURE 13. The red points are the set $\mathcal{E}(N)$. The punctured circles are boundaries of discs in $\mathcal{D}'$. The disc with boundary α belongs to $\mathcal{D}'$ but not to $\mathcal{D}$. The disc with boundary γ belongs to $\mathcal{D}$. The interiors of the shadow discs are removed and the boundaries are identified in pairs to form handles and cross-caps.

regular neighborhoods of each $\gamma \in \mathcal{H}$, has exactly one end accumulated by genus (orientable or non-orientable).

We remind the reader that while given any surface Σ with at most one end, any separating curve is null-homologous, if Σ has at least two ends, there exists separating curves that are not null-homologous. See Figure 14.

Recall that $H_1^{sep}(N;\mathbb{Z})$ denotes the subgroup of $H_1(N;\mathbb{Z})$ that is generated by homology classes that can be represented by separating simple closed curves on the surface. Notice that Lemma 4.2 of [2] is also valid for non-orientable surfaces, for the sake of completeness we enounce here this lemma.

Lemma 4.5. *If $\{N_i\}_{0\leq i<\omega}$ is a principal exhaustion of N, then*

$$H_1^{sep}(N;\mathbb{Z}) = \lim_{i\to\infty} H_1^{sep}(N_i;\mathbb{Z}).$$

In particular, there exists $0 \leq n, m < \omega$ such that every non-zero element $\nu \in H_1^{sep}(N;\mathbb{Z})$ can be written as

$$\nu = \sum_{k=1}^{m} a_k \nu_k$$

where $a_k \in \mathbb{Z}$ and ν_k can be represented by a peripheral curve on N_n.

Keeping in mind that $\widehat{N}$ is the surface obtained by “filling” all the planar ends of N, it follows from Lemma 4.5 and the fact that $H_1^{sep}(\widehat{N};\mathbb{Z})$ is a free abelian group, that there exists a collection of pairwise disjoint and non-homologous

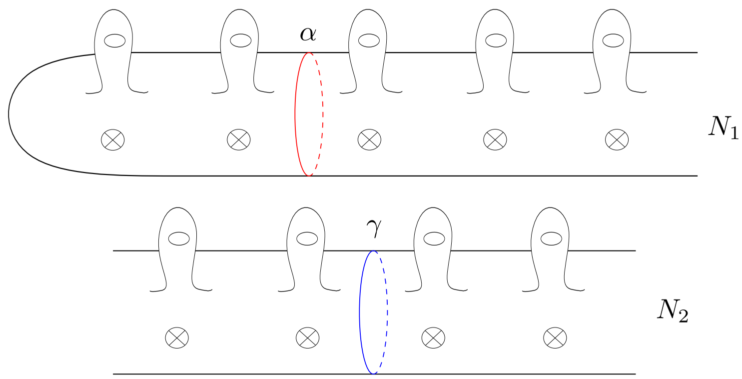

FIGURE 14. In the surface N_1, any separating curve is null-homologous. In the surface N_2, the curve γ is not null-homologous.

separating simple closed curves $\{\gamma_i\}_{0\leq i<r}$ on $\widehat{N}$ such that

$$H_1^{sep}(\widehat{N};\mathbb{Z}) = \bigoplus_{0\leq i<r} \langle \nu_i \rangle$$

where ν_i denotes the homology class of γ_i.

Remark 4.6.

(1) We can choose the curves γ_i such that they do not intersect the boundaries and planar ends of N.
(2) Notice that r could be ω, i.e. $H_1^{sep}(\widehat{N};\mathbb{Z})$ could be not finitely generated.
(3) If N' is the surface obtained by taking $\widehat{N}$ and removing pairwise disjoint regular neighborhoods of each γ_i, then each component of N' is one-ended and has infinite genus. This follows from the fact that each separating curve in N' bounds a compact surface.
(4) Recalling that we can represent N as in Subsubsection 4.5.1, we can choose each γ such that it is the boundary of a disk $\kappa \in \mathcal{D}$.

With this remark in mind, if the surface N is one of the following types

- orientable,
- even or odd non-orientable,
- non-orientable and $\mathcal{E}_\infty(N) = \mathcal{E}_-(N)$,

we define $\mathcal{H}$ as the family of pairwise disjoint and non-homologous separating simple closed curves such that their homology classes generate $H_1^{sep}(\widehat{N};\mathbb{Z})$. In these three cases, with this definition of $\mathcal{H}$ we do not have situations as in Figure 12 *b*). If N is not one of these cases, then $\mathcal{E}_-(N) \neq \emptyset$ and $\mathcal{E}_\infty(N) \neq \mathcal{E}_-(N)$. We define $\mathcal{H}$ for this type of surfaces in the following paragraphs.

Suppose the set of ends of N satisfies that $\mathcal{E}_-(N) \neq \emptyset$ and $\mathcal{E}_\infty(N) \neq \mathcal{E}_-(N)$. Consider a model for N as the one described in Subsubsection 4.5.1 and for simplicity suppose $\widehat{N} = N$. See Figure 15 for an example. We choose any non-orientable end $e_-^0 \in \mathcal{E}_-(N)$. Taking $\mathcal{D}$ as in Subsubsection 4.5.1, let $\varkappa_0 \in \mathcal{D}$ be such that $e_-^0 \in \varkappa_0$ and there exist at least one more $\varkappa \in \mathcal{D}$ with the same diameter. Let $\varkappa_1, \dots \varkappa_{r_1} \in \mathcal{D}$ be the disks different from $\varkappa_0$ but with the same diameter and denote by $\gamma_{1,1}^0, \dots, \gamma_{1,r_1}^0$ their respective boundaries. Define

$$\mathcal{H}_{\gamma,1}^0 = \{\gamma_{1,1}^0, \dots, \gamma_{1,r_1}^0\}.$$

By removing from N pairwise disjoint regular neighborhoods of each $\gamma_{1,i}^0 \in \mathcal{H}_{\gamma,1}^0$, we obtain a surface N^0 with $r_1 + 1$ connected components, each of which has infinite genus and satisfies one of the following:

a) has only orientable ends,
b) has only non-orientable ends,
c) has both orientable and non-orientable ends.

Denote each connected component of N^0 by $N_{1,i}^0$ with $1 \le i \le r_1 + 1$ and let t_0 be the number of components of N^0 that satisfy (c). Denote each component $N_{1,i}^0$, ordering the components such that the first t_0 components are those that satisfy (c).

If $t_0 < r_1 + 1$, for each subsurface $N_{1,i}^0$ with $t_0 < i \le r_1 + 1$ let $\mathcal{H}_{1,i}^0$ be a family of pairwise disjoint simple closed curves such that their homology classes generate $H_1^{sep}(\widehat{N}_{1,i}^0;\mathbb{Z})$ and such that they are boundary curves of elements in $\mathcal{D}$ (see Remark 4.6). See Figure 15 for an example. Define

$$\mathcal{H}^0 = \mathcal{H}_{\gamma,1}^0 \cup \left(\bigcup_{i=t_0+1}^{r_1+1} \mathcal{H}_{1,i}^0 \right).$$

The collection $\mathcal{H}^0$ above does not consider the surfaces satisfying c) because in this kind of surfaces we have subsurfaces as in Figure 12, so we need to be more careful when we select curves whose homology classes generate $H_1^{sep}(\widehat{N}_{i,j}^0;\mathbb{Z})$.

To avoid confusion with the subindices, let $n_1 = t_0$. If $n_1 = 0$, we are done and we define $\mathcal{H} = \mathcal{H}^0$. If $n_1 \ge 1$, for $1 \le j \le n_1$ rename each surface $N_{1,j}^0$ as N_j^1. Repeating the algorithm, we obtain the following for each N_j^1:

i) A set

$$\mathcal{H}_{\gamma,j}^1 = \{\gamma_{j,1}^1, \dots, \gamma_{j,r_j}^1\}$$

of separating simple closed curves which are boundaries of disks in $\mathcal{D}$.

ii) A family of subsurfaces $N_{j,i}^1$ with $1 \le i \le r_j + 1$, and where the first $0 \le t_j \le r_j + 1$ satisfy c) and the rest $r_j + 1 - t_j$ satisfy either a) or b).

iii) If $t_j < r_j + 1$, for each $N_{j,i}^1$ with $t_j < i \le r_j + 1$, a set $\mathcal{H}_{j,i}^1$ of pairwise disjoint simple closed curves which are boundaries of disks on $\mathcal{D}$ and their homology classes generate $H_1^{sep}(\widehat{N}_{j,i}^1;\mathbb{Z})$.

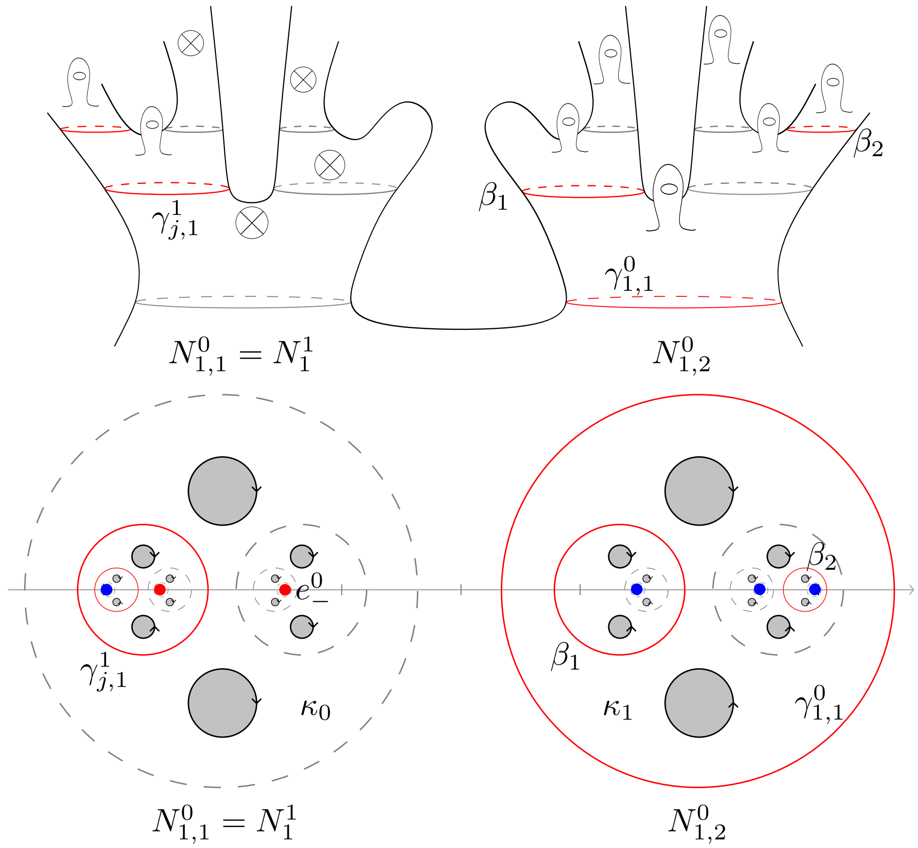

FIGURE 15. The blue points are orientable ends and the red points are non-orientable ends. $H_1^{sep}(\widehat{N}^1_{1,2}; \mathbb{Z})$ is generated by the homology classes of the curves β_1 and β_2. The red curves belong to $\mathcal{H}$.

For each $1 \leq j \leq n_1$, if $t_j < r_j + 1$ let

$$\mathcal{H}^1_j = \mathcal{H}^1_{\gamma,j} \cup \left(\bigcup_{i=t_j+1}^{r_j+1} \mathcal{H}^1_{j,i} \right)$$

and if $t_j = r_j + 1$, let $\mathcal{H}^1_j = \mathcal{H}^1_{\gamma,j}$. Define

$$\mathcal{H}^1 = \mathcal{H}^0 \cup \bigcup_{j=1}^{n_1} \mathcal{H}^1_j.$$

Let $n_2 = \sum_{j=1}^{n_1} t_j$. If $n_2 = 0$ we are done an let $\mathcal{H} = \mathcal{H}^1$. If $n_2 > 0$, rename all the subsurfaces $N^1_{j,i}$ with $1 \leq j \leq n_1$ and $1 \leq i \leq t_j$ as N^2_j, with $1 \leq j \leq n_2$. We can repeat the previous procedure to obtain a set of curves $\mathcal{H}^2$. Continuing in this way we define

$$\mathcal{H} = \bigcup_{0\le i\le n} \mathcal{H}^i$$

with $0 \le n \le \omega$. Notice that since this algorithm may not terminate in finite time, we have to allow for the possibility that $n = \omega$. If $N \neq \widehat{N}$, we define $\mathcal{H}$ as the set of curves obtained by applying the above algorithm to $\widehat{N}$.

4.5.3. Construction of $\{h_i\}_{0\le i<r}$.

In Subsubsection 4.5.2, for any infinite-type surface N we obtained a family of curves $\mathcal{H}$. In this subsection, we associate a handle-shift to each $\gamma \in \mathcal{H}$. The association is similar to the one made in the proof of Theorem 3 in [2]; for the sake of completeness, we include it here.

First denote by N' the surface obtained by taking $\widehat{N}$ and removing pairwise disjoint regular neighborhoods of each $\gamma \in \mathcal{H}$.

The idea is the following: For each γ there exist two components of N' that we denote by N_1 and N_2, and such that γ is homotopic to a boundary component b_1 of N_1 and homotopic to a boundary component b_2 of N_2. If N_1 and N_2 are both orientable, we associate to γ an orientable handle shift h_γ with attracting and repelling ends h_γ^- and h_γ^+ in N_1 and N_2 respectively. If N_1 and N_2 are both non-orientable, we associate to γ a non-orientable handle shift h_γ with h_γ^- and h_γ^+ in N_1 and N_2 respectively. Finally, if N_1 is non-orientable and N_2 is orientable, we associate to γ a semi-orientable handle shift h_γ with h_γ^- and h_γ^+ in N_1 and N_2 respectively.

We have the following observations:

i) Each closed curve $\gamma \in \mathcal{H}$ is the boundary of a disk $\varkappa_\gamma \in \mathcal{D}$.
ii) The homology classes of the closed curves of $\mathcal{H}$ are linearly independent and generate $H_1^{sep}(\widehat{N};\mathbb{Z})$.
iii) Each separating curve in N' bounds a compact surface.
iv) Each connected component of N' has one end accumulated by genus.
v) If N has at least two non-orientable ends, let N_1 be a connected component of N' with non-orientable end. Then, there exist a connected component N_2 of N', different from N_1, and a curve $\gamma \in \mathcal{H}$ such that:
   (a) N_2 has a non-orientable end and,
   (b) N_1 and N_2 have a boundary isotopic to γ in N.

   This happens by construction of the familly $\mathcal{H}$.

For simplicity, suppose $N = \widehat{N}$ and index the curves in $\mathcal{H}$, in other words $\mathcal{H} = \{\gamma_i\}_{0\le i<r}$ where $1 \le r \le \omega$. Recall that every connected component of N' is classified up to homeomorphism by its orientability class and the number of boundary components (see [16]).

As is described in [2], let Y be the surface obtained from $[0,1]\times[1,\infty) \subset \mathbb{R}^2$ by attaching periodically infinitely many tori like in the construction of $o\Sigma$. Similarly, let Y' be the surface obtained from $[0,1] \times [-1,-\infty) \subset \mathbb{R}^2$ by attaching periodically infinitely many projective planes.

On the other hand, consider $\mathbb{R}^2$ and remove n open disks centered along the horizontal axis. Attaching an infinite number of tori periodically and vertically above each removed disk, we obtain the one-ended infinite-genus orientable surface with n boundary components that in [2] is denoted by Z_n. If below each removed disk we also attach, periodically and vertically, an infinite number of projective planes, we obtain the one-ended infinitely non-orientable surface with n boundary components, W_n. If instead of attaching an infinite number of projective planes to Z_n we attach an even (respectively odd) number, we obtain the one-ended infinite-genus even non-orientable (respectively odd non-orientable) surface with n boundary components that we denote by eZ_n (respectively oZ_n).

Observe that there are n disjoint embeddings of Y into Z_n, W_n, eZ_n and oZ_n, respectively such that in each one the image of $[0,1] \times \{1\} \subset Y$ is contained in a unique boundary component of the surface under consideration and each boundary component contains only one of such images. In the surface W_n, we also have n disjoint embeddings of Y' satisfying similar conditions.

Fix $\gamma_i \in \{\gamma_i\}_{0 \leq i < r}$ and let N_1 and N_2 be the two connected components of N' that have a boundary component homotopic to γ_i in N. Denote these boundaries by $b_1 \subset N_1$ and $b_2 \subset N_2$. We have the following three cases depending on the orientability of the ends of N_1 and N_2.

(1) If the end of N_1 and N_2 are both orientable, as is done in [2], let Y_j denote the image of Y in N_j intersecting b_j, for $j = 1, 2$. The intervals $Y_1 \cap b_1$ and $Y_2 \cap b_2$ can be connected with a strip $T \cong [0,1] \times [0,1]$ in the regular neighborhood of γ_i. The surface $o\Sigma$ can be embedded in N with image $o\Sigma_i = Y_1 \cup T \cup Y_2$. We have two orientable handle-shifts supported in $o\Sigma_i$, choose one and denote it by h_i.
(2) If the end of N_1 is orientable and the end of N_2 is non-orientable (or vice versa), we also have an embedding of $o\Sigma$ with image $o\Sigma_i = Y_1 \cup T \cup Y_2$ but now we have two semi-orientable handle-shifts supported in $o\Sigma_i$, again choose one and denoted it by h_i.
(3) If the end of N_1 and N_2 are both non-orientable, denote by Y'_j the image of Y' in N_j intersecting b_j, $j = 1, 2$. Proceeding as in the previous cases we have now an embedding of $n\Sigma$ in N with image $n\Sigma_i = Y'_1 \cup T \cup Y'_2$ and in consequence we have two non-orientable handle-shifts supported in $n\Sigma_i$, choose one and denote it by h_i.

To each $\gamma_i \in \{\gamma_i\}_{0 \leq i < r}$ we assign it the corresponding handle-shift h_i. Notice that:

(1) The support of h_i intersect γ_j if and only if $i = j$, hence $\langle h_i, h_j \rangle$ is a free abelian group, and its rank is 2 if and only if $i \neq j$.
(2) In the case 3, there are also two pseudo-orientable handle-shifts (see Figure 16), however we do not choose any one of them because it is not clear that doing that we can generate all PMap(N).

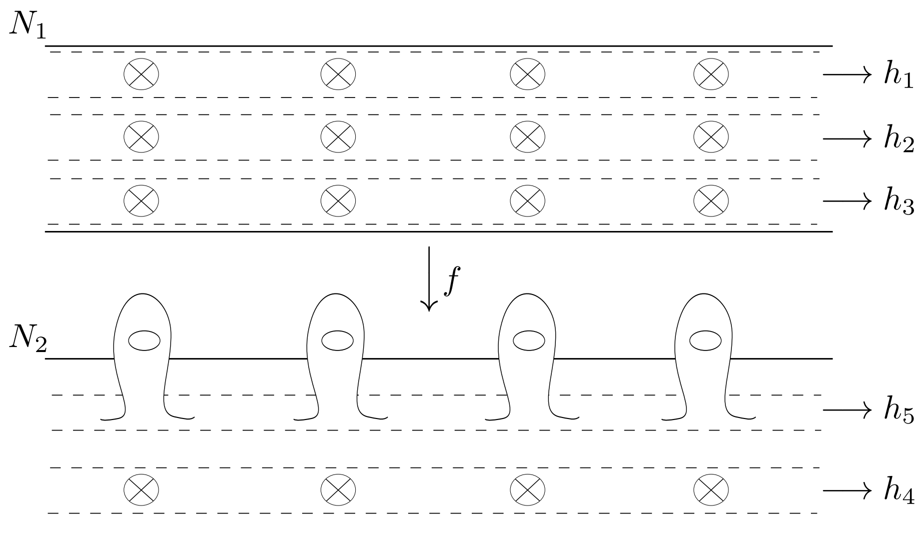

FIGURE 16. $[h_1h_2h_3] = f^{-1}[h_5h_4]f$.

(3) For each h_i, the ends h_i^+ and h_i^- span an edge of $EG(N)$ which by abuse of notation we also denote by h_i. The set of all such ends and edges form a **maximal tree** of $EG(N)$ denoted by $TEG(N)$. Even more, we give an arbitrary orientation to $EG(N)$ with the condition that every edge h_i in $TEG(N)$ has initial vertex $\iota(h_i) = h_i^-$ and terminal vertex $\tau(h_i) = h_i^+$.

(4) The set of vertices of $EG(N)$ and edges h_i that correspond to non-orientable ends and non-orientable handle-shifts respectively, form a subgraph of $TEG(N)$ that we denote by $nTEG(N)$. By construction and the previous points, $nTEG(N)$ is a subtree.

Consider the abelian subgroup topologically generated by the handle-shifts $\{h_i\}_{0\le i<r}$ with the subgroup topology. Note that using the permutation topology, this group is homeomorphic to the group $\mathbb{Z}^r$ with the product topology.

4.6. Proof of Theorem E. Before we start the proof of Theorem E, consider the homeomorphic surfaces N_1 and N_2 that are shown in Figure 16. Let f be the homeomorphism that sends a neighborhood of each column of three cross-caps on N_1 to a neighborhound of a handle and a cross-cap in N_2 (the ones that are in the correponding column in Figure 16). It is not hard to convince yourself that

$$[h_1\circ h_2\circ h_3] = f^{-1}[h_5\circ h_4]f. \tag{4.1}$$

Note that in N_1, $[f^{-1}\circ h_5\circ f]$ is a pseudo-orientable handle-shift and $[f^{-1}\circ h_4\circ f]$ is a non-orientable handle-shift.

By Theorem 4.3 $\overline{\langle \mathrm{PMap}_c(N)\cup\{h_i\}_{0\le i<r}\rangle} \subseteq \overline{\langle \mathrm{PMap}_c(N)\cup H\rangle}$, then to prove Theorem E it is enough to prove that H is in $\overline{\langle \mathrm{PMap}_c(N)\cup\{h_i\}_{0\le i<r}\rangle}$. Given

$h \in H$, the idea is to use $\overline{\mathrm{PMap}_c(N)}$ and $\{h_i\}_{0\leq i<r}$ to build a handle-shift $\overline{h}$ with the same ends that h, then using $\overline{\mathrm{PMap}_c(N)}$ modify such $\overline{h}$ to get h. We do this in four cases.

Non-orientable handle-shifts:: Suppose $h \in H$ is non-orientable. The attracting and repelling ends h^+ and h^- are non-orientable. Then, in $nTEG(N)$ there exists a path γ that goes from h^- to h^+. Using the handle-shifts corresponding to edges of γ and $\overline{\mathrm{PMap}_c(N)}$, we can construct a non-orientable handle-shift $\overline{h}$ such that $\overline{h}^+ = h^+$ and $\overline{h}^- = h^-$. Finally, using $\overline{\mathrm{PMap}_c(N)}$ we can modify $\overline{h}$ to obtain h via conjugation (approximating the support of $\overline{h}$ to the support of h) and compositions. Therefore, $h \in \overline{\langle \mathrm{PMap}_c(N) \cup \{h_i\}_{0\leq i<r}\rangle}$.

Pseudo-orientable handle-shifts:: Suppose $h \in H$ is pseudo-orientable. The attracting and repelling ends h^+ and h^- are non-orientable. We can build h by using non-orientable handle-shifts with ends h^+ and h^-, $\overline{\mathrm{PMap}_c(N)}$ and relations of the type (4.1) above. As, by the previous case, all non-orientable handle-shifts are in $\overline{\langle \mathrm{PMap}_c(N) \cup \{h_i\}_{0\leq i<r}\rangle}$, we have that $h \in \overline{\langle \mathrm{PMap}_c(N) \cup \{h_i\}_{0\leq i<r}\rangle}$.

Semi-orientable handle-shifts:: Suppose $h \in H$ is semi-orientable. Without loss of generality we can suppose that h^+ is orientable and h^- is non-orientable. Due to $nTEG(N)$ being connected, in $TEG(N)$ there exists a path γ from h^- to h^+ that first consists of non-orientable handle-shifts in $\{h_i\}_{0\leq i<r}$, followed by one semi-orientable handle-shift $h_\gamma \in \{h_i\}_{0\leq i<r}$ and finally followed by orientable handle-shifts in $\{h_i\}_{0\leq i<r}$. Suppose that h_γ^- is non-orientable (if not take h_γ^{-1}). Using $\overline{\mathrm{PMap}_c(N)}$, the orientable handle-shifts of γ, h_γ and pseudo-orientable handle-shifts with ends h^- and h_γ^- we can construct a semi-orientable handle-shift $\overline{h}$ such that $\overline{h}^+ = h^+$ and $\overline{h}^- = h^-$. Finally, using $\overline{\mathrm{PMap}_c(N)}$ we can modify $\overline{h}$ to get h. Therefore, $h \in \overline{\langle \mathrm{PMap}_c(N) \cup \{h_i\}_{0\leq i<r}\rangle}$.

Orientable handle-shifts:: Suppose $h \in H$ is orientable. Let γ be a path in $TEG(N)$ from h^- to h^+. If all the vertices of γ are orientable then all the edges correspond to orientable handle-shifts. By using these handle-shifts and $\overline{\mathrm{PMap}_c(N)}$, we can construct an orientable handle-shift $\overline{h}$ such that $\overline{h}^+ = h^+$ and $\overline{h}^- = h^-$. By using $\overline{\mathrm{PMap}_c(N)}$, we can modify $\overline{h}$ to get h. If γ has non-orientable vertices, then it has two semi-orientable handle-shift $h_{\gamma 1}$ and $h_{\gamma 2}$. Using $h_{\gamma 1}$ and $h_{\gamma 2}$, pseudo-orientable handle-shifts, and $\overline{\mathrm{PMap}_c(N)}$, we can build a handle-shift $\overline{h}$ with the same "endpoints" as h. Then, as in the non-orientable handle-shift case, $\overline{h}$ can be modified, using $\overline{\mathrm{PMap}_c(N)}$, to get h. Therefore, $h \in \overline{\langle \mathrm{PMap}_c(N) \cup \{h_i\}_{0\leq i<r}\rangle}$.

5. Proofs of Theorem G and Corollary H

This section is dedicated to the proofs of Theorem G and Corollary H, which we recall here.

Theorem (G). *Let N be a connected (possibly non-orientable) surface of infinite topological type with at least two ends accumulated by genus. Then, we have that:*

$$\mathrm{PMap}(N) = \overline{\mathrm{PMap}_c(N)} \rtimes \prod_{0\le i<r} \langle h_i \rangle.$$

Corollary (H). *Let N be a connected (possibly non-orientable) surface of infinite topological type with genus at least 3. If N has at most one end accumulated by genus, then $H^1(\mathrm{PMap}(N);\mathbb{Z})$ is trivial. If N has at least two ends accumulated by genus, then*

$$H^1(\mathrm{PMap}(N);\mathbb{Z}) = H^1(\mathbb{Z}^r;\mathbb{Z}) = \bigoplus_{0\le i<r} \mathbb{Z}.$$

For Theorem G, we use the collection of handle-shifts $\{h_i\}_{0\le i<r}$ (constructed in the previous section) to define a group homomorphism

$$\overline{\varphi} : \overline{\langle \mathrm{PMap}_c(N) \cup \{h_i\}_{0\le i<r} \rangle} \longrightarrow \mathbb{Z}^r.$$

The proof of Theorem G is a corollary of the fact that $\overline{\varphi}$ induces a short exact sequence that splits.

Afterwards, we use the same argument used in [2] to prove Corollary H.

5.1. The homomorphism $\overline{\varphi}$.

In the following, we continue using the same set of curves $\{\gamma_i\}_{0\le i<r}$ and handle-shifts $\{h_i\}_{0\le i<r}$ that were defined in the previous section.

Let $F(\mathrm{PMap}_c(N) \cup \{h_i\}_{0\le i<r})$ be the free group generated by $\mathrm{PMap}_c(N) \cup \{h_i\}_{0\le i<r}$, and let $e_i \in \mathbb{Z}^r$ be the sequence with 1 in the i-th coordinate and 0 everywhere else. We define a homomorphism

$$\widetilde{\varphi} : F(\mathrm{PMap}_c(N) \cup \{h_i\}_{0\le i<r}) \to \mathbb{Z}^r,$$

defining $\widetilde{\varphi}(f) = \vec{0}$ for all $f \in \mathrm{PMap}_c(N)$, $\widetilde{\varphi}(h_i) = e_i$ for all $0 \le i < r$, and extending via the universal property of free groups.

The next lemma is needed to prove that $\widetilde{\varphi}$ induces a homomorphism

$$\varphi : \langle \mathrm{PMap}_c(N) \cup \{h_i\}_{0\le i<r} \rangle \longrightarrow \mathbb{Z}^r.$$

Lemma 5.1. *Let $\pi : F(\mathrm{PMap}_c(N) \cup \{h_i\}_{0\le i<r}) \to \langle \mathrm{PMap}_c(N), \{h_i\}_{0\le i<r} \rangle$ be the canonical projection and let w be a reduced word in $F(\mathrm{PMap}_c(N) \cup \{h_i\}_{0\le i<r})$. If $\pi(w) = \mathrm{id}$, then $\widetilde{\varphi}(w) = \vec{0}$.*

Proof. Due to $\mathrm{Ker}(\pi)$ being a normal subgroup and that $w \in \mathrm{Ker}(\pi)$, we can assume that $w = w_0 h_{i_1}^{\varepsilon_1} w_1 \cdots w_n h_{i_n}^{\varepsilon_n} w_{n+1}$ with $w_i \in \mathrm{PMap}_c(N)$ and $\varepsilon_i = \pm 1$.

Since w has finite length, there exists a compact surface Σ that contains the supports of all w_i's, such that $\pi(w)|_{N\setminus\Sigma} = (h_{i_1}^{\varepsilon_1} \circ \cdots \circ h_{i_n}^{\varepsilon_n})|_{N\setminus\Sigma}$. And given that

$\pi(w) = \mathrm{id}$, we have then that $(h_{i_1}^{\varepsilon_1} \circ \cdots \circ h_{i_n}^{\varepsilon_n})|_{N\setminus\Sigma} = \mathrm{id}|_{N\setminus\Sigma}$. The elements of $\{h_i\}_{0\leq i<r}$ have pairwise disjoint support. Hence, if for some i_m we have that $\varepsilon_{i_m} = \pm 1$ (that is, we are "shifting forward/backwards" in the support of h_{i_m}), then there exists some i_k for which $\varepsilon_{i_k} = \mp 1$ (that is, we are "shifting backwards/forward" in the support of h_{i_k}) and the support of h_{i_m} is equal to the suport of h_{i_k} (that is $h_{i_m} = h_{i_k}$).

Thus, we have the following:

$$\begin{aligned} \widetilde{\varphi}(w) &= \widetilde{\varphi}(w_0) + (\varepsilon_1)\widetilde{\varphi}(h_{i_1}) + \cdots + (\varepsilon_n)\widetilde{\varphi}(h_{i_n}) + \widetilde{\varphi}(w_n) \\ &= (\varepsilon_1)\widetilde{\varphi}(h_{i_1}) + \cdots + (\varepsilon_n)\widetilde{\varphi}(h_{i_n}) \\ &= \vec{0}. \end{aligned}$$

□

Due to the previous lemma, we have that $\widetilde{\varphi}$ descends to a homomorphism:

$$\varphi : \langle \mathrm{PMap}_c(N) \cup \{h_i\}_{0\leq i<r}\rangle \to \mathbb{Z}^r.$$

Now, for each $0 \leq i < r$, let $\psi_i := \pi_i \circ \varphi$, where π_i is the canonical projection to the i-th coordinate. This is obviously a homomorphism, but before we prove it is continuous we need an auxiliary lemma and a definition.

Lemma 5.2. *Let $0 \leq i < r$ be fixed, and A be a finite set of curves such that A does not separate the ends of N corresponding to the ends of* $\mathrm{supp}(h_i)$*. Then for any $g \in \mathrm{PMap}(N)$, there exists $f \in \mathrm{PMap}_c(N)$ such that $h_i|_{g(A)} = f|_{g(A)}$.*

Proof. Since A does not separate the ends of N corresponding to the ends of $\mathrm{supp}(h_i)$ and $g \in \mathrm{PMap}(N)$, we have that $g(A)$ does not separate them too. If $g(A)$ is disjoint from $\mathrm{supp}(h_i)$, then $f = \mathrm{id}$. So, suppose that $g(A)$ does intersect $\mathrm{supp}(h_i)$.

Let K be a compact subset in $\overline{\mathrm{supp}(h_i)}$ that contains the intersection of $g(A)$ with $\mathrm{supp}(h_i)$ and has exactly one boundary component, and let Σ_1 and Σ_2 be two subsurfaces of $\mathrm{supp}(h_i)$ homeomorphic to either a Möbius strip or a torus with a boundary component (depending if h_i is non-orientable or not), such that Σ_1 and Σ_2 are both disjoint from K. Finally, let a be an arc with endpoints in the boundary components of Σ_1 and Σ_2, such that a is disjoint from $g(A)$. See Figure 17 for an example.

We define f as follows: slide Σ_2 through a while shifting Σ_1 once in the direction that had originally Σ_2. See Figure 17. Thus, $f \in \mathrm{PMap}_c(N)$ and $h_i|_{g(A)} = f|_{g(A)}$ as desired. □

Recall that given a separating curve α in a finite-type surface Σ, the *genus of α* is the minimum of the genus of the connected components of $\Sigma \setminus \alpha$.

Lemma 5.3. *For each $0 \leq i < r$, the homomorphism $\psi_i : \langle \mathrm{PMap}_c(N) \cup \{h_j\}_{0\leq j<r}\rangle \to \mathbb{Z}$ is continuous.*

Proof. To prove that ψ_i is continuous, it suffices to prove that $\mathrm{Ker}(\psi_i)$ is an open set in $\langle \mathrm{PMap}_c(N) \cup \{h_j\}_{0\leq j<r}\rangle$ with the subspace topology. To prove that $\mathrm{Ker}(\psi_i)$ is open is suffices to prove that there exists some open neighbourhood

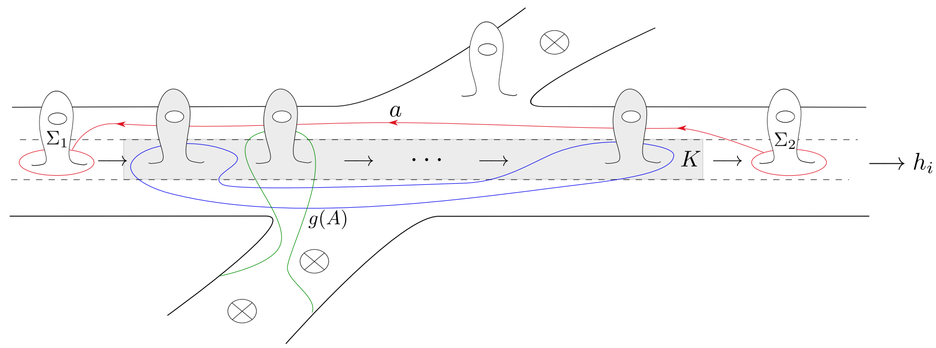

FIGURE 17. How to substitute h_i with an element of $\mathrm{PMap}_c(N)$.

V of the identity with $V \subset \mathrm{Ker}(\psi_i)$, since then for every $f \in \mathrm{Ker}(\psi_i)$ we have that fV is an open neighbourhood of f contained in $f\,\mathrm{Ker}(\psi_i) = \mathrm{Ker}(\psi_i)$.

Let $V = [\![\gamma_i, V(\gamma_i)]\!] \cap \langle \mathrm{PMap}_c(N) \cup \{h_j\}_{0\le j<r}\rangle$, where γ_i is the curve used to define h_i in Subsubsection 4.5.3 and $V(\gamma_i)$ is a regular neighborhood of γ_i. We need to prove that V is an open neighborhood of the identity contained in $\mathrm{Ker}(\psi_i)$.

Note that V is open in the subset topology, since $[\![\gamma_i, V(\gamma_i)]\!]$ is a sub-basic open set of $\mathrm{PMap}(N)$.

Now, let $f \in V$. If $f \in \mathrm{PMap}_c(N)$ or $f = h_j$ with $j \neq i$, it is obvious that $\psi_i(f) = 0$.

Hence, let f be of the form

$$f = w_0 \circ h_{i_1}^{\varepsilon_1} \circ w_1 \circ \cdots \circ w_{n-1} \circ h_{i_n}^{\varepsilon_n} \circ w_n$$

with $w_1, \cdots, w_n \in \mathrm{PMap}_c(N)$ and $\varepsilon_j = \pm 1$ for all j. To simplify proving that $f \in \mathrm{Ker}(\psi_i)$, we "get rid" of the handle-shifts that are not apporting anything to $\psi_i(f)$: Given that γ_i does not separate the ends of the support of h_j for $j \neq i$, by a repeated use of Lemma 5.2 we can define $\widetilde{f}$ by substituting each $h_{i_m} \neq h_i$ with an element of $\mathrm{PMap}_c(N)$, obtaining the following properties:

(1) $\widetilde{f}$ has the form $\widetilde{w}_0 \circ h_i^{\varepsilon_1} \circ \cdots \circ h_i^{\varepsilon_k} \circ \widetilde{w}_k$ with $\epsilon_j = \pm 1$ for all j.
(2) $\psi_i(\widetilde{f}) = \psi_i(f)$.
(3) $f(\gamma_i) = \widetilde{f}(\gamma_i)$.

Note that if $k = 0$, we have that $\widetilde{f} \in \mathrm{PMap}_c(N)$, and by (2) above we have that $\psi_i(f) = 0$. So, we suppose that $k > 0$.

Since for any $w \in \mathrm{PMap}_c(N)$ we have that $h_i^{\epsilon} \circ w \circ h_i^{-\epsilon} = w'$ for some $w' \in \mathrm{PMap}_c(N)$, we can assume that the form in (1) above satisfies that for all $1 \le j \le k$, $\epsilon = \epsilon_j = \epsilon_{j+1}$. Thus, $|\psi_i(\widetilde{f})| = k$.

We claim that $k \neq 0$ implies that $\widetilde{f} \notin V$ (and by (3) above $f \notin V$). Note that this claim immediately implies the lemma.

We prove this claim using induction on $|k|$:

If $k = 1$ and $\widetilde{f} \in V$, we have that $h_i^{\epsilon} \circ \widetilde{w}_1(\gamma_i) = \widetilde{w}_0^{-1}(\gamma_i)$. Let K be a compact subsurface that contains γ_i and $\widetilde{w}_0^{-1}(\gamma_i)$ as essential curves, and contains $\text{supp}(\widetilde{w}_0)$, $\text{supp}(\widetilde{w}_1)$ and $h_i^{\epsilon}(\text{supp}(\widetilde{w}_1))$. Then γ_i, $\widetilde{w}_0^{-1}(\gamma_i)$ and $\widetilde{w}_1(\gamma_i)$ have the same genus in K. However, by the definitions of the γ_i and h_i, $\widetilde{w}_1(\gamma_i)$ separates the ends of the support of h_i. Thus, the difference between the genera of $h_i^{\epsilon}(\widetilde{w}_1(\gamma_i))$ and $\widetilde{w}_1(\gamma_i)$ is 1. In particular, $h_i^{\epsilon}(\widetilde{w}_1(\gamma_i))$ and $\widetilde{w}_0^{-1}(\gamma_i)$ cannot have the same genus in K, reaching a contradiction. Therefore, if $k = 1$, $\widetilde{f} \notin V$.

Using an analogous observation as in the base case, we have as induction hypothesis that $k = j \geq 1$ implies that $\widetilde{f} \notin V$. Suppose that $k = j + 1$ and $\widetilde{f} \in V$. Let K be a compact subsurface that contains

$$\{\widetilde{w}_0^{-1}(\gamma_i), \gamma_i, \widetilde{w}_{j+1}(\gamma_i), h_i^{\epsilon} \circ \widetilde{w}_{j+1}(\gamma_i), \dots, h_i^{\epsilon} \circ \widetilde{w}_1 \circ \cdots \circ \widetilde{w}_{j+1}(\gamma_i)\}$$

as essential curves, and contains $\text{supp}(\widetilde{w}_0)$, …, $\text{supp}(\widetilde{w}_{j+1})$ and all possible translations of them by the elements $\{h_i^{\epsilon}, \widetilde{w}_j \circ h_i^{\epsilon}, \dots, h_i^{\epsilon} \circ \widetilde{w}_1 \circ \cdots \circ \widetilde{w}_j\}$. By the induction hypothesis, the genera of $\widetilde{w}_0^{-1}(\gamma_i)$ and $\widetilde{w}_1 \circ \cdots \circ \widetilde{w}_{j+1}(\gamma_i)$ are different; moreover, the difference between their genus is j. Since $\widetilde{w}_1 \circ \cdots \circ \widetilde{w}_{j+1}(\gamma_i)$ separates the ends of the support of h_i, we obtain that the difference between the genus of $h_i^{\epsilon} \circ \widetilde{w}_1 \circ \cdots \circ \widetilde{w}_{j+1}(\gamma_i)$ and $\widetilde{w}_1 \circ \cdots \circ \widetilde{w}_{j+1}(\gamma_i)$ is 1, increasing the difference between the genus of $h_i^{\epsilon} \circ \widetilde{w}_1 \circ \cdots \circ \widetilde{w}_{j+1}(\gamma_i)$ and $\widetilde{w}_0^{-1}(\gamma_i)$ to $j + 1$ (this is because all the h_i have the same power). Thus, we reach a contradiction.

This finishes the proof of the claim. □

The next step is to extend the homomorphism ψ_i, for all $0 \leq i < r$, to the closure of $\langle \text{PMap}_c(N) \cup \{h_i\}_{0 \leq i < r} \rangle$. This is done in the following lemma.

Lemma 5.4. *For each $0 \leq i < r$, ψ_i extends to a continuous group homomorphism:*

$$\overline{\psi}_i : \overline{\langle \text{PMap}_c(N) \cup \{h_i\}_{0 \leq i < r} \rangle} \to \mathbb{Z}.$$

Proof. Let $f \in \overline{\langle \text{PMap}_c(N) \cup \{h_i\}_{0 \leq i < r} \rangle}$ and $(f_j)_{j=0}^{\infty}$ be a sequence of elements of the group $\langle \text{PMap}_c(N) \cup \{h_i\}_{0 \leq i < r} \rangle$ such that

$$\lim_{j \to \infty} f_j = f.$$

Let $N(f(\gamma_i))$ be a regular neighborhood of $f(\gamma_i)$ and consider the open set $[\![\gamma_i, N(f(\gamma_i))]\!]$ in $\overline{\langle \text{PMap}_c(N) \cup \{h_i\}_{0 \leq i < r} \rangle}$. Fix $M \geq 0$ such that for all $j \geq M$ we have that $f_j \in [\![\gamma_i, N(f(\gamma_i))]\!]$; note this implies that for all $j \geq M$, $f_j(\gamma_i) = f(\gamma_i)$. Then, for all $j \geq M$

$$\begin{aligned} [\![\gamma_i, N(f(\gamma_i))]\!] &= [\![\gamma_i, N(f_j(\gamma_i))]\!] \\ &= f_j \cdot [\![\gamma_i, N(\gamma_i)]\!]. \end{aligned}$$

In particular, $[\![\gamma_i, N(f(\gamma_i))]\!] = f_M \cdot [\![\gamma_i, N(\gamma_i)]\!]$. As seen in Lemma 5.3, $[\![\gamma_i, N(\gamma_i)]\!] \cap \langle \text{PMap}_c(N) \cup \{h_i\}_{0 \leq i < r} \rangle \subset \text{Ker}\, \psi_i$, implying that:

$$f_M \cdot [\![\gamma_i, N(\gamma_i)]\!] \cap \langle \text{PMap}_c(N) \cup \{h_i\}_{0 \leq i < r} \rangle \subset f_M \cdot \text{Ker}\, \psi_i.$$

Then, for all $j \geq M$ the class f_j is in $f_M \cdot \operatorname{Ker}\psi_i$. Therefore, $\psi_i(f_j) = \psi_i(f_M)$ for all $j \geq M$, and consequently $(\psi_i(f_j))_{j=0}^{\infty}$ converges to $\psi_i(f_M)$.

We define $\overline{\psi_i}(f)$ as $\psi_i(f_M)$. We claim that the definition of $\overline{\psi_i}$ is independent of the the sequence and thus, restricted to $\langle \mathrm{PMap}_c(N) \cup \{h_i\}_{0\leq i<r}\rangle$ is ψ_i: If $(g_j)_{j=0}^{\infty}$ is another sequence of $\langle \mathrm{PMap}_c(N)\cup\{h_i\}_{0\leq i<r}\rangle$ that converges to f, then it is eventually contained in $[\![\gamma_i, N(f(\gamma_i))]\!]$ by the same argument as above. This implies that the sequence $\{g_j^{-1}f_j\} \subset \langle \mathrm{PMap}_c(N) \cup \{h_i\}_{0\leq i<r}\rangle$ is eventually contained in $[\![\gamma_i, N(\gamma_i)]\!] \cap \langle \mathrm{PMap}_c(N) \cup \{h_i\}_{0\leq i<r}\rangle \subset \operatorname{Ker}\psi_i$. Thus, the sequences $(\psi_i(g_j))_{j=0}^{\infty}$ and $\{\psi_i(f_j)\}_{j=0}^{\infty}$ are eventually equal, which implies their limits are the same.

So, we have defined a sequentially continuous function $\overline{\psi_i}$. Given that both $\overline{\langle \mathrm{PMap}_c(N) \cup H\rangle}$ and $\langle h_i\rangle$ are metrizable, we have that $\overline{\psi_i}$ is continuous. The fact that it is a group homomorphism is a well-known fact of topological groups. □

Finally, we define:

$$\begin{array}{rccc} \overline{\varphi} \ : & \mathrm{PMap}(N) = \overline{\langle \mathrm{PMap}_c(N) \cup \{h_i\}_{0\leq i<r}\rangle} & \longrightarrow & \mathbb{Z}^r \\ & f & \longmapsto & \left(\overline{\psi_i}(f)\right)_{0\leq i<r}. \end{array}$$

This is a continuous group homomorphism since $\mathbb{Z}^r$ has the product topology.

5.2. The semi-direct product. With $\overline{\varphi}$ defined, we can have a short exact sequence (which naturally splits!), but we need the following lemma to determine exactly who $\operatorname{Ker}(\overline{\varphi})$ is.

Lemma 5.5. *The kernel of $\overline{\varphi}$ is exactly $\overline{\mathrm{PMap}_c(N)}$.*

Proof. Some key observations for the proof of this lemma are the following:

(1) $\mathrm{PMap}_c(N)$ is a normal subgroup of $\mathrm{Map}(N)$.
(2) Since the handle shifts $\{h_i\}_{0\leq i<r}$ have disjoint support, they commute.
(3) Due to (1) and (2) above, for any i, $w_j \in \mathrm{PMap}_c(N)$ and $\epsilon = \pm 1$, we have that

$$h_i^{\epsilon}\circ w_0\circ h_{i_0}^{n_0}\circ \cdots \circ w_{k-1}\circ h_{i_k}^{n_k}\circ w_k\circ h_i^{-\epsilon} = \widetilde{w}_0\circ h_{i_0}^{n_0}\circ \cdots \circ\widetilde{w}_{k-1}\circ h_{i_k}^{n_k}\circ\widetilde{w}_k$$

for some $\widetilde{w}_j \in \mathrm{PMap}_c(N)$.

Thus, if $f \in \operatorname{Ker}(\psi_i)$ we can write f as a word in $(\mathrm{PMap}_c(N)\cup\{h_j\}_{0\leq j<r})\setminus\{h_i\}$.

Now, by construction of $\overline{\varphi}$ we have that $\overline{\mathrm{PMap}_c(N)} \subset \operatorname{Ker}(\overline{\varphi})$, so we only need to prove that $\operatorname{Ker}(\overline{\varphi}) \subset \overline{\mathrm{PMap}_c(N)}$.

Let $f \in \operatorname{Ker}(\overline{\varphi})$, and $(f_j)_{j=0}^{\infty} \subset \langle \mathrm{PMap}_c(N) \cup \{h_i\}_{0\leq i<r}\rangle$ be a sequence converging to f; we prove that $f \in \overline{\mathrm{PMap}_c(N)}$ by finding a sequence $(\widetilde{f}_j)_{j=0}^{\infty} \subset$

$\mathrm{PMap}_c(N)$ that converges to f, where intuitively the $\widetilde{f}_j$ are obtained from the f_j by conveniently substituting handle-shifts by elements of $\mathrm{PMap}_c(N)$.

For each $0 \leq i < r$ we define the following open neighborhoods of f:

- $U_0 := \mathrm{Ker}(\overline{\psi}_0)$.
- For $i > 0$, $U_i := U_{i-1} \cap \mathrm{Ker}(\overline{\psi}_i)$.

Since $f_j \to f$ as $j \to \infty$, for each i there exists $L_i \geq 0$ such that for all $j \geq L_i$ we have that $f_j \in U_i$. Thus, we have that for $L_i \leq j < L_{i+1}$ we can write f_j as a word in $\mathrm{PMap}_c(N) \cup \{h_m\}_{i+1 \leq m < r}$.

Let $\Sigma_0 \subset \Sigma_1 \subset \cdots \subset N$ be a principal exhaustion of N (see Section 1 to recall the definition) such that for all $j \geq 0$ and all $i > j$, $\Sigma_j \cap \mathrm{supp}(h_i) = \varnothing$. For each $j \geq 0$, let A_j be a finite set of curves such that $A_j \cap \Sigma_j$ is a collection of arcs and curves in Σ_j that satisfies the hypotheses for the Alexander method for finite-type surfaces (see Theorem 2.1); that is, for any $g, h \in \mathrm{Map}(N)$, we have that if $g|_{A_j} = h|_{A_j}$, then $g|_{\Sigma_j} = h|_{\Sigma_j}$. For each $j \geq 0$, define $V_j = \cap_{\alpha \in A_j} [\![\alpha, N(f(\alpha))]\!]$, where $N(\alpha)$ is a regular neighborhood of α. Since every A_j is a finite set, V_j is an open set.

For each $i \geq 0$, we define the open set $W_i = U_i \cap V_i$. For each i there exists $M_i \geq L_i \geq 0$ such that for all $j \geq M_i$, we have that $f_j \in W_i$. This implies that $f_j|_{\Sigma_i} = f|_{\Sigma_i}$, and $f_j = w_{j,0} \circ \cdots \circ w_{j,k_j}$, where $w_{j,n} \in \mathrm{PMap}_c(N) \cup \{h_m\}_{i+1 \leq m < r}$ (this is becase, as mentioned above, f_j can be written as a word in $\mathrm{PMap}_c(N) \cup \{h_m\}_{i+1 \leq m < r}$).

Now, for each $i \geq 0$ we take $f_{M_i} = w_{M_i,0} \circ \cdots \circ w_{M_i,k_{M_i}}$. We can assume that $w_{M_i,k_{M_i}} = \mathrm{id}$ without any loss in generality, and we perform the following algorithm to obtain $\widetilde{f}_i$:

- We start a cycle with $j = k_{M_i}$, at the end of the instructions decrease j by one and the cycle ends when $j = -1$.
  - If $w_{M_i,j} \in \mathrm{PMap}_c(N)$: We define $\widetilde{w}_{i,j} := w_{M_i,j}$.
  - Else: We have that $w_{M_i,j} \in \{h_m, h_m^{-1}\}$ for some $m > i$. Recalling that A_i does not separate the ends of $\mathrm{supp}(h_m)$, by Lemma 5.2 there exists $h \in \mathrm{PMap}_c(N)$ such that
  $$h|_{\widetilde{w}_{i,j+1} \circ \cdots \circ \widetilde{w}_{i,k_{M_i}}(A_i)} = w_{M_i,j}|_{w_{M_i,j+1} \circ \cdots \circ w_{M_i,k_{M_i}}(A_i)}.$$
  Then we define $\widetilde{w}_{i,j} := h$. This implies that
  $$\widetilde{w}_{i,j} \circ \cdots \circ \widetilde{w}_{i,k_{M_i}}|_{A_i} = w_{M_i,j} \circ \cdots \circ w_{M_i,k_{M_i}}|_{A_i}.$$
- Define $\widetilde{f}_i := \widetilde{w}_{i,0} \circ \cdots \circ \widetilde{w}_{i,k_{M_i}}$.

Note that $\widetilde{f}_i \in \mathrm{PMap}_c(N)$ and $\widetilde{f}_i|_{A_i} = f_{M_i}|_{A_i} = f|_{A_i}$, which implies that $\widetilde{f}_i|_{\Sigma_i} = f|_{\Sigma_i}$. Hence, $\widetilde{f}_i \to f$, and $f \in \overline{\mathrm{PMap}_c(N)}$. □

So, we have the following short exact sequence of groups

$$1 \longrightarrow \overline{\mathrm{PMap}_c(N)} \longrightarrow \mathrm{PMap}(N) \longrightarrow \mathbb{Z}^r \cong \overline{\langle h_i : 0 \leq i < r \rangle} \longrightarrow 1,$$

which naturally splits as the product topology of $\mathbb{Z}^r$ coincides with the subgroup topology of $\prod_{i=1}^{r}\langle h_i\rangle$. Therefore,

$$\mathrm{PMap}(N) = \overline{\mathrm{PMap}_c(N)} \rtimes \prod_{i=1}^{r}\langle h_i\rangle,$$

finishing the proof of Theorem G.

5.3. The first integral cohomology group. As mentioned at the beginning of the section, the argument to prove Corollary H is analogous to the one presented in [2] for the orientable case. For the sake of completeness we present it anyway:

Let N be an infinite-type surface, $\Sigma_0 \subset \Sigma_1 \subset \cdots \subset N$ be a principal exhaustion, and $\phi : \mathrm{PMap}(N) \to \mathbb{Z}$ be a homomorphism. We know that $\mathrm{PMap}_c(N) = \langle \bigcup_{0\le j<\omega} \mathrm{PMap}(\Sigma_j) \rangle$. This implies that $\mathrm{PMap}_c(N)_{ab} = \langle \bigcup_{0\le j<\omega} \mathrm{PMap}(\Sigma_j)_{ab} \rangle$. By the results of Stukow in [19], for all $j \ge 0$ we have that $\mathrm{PMap}(\Sigma_j)_{ab}$ is a torsion group. Thus, $\mathrm{PMap}_c(N)_{ab}$ is generated by torsion elements, which implies that $\phi|_{\mathrm{PMap}_c(N)} \equiv 0$.

Now, since ϕ is a homomorphism from a Polish group to $\mathbb{Z}$, then by Theorem 1 in [7] we have that ϕ is continuous. Since the restriction of ϕ to $\mathrm{PMap}_c(N)$ is constantly 0, we have that $\phi|_{\overline{\mathrm{PMap}_c(N)}} \equiv 0$.

Then, we have two possible cases depending on the number of ends of N accumulated by genus:

- If N has at most one end accumulated by genus, by Theorem E we have that $\overline{\mathrm{PMap}_c(N)} = \mathrm{PMap}(N)$. So $H^1(\mathrm{PMap}(N);\mathbb{Z}) = \mathrm{Hom}(\mathrm{PMap}(N),\mathbb{Z})$ is trivial.
- If N has at least two ends accumulated by genus, by Theorem G we have that $\mathrm{PMap}(N) = \overline{\mathrm{PMap}_c(N)} \rtimes \prod_{0\le i<r}\langle h_i\rangle$. Thus, we have an isomorphism from $H^1(\mathrm{PMap}(N);\mathbb{Z}) = \mathrm{Hom}(\mathrm{PMap}(N),\mathbb{Z})$ to $\mathrm{Hom}(\mathbb{Z}^r,\mathbb{Z})$ which, by the results from Specker in [18] and the arguments from Blass and Göbel in [6], is known to be isomorphic to the free abelian group of rank r $\bigoplus_{0\le i<r}\mathbb{Z}$.

References

[1] Abbott, Carolyn R.; Miller, Nicholas; Patel, Priyam. Infinite-type loxodromic isometries of the relative arc graph. *Algebr. Geom. Topol.* To appear. arXiv:2109.06106. 1725

[2] Aramayona, Javier; Patel, Priyam; Vlamis, Nicholas G. The first integral cohomology of pure mapping class groups. *Int. Math. Res. Not. IMRN* (2020), no. 22, 8973–8996. MR4216709, Zbl 1462.57020, arXiv:1711.03132, doi:10.1093/imrn/rnaa229. 1706, 1707, 1708, 1721, 1722, 1724, 1730, 1731, 1733, 1737, 1738, 1741, 1747

[3] Aramayona, Javier; Vlamis, Nicholas G. Big mapping class groups: An overview. *In the tradition of Thurston—geometry and topology* 459–496. *Springer, Cham*, 2020. MR4264585, Zbl 1479.57037, arXiv:2003.07950, doi: 10.1007/978-3-030-55928-1_12. 1707, 1711, 1721, 1722

[4] ATALAN, FERIHE; KORKMAZ, MUSTAFA. Automorphisms of curve complexes on nonorientable surfaces. *Groups Geom. Dyn.* **8** (2014), no. 1, 39–68. MR3209702, Zbl 1306.57013, doi: 10.4171/GGD/216. 1706, 1720

[5] BAVARD, JULIETTE; DOWDALL, SPENCER; RAFI, KASRA. Isomorphisms between big mapping class groups. *Int. Math. Res. Not. IMRN* (2020), no. 10, 3084–3099. MR4098634, Zbl 1458.57014. arXiv:1708.08383, doi: 10.1093/imrn/rny093. 1706, 1707

[6] BLASS, ANDREAS; GOEBEL, RÜDIGER. Subgroups of the Baer–Specker group with few endomorphisms but large dual. *Fund. Math.* **149** (1996), no. 1, 19–29. MR1372355, Zbl 0851.20052, arXiv:math/9405206. 1709, 1747

[7] DUDLEY, RICHARD M. Continuity of homomorphisms. *Duke Math. J.* **28** (1961), 587–594. MR0136676, Zbl 0103.01702, doi: 10.1215/S0012-7094-61-02859-9. 1709, 1747

[8] FARB, BENSON; MARGALIT, DAN. A primer on mapping class groups. Princeton Mathematical Series, 49. *Princeton University Press, Princeton, NJ*, 2012. xiv+472 pp. ISBN:978-0-691-14794-9. MR2850125, Zbl 1245.57002, doi: 10.23943/princeton/9780691147949.003.0001. 1706, 1709, 1713

[9] HERNÁNDEZ HERNÁNDEZ, JESÚS; MORALES, ISRAEL; VALDEZ, FERRÁN. Isomorphisms between curve graphs of infinite-type surfaces are geometric. *Rocky Mountain J. Math.* **48** (2018), no. 6, 1887–1904. MR3879307, Zbl 1481.57037, arXiv:1706.03697 doi: 10.1216/RMJ-2018-48-6-1887. 1706, 1707, 1717, 1718, 1720

[10] HERNÁNDEZ HERNÁNDEZ, JESÚS; MORALES, ISRAEL; VALDEZ, FERRÁN. The Alexander method for infinite-type surfaces. *Michigan Math. J.* **68** (2019), no. 4, 743–753. MR4029627, Zbl 1481.57038, arXiv:1703.00407, doi: 10.1307/mmj/1561773633. 1706, 1707, 1713, 1715

[11] IVANOV, NIKOLAI V. Automorphism of complexes of curves and of Teichmüller spaces. *Internat. Math. Res. Notices* (1997), no. 14, 651–666. MR1460387, Zbl 0890.57018, doi: 10.1155/S1073792897000433. 1706, 1720

[12] KORKMAZ, MUSTAFA. Automorphisms of complexes of curves on punctured spheres and on punctured tori. *Topology Appl.* **95** (1999), no. 2, 85–111. MR1696431, Zbl 0926.57012, doi: 10.1016/S0166-8641(97)00278-2. 1706, 1720

[13] LUO, FENG. Automorphisms of the complex of curves. *Topology* **39** (2000), no. 2, 283–298. MR1722024, Zbl 0951.32012, arXiv:math/9904020, doi: 10.1016/S0040-9383(99)00008-7. 1706, 1720

[14] PARIS, LUIS. Mapping class groups of non-orientable surfaces for beginners. *Winter Braids Lect. Notes* **1** (2014), *Winter Braids IV* (Dijon, 2014). Exp. No. 3, 17 pp. MR3703250, Zbl 1422.57049, arXiv:1410.1123, doi: 10.5802/wbln.4. 1706

[15] PATEL, PRIYAM; VLAMIS, NICHOLAS G. Algebraic and topological properties of big mapping class groups. *Algebr. Geom. Topol.* **18** (2018), no. 7, 4109–4142. MR3892241, Zbl 1490.57016, arXiv:1703.02665, doi: 10.2140/agt.2018.18.4109. 1708, 1721, 1723, 1728, 1729

[16] RICHARDS, IAN. On the classification of noncompact surfaces. *Trans. Amer. Math. Soc.* **106** (1963), 259–269. MR0143186, Zbl 0156.22203, doi: 10.1090/S0002-9947-1963-0143186-0. 1711, 1712, 1731, 1732, 1737

[17] SHAPIRO, ROBERTA. An Alexander method for infinite-type surfaces. *New York J. Math.* **28** (2022), 1137–1151. MR4474179, Zbl 1498.57015, arXiv:2107.06909. 1707

[18] SPECKER, ERNST. Additive Gruppen von Folgen ganzer Zahlen. *Portugal. Math.* **9** (1950), 131–140. MR0039719, Zbl 0041.36314. 1709, 1747

[19] STUKOW, MICHAŁ. Generating mapping class groups of nonorientable surfaces with boundary. *Adv. Geom.* **10** (2010), no. 2, 249–273. MR2629814, Zbl 1207.57030, arXiv:0707.3497, doi: 10.1515/advgeom.2010.010. 1706, 1709, 1747

[20] VON KERÉKJÁRTÓ, BÉLA. Vorlesungen über Topologie: I, Flächentopologie. Die Grundlehren der Mathematischen Wissenschaften, Bd. 8. *Springer-Verlag, Berlin, Heidelberg*, 1923. Zbl 49.0396.07, doi: 10.1007/978-3-642-50825-7. 1711

(Jesús Hernández Hernández) Centro de Ciencias Matemáticas. Universidad Nacional Autónoma de México. Morelia, Mich. 58190, México
jhdez@matmor.unam.mx

(Cristhian E. Hidber) Centro de Ciencias Matemáticas. Universidad Nacional Autónoma de México. Morelia, Mich. 58190, México
hidber@matmor.unam.mx

This paper is available via http://nyjm.albany.edu/j/2024/30-73.html.