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Joint projective spectrum of D_∞^n

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ABSTRACT. In this paper, we compute the joint spectrum of D_∞^n with respect to the left regular representation and identify n generators of the first de Rham cohomology group of joint resolvent set, which is induced by several central linear functionals. Through action of D_∞^n on $2n$-ary trees, we obtain a self-similar realization of the group C^*-algebra of D_∞^n.

CONTENTS

1. Introduction

In classical Banach algebra theory, the Gelfand theory gives a comprehensive description of spectra of operators. However, in the case of several Banach algebra elements, the (joint) spectral theory is more complicated. It is noteworthy that not only the commutativity of the tuple will influence the study, there is also a distinction between algebraic and spatial joint spectra.

If the tuple $A = (A_1, A_2, \cdots, A_n)$ is a commutative tuple, i.e. $A_iA_j = A_jA_i$, $1 \le i, j \le n$, J. L. Taylor defines the Taylor spectrum of the tuple by Koszul complex [19, 23]. We refer the reader to [8, 10] for its applications in operator theory and sheaf theory.

The matter becomes difficult when the tuple is non-commuting. In [25], R. Yang considered the invertibility of the linear pencil

$$A(z) = z_1A_1 + z_2A_2 + \cdots + z_nA_n.$$

In fact, there has been increasing interest of the invertibility of $A(z)$ in fields of algebraic geometry, group theory, mathematical physics, PDEs and operator

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theory. We refer the reader to [1, 2, 22, 24] for more information. This gives rise to the following notion of projective joint spectrum.

Definition 1.1. *For a tuple $A = (A_1, A_2, \cdots, A_n)$ of elements in a unital Banach algebra $\mathcal{B}$, its projective joint spectrum $P(A)$ consists of $z \in \mathbb{C}^n$ such that $A(z) = z_1A_1 + z_2A_2 + \cdots + z_nA_n$ is not invertible in $\mathcal{B}$.*

In contrast to other notions of joint spectrum, for example, Taylor spectrum, the projective joint spectrum is novel in the sense that it is "base free". Instead of considering the invertibility of

$$(A_1 - z_1I, A_2 - z_2I, \cdots, A_n - z_nI),$$

it centers on the homogeneous multiparameter pencil $A(z)$. This simplifies the study in many cases. Moreover, by the homogeneity of $A(z)$, we can consider the projective joint spectrum $p(A)$ in the complex projective space $\mathbb{P}^{n-1} = \mathbb{C}^n/\sim$ defined by $p(A) = P(A)/\sim$. In [25], it is proved by Hartogs extension theorem that $p(A)$ is a non-trivial compact subset in $\mathbb{P}^{n-1}$.

If the Banach algebra $\mathcal{B}$ is finite dimensional, such as a matrix algebra, then the projective joint spectrum is the hypersurface $\{\det A(z) = 0\}$. If the tuple is commutative, the projective joint spectrum is a union of hyperplanes, which is closely related to the case of Taylor spectrum.

The projective resolvent set $P^c(A) = \mathbb{C}^n \setminus P(A)$ and the spectrum itself have many properties similar to those in the single-operator case. For example, it is proved that every path-connected component of $P^c(A)$ is a domain of holomorphy [18]. We also refer the reader to [6, 11, 18, 20, 25] on its connections with Hermitian metrics, hyperinvariant subspace problem and cyclic cohomology.

Now consider a finitely generated group G[1] with the generating set $S = \{g_1, g_2, \cdots, g_n\}$. Let ρ be a unitary representation of G on a Hilbert space H, which will be denoted by (ρ, H). Let $C^*_\rho(G)$ denote the C^*-algebra generated by $A_i = \rho(g_i)$, for $i = 1, 2, \ldots, n$. The projective joint spectrum of G with respect to ρ, denoted by $P(A_\rho)$, is the projective joint spectrum of the tuple $A_\rho = (I, A_1, A_2, \cdots, A_n)$.

Given two representations (ρ_1, H_1) and (ρ_2, H_2) of G, they are said to be (unitarily) equivalent if there exists a unitary map $U : H_1 \to H_2$ such that

$$\rho_2(g) = U\rho_1(g)U^{-1}, \qquad \forall g \in G.$$

Apparently, the projective joint spectrum is invariant for equivalent representations. Moreover, it is invariant under the weak equivalence of representations. Let (π, H) and (ρ, K) be unitary representations of group G. We say that π is weakly contained in ρ if for every $\xi \in H$, every compact subset Q of G and every $\varepsilon > 0$, there exists $\eta_1, \cdots, \eta_n \in K$ such that,

$$|\langle \pi(x)\xi, \xi\rangle - \sum_{i=1}^{n}\langle \rho(x)\eta_i, \eta_i\rangle| < \varepsilon, \quad \forall x \in Q.$$

[1]In this paper, all groups are discrete locally compact groups unless otherwise stated.

We shall write $\pi \prec \rho$ for this relation. If $\pi \prec \rho$ and $\rho \prec \pi$, we say that π and ρ are weakly equivalent and denote this relation by $\pi \sim \rho$. Generally speaking, it is difficult to determine whether two representations are weakly equivalent from definition. However, in [9], it is proved that $\pi \prec \rho$ if and only if the canonical homomorphism $\rho(g) \mapsto \pi(g), g \in G$, extends to a unital $*$-homomorphism from $C^*_\rho(G)$ onto $C^*_\pi(G)$. It implies that if $\rho(g)$ is invertible in $C^*_\rho(G)$, then $\pi(g)$ is invertible in $C^*_\pi(G)$. Moreover, $\pi \prec \rho$ implies $P(A_\pi) \subset P(A_\rho)$. So $P(A_\pi) = P(A_\rho)$ if these two representations are weakly equivalent.

Conversely, it is a natural question whether the projective joint spectrum determines the representation up to weak equivalence. It is often not the case. We will give some examples later in this paper. The infinite dihedral group D_∞ is defined as the group generated by rotations and reflections of the plane that preserves the origin. Grigorchuk and Yang give a detailed description on the joint projective spectrum of D_∞ [16].

Throughout this article, D^n_∞ will denote the group $\mathbb{Z}_n \times D_\infty$, which is isomorphic to $\mathbb{Z}_n \times (\mathbb{Z}_2 * \mathbb{Z}_2)$ and has the presentation

$$D^n_\infty = \left\langle a, t, \tau | a^2 = t^2 = \tau^n = 1, a\tau = \tau a, t\tau = \tau t \right\rangle. \quad (1)$$

In particular, D^2_∞ can be realized as the group of rigid motions in 3-space consisting of rotations and reflections of the plane that preserves the origin, together with a reflection τ through the origin. For example, $a(x, y, z) = (x, -y, z)$, $\tau(x, y, z) = -(x, y, z)$, and t is chosen to be an involution that makes at an irrational rotation in the plane Oxy. We refer the readers to [14] for more information about this group and its application on the electronic wave functions of molecules.

First, we will compute the projective joint spectrum of D^n_∞ with respect to its left regular representation.

Theorem 1.2. *If we define $P(R_\lambda)$ as the projective joint spectrum of*

$$R_\lambda(z) = z_0\lambda(e) + z_1\lambda(a) + z_2\lambda(t) + z_3\lambda(\tau),$$

then

$$P(R_\lambda) = \bigcup_{k=1}^{n} \bigcup_{-1 \le x \le 1} \{z \in \mathbb{C}^4 : (z_0 + \omega_k z_3)^2 - z_1^2 - z_2^2 - 2z_1z_2x = 0\},$$

where λ is the left regular representation of D^n_∞ and $\{\omega_0, \cdots, \omega_{n-1}\}$ is the set of n-th roots of unity.

A linear functional ϕ on a unital Banach algebra $\mathcal{B}$ is said to be central if $\phi(xy) = \phi(yx)$ for all $x, y \in \mathcal{B}$. In Section 3, we will concentrate on the 1-forms generated by central functionals and the Maurer-Cartan form. Let Tr and tr be the canonical traces on $C^*(D^n_\infty)$ and $C^*(D_\infty)$, respectively, and ϕ_α be the central linear functional on $C^*(\mathbb{Z}_n)$ defined by

$$\phi_\alpha(\lambda_{\mathbb{Z}_n}(\tau^\beta)) = \begin{cases} 1 & \text{if} \quad \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Then we will get the following theorem.

Theorem 1.3. *The set* $\{\widetilde{\phi_\alpha}\otimes tr\ :\ 1\le\alpha\le n-1\}$ *induces* $n-1$ *different elements besides* $Tr(\omega_R(z))$ *in the cohomology group* $H^1_{de}(P^c(R_\lambda),\mathbb{C})$, *where* $\widetilde{\phi_\alpha}\triangleq\phi_\alpha-\phi_0$ *for* $1\le\alpha\le n-1$, *and* $\widetilde{\phi}_0\triangleq\phi_0$.

2. Projective joint spectrum of D_∞^n

In this section, we will compute the projective joint spectrum of D_∞^n with respect to the left regular representation.

For a discrete group G, the group algebra $\mathbb{C}[G]$ is the complex algebra generated by elements in G, i.e.

$$\mathbb{C}[G]=\{f\,|\,f=\sum_{g\in G}a_g g,\,a_g\in\mathbb{C},\text{only finitely many } a_g \text{ non zero}\}.$$

It is a $*$ -algebra under the conjugate operation defined by

$$f^*=(\sum_{g\in G}a_g g)^*=\sum_{g\in G}\overline{a_g}g^{-1},$$

where "-" represents complex conjugation. Let e denote the unit of G. Consider a positive definite function tr on $\mathbb{C}[G]$ defined by $tr(f)=a_e$. Through the GNS construction for groups [4], a GNS triple (π_{tr},H_{tr},e_{tr}) can be obtained, where $e_{tr}=e$ and π_{tr} is defined by $\pi_{tr}(g_1)g_2=g_1g_2$. Let $U\ :\ H_{tr}\to l^2(G)$ be the unitary map defined by

$$U(g)=\delta_g,\quad\forall g\in G,$$

where δ_g is the function that takes value 1 at g and 0 otherwise. The left regular representation of group G on $l^2(G)$, denoted by λ_G, is defined by

$$\lambda_G(g)f(t)=f(g^{-1}t),\quad\forall s\in G,g\in l^2(G).$$

It is well known that π_{tr} is unitarily equivalent to the left regular representation λ_G of group G via the map U.

According to the presentation (1), D_∞^n consists of elements that have the form of $\tau^k(at)^j$, $\tau^k t(at)^j$ for $0\le k\le n-1$ and $j\in\mathbb{Z}$. From the GNS construction mentioned above, the Hilbert space H_{tr} can be decomposed as $H_{tr}=\bigoplus_{k=0}^{n-1}\tau^k(H\oplus tH)$, where

$$H=\{f=\sum_{j=-\infty}^{\infty}\alpha_j(at)^k\ :\ \sum_{j=-\infty}^{\infty}\left|\alpha_j\right|^2<\infty\}.$$

Multiplication by at on the Hilbert space H will be denoted by T in the sequel. Through the unitary map $V\left((at)^k\right)=e^{ik\theta}$, H is isomorphic to $L^2(\mathbb{T},\frac{1}{2\pi}d\theta)$ and T is unitarily equivalent to the bilateral shift operator defined by multiplication by $e^{i\theta}$.

Since projective joint spectrum is invariant with respect to unitary equivalence, we will omit writing the unitary operator V in the sequel.

Theorem 2.1. *If we define $P(R_\lambda)$ as the projective joint spectrum of*

$$R_\lambda(z) = z_0\lambda(e) + z_1\lambda(a) + z_2\lambda(t) + z_3\lambda(\tau),$$

then

$$P(R_\lambda) = \bigcup_{k=1}^{n} \bigcup_{-1\le x\le 1} \{z \in \mathbb{C}^4 : (z_0 + \omega_k z_3)^2 - z_1^2 - z_2^2 - 2z_1z_2x = 0\},$$

where λ is the left regular representation of D_∞^n and $\{\omega_0, \cdots, \omega_{n-1}\}$ is the set of n-th roots of unity.

Proof. Using the orthogonal direct sum $H_{tr} = \bigoplus_{k=0}^{n-1} \tau^k(H \oplus tH)$ and the unitary map $W : \bigoplus_{k=0}^{n-1} \tau^k(H \oplus tH) \to \bigoplus_{k=0}^{n-1}(H \oplus H)$ defined by

$$W = diag[1, t, \eta, t\eta, \eta^2, t\eta^2, \cdots, \eta^{n-1}, t\eta^{n-1}],$$

we can easily compute that

$$R_\lambda(z) \overset{W}{\simeq} \begin{pmatrix} A(z) & 0 & 0 & \cdots & 0 & z_3I_2 \\ z_3I_2 & A(z) & 0 & \cdots & 0 & 0 \\ 0 & z_3I_2 & A(z) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A(z) & 0 \\ 0 & 0 & 0 & \cdots & z_3I_2 & A(z) \end{pmatrix}. \tag{2}$$

Here, $A \overset{W}{\simeq} B$ means A and B is unitarily equivalent under W and $A(z) = \begin{pmatrix} z_0 & z_1T + z_2 \\ z_1T^* + z_2 & z_0 \end{pmatrix}$. We divide the argument into two cases.

Case I. If $z_3 = 0$, it is trivial since $R_\lambda(z)$ is invertible if and only if $A(z)$ is invertible. In this case, the projective joint spectrum $P(R_\lambda)$ equals

$$P(R_\lambda) = \bigcup_{-1\le x\le 1} \{z \in \mathbb{C}^4 : z_0^2 - z_1^2 - z_2^2 - 2z_1z_2x = 0, z_3 = 0\}$$

by [16, Theorem 1.1].

Case II. If $z_3 \neq 0$, by multiplying $\begin{pmatrix} 0 & I_{2n-2} \\ I_2 & 0 \end{pmatrix}$ on the right, we turn $R_\lambda(z)$ into

$$\widetilde{R(z)} = \begin{pmatrix} z_3I_2 & A(z) & 0 & \cdots & 0 & 0 \\ 0 & z_3I_2 & A(z) & \cdots & 0 & 0 \\ 0 & 0 & z_3I_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z_3I_2 & A(z) \\ A(z) & 0 & 0 & \cdots & 0 & z_3I_2 \end{pmatrix}. \tag{3}$$

By the Schur complement trick in [17], $R_\lambda(z)$ is invertible if and only if $z_3^n - (-1)^n A(z)^n$ is invertible.

Using the spectral theorem for normal operators in [7, Theorem 9.2.2], we can write

$$T = \int_{\mathbb{T}} \lambda dE(\lambda), \tag{4}$$

where $dE(\lambda)$ is the spectral measure of T. Noting that

$$z_3^n - (-1)^n A(z)^n = (-1)^{n+1} \prod_{k=1}^{n} (z_3\omega_k + A(z))$$

and $z_3\omega_k + A(z)$ mutually commutes for $1 \leq k \leq n$, it follows that $z_3^n - (-1)^n A(z)$ is invertible if and only if $z_3\omega_k + A(z)$ is invertible for each k, or more precisely, by (4)

$$(z_0 + \omega_k z_3)^2 - z_1^2 - z_2^2 - z_1 z_2(\lambda + \bar{\lambda}) \neq 0, \quad \forall \lambda \in \mathbb{T}, 1 \leq k \leq n.$$

Letting $\lambda = e^{i\theta}$ for $\theta \in [0, 2\pi)$, $x = \cos\theta$, we have

$$P^c(R_\lambda) = \bigcap_{k=1}^{n} \bigcap_{-1 \leq x \leq 1} \{z \in \mathbb{C}^4 : (z_0 + \omega_k z_3)^2 - z_1^2 - z_2^2 - 2z_1 z_2 x \neq 0\},$$

and the theorem is proved by taking its complement. □

In the sequel, we set

$$G_\theta^k(z) = (z_0 + \omega_k z_3)^2 - z_1^2 - z_2^2 - 2z_1 z_2 \cos\theta. \tag{5}$$

Remark. It is proved in [16] that the Koopman representation and the left regular representation of D_∞ are weakly equivalent. Without much effort, we can obtain by Theorem [4, Proposition F.3.2] that the result also holds for D_∞^n.

3. Trace of Maurer-Cartan form and de Rham cohomology group

In [25], for a tuple $A = (A_1, A_2, \dots, A_n)$, the Maurer-Cartan form ω_A is an operator-valued 1-form defined by

$$\omega_A(z) = A(z)^{-1} dA(z) = \sum_{i=1}^{n} A(z)^{-1} A_i dz_i, \qquad \forall z \in P^c(A).$$

A linear functional ϕ on a unital Banach algebra $\mathcal{B}$ is said to be central if $\phi(xy) = \phi(yx)$ for any $x, y \in \mathcal{B}$. In [25, Theorem 3.2], it is proved that if ϕ is central and $\phi(I) \neq 0$, then $\phi(\omega_A)$ is a non-trivial element in the de Rham cohomology group $H_{de}^1(P^c(A), \mathbb{C})$. This section is devoted to studying 1-forms induced by different central linear functionals.

For a discrete group G, it is well known that its reduced group C^*-algebra $C_r^*(G)$ admits a canonical tracial state

$$tr(a) = \langle a\delta_e, \delta_e \rangle, \qquad \forall a \in C_r^*(G). \tag{6}$$

In [5, Corollary 4.3], it is proved that the reduced group C^*-algebra $C_r^*(G)$ has only one tracial state if and only if G is amenable or none of its normal subgroup are amenable. Since D_∞^n itself is amenable, the canonical trace Tr is the unique tracial state on $C_r^*(D_\infty^n)$.

Since $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}_2$ and $D_\infty^n = \mathbb{Z}_n \times D_\infty$, the inclusion maps

$$\mathbb{Z} \hookrightarrow D_\infty \hookrightarrow D_\infty^n$$

induce the inclusion maps of group C^*-algebras. Based on this observation, we have the following proposition.

Proposition 3.1. *The canonical traces on $C_r^*(\mathbb{Z}), C_r^*(D_\infty)$ and $C_r^*(D_\infty^n)$ coincide by restriction.*

Proof. We first denote the canonical traces on these C^*-algebras by $tr_{\mathbb{Z}}$, tr and Tr.

By the form of elements in D_∞, $l^2(D_\infty) \cong l^2(\mathbb{Z}) \oplus l^2(t\mathbb{Z})$. This implies that any $a \in C_r^*(\mathbb{Z})$ can be treated as $a \oplus 0 \in C_r^*(D_\infty)$. Thus

$$tr_{\mathbb{Z}}(a) = \left\langle a\delta_{e_{\mathbb{Z}}}, \delta_{e_{\mathbb{Z}}} \right\rangle = \left\langle (a \oplus 0)\delta_{e_{D_\infty}}, \delta_{e_{D_\infty}} \right\rangle = tr(a \oplus 0),$$

which leads to $tr|_{\mathbb{Z}} = tr_{\mathbb{Z}}$.

Since

$$C_r^*(D_\infty^n) = \overline{C_r^*(\mathbb{Z}_n) \otimes C_r^*(D_\infty)}^{B(l^2(D_\infty^n))}$$

and $\delta_{e_{D_\infty^n}} = \delta_{e_{\mathbb{Z}_n}} \otimes \delta_{e_{D_\infty}}$, for any $a \in C_r^*(D_\infty)$,

$$\begin{aligned} tr(a) &= \left\langle a\delta_{e_{D_\infty}}, \delta_{e_{D_\infty}} \right\rangle = \left\langle a\delta_{e_{D_\infty}}, \delta_{e_{D_\infty}} \right\rangle \left\langle \delta_{e_{\mathbb{Z}_n}}, \delta_{e_{\mathbb{Z}_n}} \right\rangle \\ &= \left\langle (a \otimes 1)\delta_{e_{D_\infty^n}}, \delta_{e_{D_\infty^n}} \right\rangle \\ &= Tr(a \otimes 1) \end{aligned}$$

Therefore, $Tr|_{D_\infty} = tr$. □

Due to this proposition, we will not distinguish the canonical traces on these groups and denote them all by tr.

The group Von Neumann algebra $L(G)$ is the closure of $\lambda_G(\mathbb{C}[G])$ with respect to the weak operator topology in the Hilbert space. Thus, formula (6) can be naturally extended to $L(G)$. In the previous section, we obtained the matrix representation of $R_\lambda(z)$ by (2). In order to compute the trace of the Maurer-Cartan form, we can canonically define the extended trace $\widetilde{tr}$ on $2n \times 2n$ matrices with $L(\mathbb{Z})$ entries by

$$\widetilde{tr}((a_{ij})_{i,j=1}^{2n}) := \frac{1}{2n} tr(\sum_{i=1}^{2n} a_{ii}). \tag{7}$$

The sign "$\sim$" will be omitted if there is no confusion.

By the observation in [16], we have

$$tr(dE(\lambda)) = tr(dE(e^{i\theta})) = dtr(E(e^{i\theta})) = \frac{1}{2\pi} d\theta, \tag{8}$$

where $E(\lambda)$ is the spectral measure in (4).

For D_∞^n, the Maurer-Cartan form is

$$\omega_R(z) = R^{-1}(z)dR(z) = R^{-1}(z)(dz_0 + adz_1 + tdz_2 + \tau dz_3), \tag{9}$$

where $R(z)$ is defined in (2).

Lemma 3.2. *Suppose $z = (z_0, z_1, z_2, z_3) \in \mathbb{C}^4$, and let f and g denote $z_1 e^{i\theta} + z_2$ and $z_1 e^{-i\theta} + z_2$, respectively. If*

$$a_{11}^{(n)} = \sum_{k=0}^{[\frac{n}{2}]} z_0^{n-2k} \binom{n}{n-2k} f^k g^k$$

and

$$a_{12}^{(n)} = \sum_{k=0}^{[\frac{n-1}{2}]} z_0^{n-2k-1} \binom{n}{n-2k-1} f^k g^k,$$

where $\binom{n}{m} = \frac{n!}{(n-m)!m!}$, then we have the following factorization:

$$(z_3^n - (-1)^n a_{11}^{(n)})^2 - (a_{12}^{(n)})^2 fg = \prod_{k=1}^{n} G_\theta^k(z).$$

Proof. Using direct computation,

$$\begin{aligned}
LHS &= \Big(z_3^n - (-1)^n (a_{11}^{(n)} + a_{12}^{(n)}\sqrt{fg})\Big)\Big(z_3^n - (-1)^n (a_{11}^{(n)} - a_{12}^{(n)}\sqrt{fg})\Big) \\
&= \Big((a_{11}^{(n)} + a_{12}^{(n)}\sqrt{fg}) - (-z_3)^n\Big)\Big((a_{11}^{(n)} - a_{12}^{(n)}\sqrt{fg}) - (-z_3)^n\Big) \\
&= \Big((z_0 + \sqrt{fg})^n - (-z_3)^n\Big)\Big((z_0 - \sqrt{fg})^n - (-z_3)^n\Big) \\
&= \prod_{k=0}^{n-1} (z_0 + \sqrt{fg} + z_3\omega_k)(z_0 - \sqrt{fg} + z_3\omega_k) \\
&= \prod_{k=0}^{n-1} \big((z_0 + z_3\omega_k)^2 - z_1^2 - z_2^2 - 2z_1z_2\cos\theta\big) \\
&= RHS
\end{aligned}$$

□

Proposition 3.3. *The trace of Maurer-Cartan form $\omega_R(z)$ is*

$$\widetilde{tr}(\omega_R(z)) = d\Big(\frac{1}{4n\pi}\int_0^{2\pi} \log(\prod_{k=1}^{n} G_\theta^k(z))d\theta\Big),\ z \in P^c(R_\lambda),$$

where $G_\theta^k(z)$ is defined in (5), and d stands for

$$\frac{\partial}{\partial z_0}dz_0 + \frac{\partial}{\partial z_1}dz_1 + \frac{\partial}{\partial z_2}dz_2 + \frac{\partial}{\partial z_3}dz_3.$$

Proof. As in the proof of Theorem 2.1, the argument will be divided into two parts.

Case I. If $z_3 = 0$, then $\omega_R(z) = (z_0 + z_1 a + z_2 t)^{-1}(dz_0 + a dz_1 + t dz_2)$. In this case, we can directly use [16, Proposition 3.2] to get

$$\widetilde{tr}(\omega_R(z)) = d\left(\frac{1}{4\pi}\int_0^{2\pi} \log(z_0^2 - z_1^2 - z_2^2 - 2z_1 z_2 \cos\theta) d\theta\right).$$

Case II. For $z_3 \neq 0$, we will first compute the inverse of $\widetilde{R(z)}$ defined in (3).

Using the Schur complement as in the proof of Theorem 2.1, we define a $2(n-k)\times 2(n-k)$ matrix $S_{n-k}(z)$ by

$$S_{n-k}(z) = \begin{pmatrix} z_3 I_2 & A(z) & 0 & \cdots & 0 & 0 \\ 0 & z_3 I_2 & A(z) & \cdots & 0 & 0 \\ 0 & 0 & z_3 I_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z_3 I_2 & A(z) \\ \frac{(-1)^k}{z_3^k} A(z)^{k+1} & 0 & 0 & \cdots & 0 & z_3 I_2 \end{pmatrix}.$$

Thus, $S_{n-1}(z)$ is the Schur complement of $\widetilde{R(z)}$, and $S_{n-k}(z)$ is that of $S_{n-k+1}(z)$.

To perform the computation, we should first focus on $S_1(z)^{-1}$ and $A(z)^n$. We obtain by induction that $A(z)^n = \begin{pmatrix} a_{11}^{(n)} & a_{12}^{(n)} f \\ a_{12}^{(n)} g & a_{11}^{(n)} \end{pmatrix}$, where $a_{11}^{(n)}$ and $a_{12}^{(n)}$ are defined in Lemma 3.2.

Using Lemma 3.2, for $S_1(z) = \frac{1}{z_3^{n-2}}(z_3^n - (-1)^n A(z)^n)$, we have

$$\begin{aligned} S_1(z)^{-1} &= \frac{z_3^{n-2}}{(z_3^n - (-1)^n a_{11}^{(n)})^2 - (a_{12}^{(n)})^2 fg} \begin{pmatrix} z_3^n - (-1)^n a_{11}^{(n)} & (-1)^n a_{12}^{(n)} f \\ (-1)^n a_{12}^{(n)} g & z_3^n - (-1)^n a_{11}^{(n)} \end{pmatrix} \\ &= \frac{z_3^{n-2}}{\prod_{k=1}^{n} G_\theta^k(z)} \begin{pmatrix} z_3^n - (-1)^n a_{11}^{(n)} & (-1)^n a_{12}^{(n)} f \\ (-1)^n a_{12}^{(n)} g & z_3^n - (-1)^n a_{11}^{(n)} \end{pmatrix}. \end{aligned}$$

By induction, we have $S_k(z)^{-1} = (s_{ij})_{i,j=1}^k$ with

$$\begin{aligned} s_{11} &= z_3^{-1} + \frac{(-1)^n}{z_3^n} S_1(z)^{-1} A(z)^n, \\ s_{1k} &= \frac{(-1)^{k-1}}{z_3^{k-1}} S_1(z)^{-1} A(z)^{k-1}, \\ s_{1j} &= \frac{(-1)^{j-1}}{z_3^j} A(z)^{j-1} + \frac{(-1)^{n+j-1}}{z_3^{n+j-1}} S_1(z)^{-1} A(z)^{n+j-1}, \quad 2 \le j \le k-1 \\ s_{i1} &= \frac{(-1)^{n-i+1}}{z_3^{n-i+1}} S_1(z)^{-1} A(z)^{n-i+1}, \quad i > 1 \end{aligned}$$

and $(s_{ij})_{i,j=2}^k = S_{k-1}(z)^{-1}$.

Thus $\widetilde{R(z)}^{-1} = (r_{ij})_{i,j=1}^n$ with

$$r_{11} = z_3^{-1} + \frac{(-1)^n}{z_3^n} S_1(z)^{-1} A(z)^n,$$

$$r_{1n} = \frac{(-1)^{n-1}}{z_3^{n-1}} S_1(z)^{-1} A(z)^{n-1},$$

$$r_{1j} = \frac{(-1)^{j-1}}{z_3^j} A(z)^{j-1} + \frac{(-1)^{n+j-1}}{z_3^{n+j-1}} S_1(z)^{-1} A(z)^{n+j-1}, \quad 2 \le j \le n-1$$

$$r_{i1} = \frac{(-1)^{n-i+1}}{z_3^{n-i+1}} S_1(z)^{-1} A(z)^{n-i+1}, \quad i > 1$$

and $(r_{ij})_{i,j=2}^n = S_{n-1}(z)^{-1}$.

Therefore, by (7), (8) and the matrices of $A(z)^{n-1}$ and $S_1(z)^{-1}$,

$$\widetilde{tr}(R(z)^{-1}) = \frac{1}{2n} tr\left(\frac{n(-1)^{n-1} A(z)^{n-1} S_1(z)^{-1}}{z_3^{n-1}}\right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{(-1)^{n-1}((z_3^n - (-1)^n a_{11}^{(n)}) a_{11}^{(n-1)} + (-1)^n a_{12}^{(n)} a_{12}^{(n-1)} fg)}{\prod_{k=1}^n G_\theta^k(z)} d\theta$$

$$= \frac{1}{4n\pi} \int_0^{2\pi} \sum_{k=1}^n \frac{2(z_0 + \omega_k z_3)}{G_\theta^k(z)} d\theta.$$

Similarly, we have that

$$\widetilde{tr}(R^{-1}(z)a) = -\frac{1}{4n\pi} \int_0^{2\pi} \sum_{k=1}^n \frac{2(z_1 + z_2 cos\theta)}{G_\theta^k(z)} d\theta,$$

$$\widetilde{tr}(R^{-1}(z)t) = -\frac{1}{4n\pi} \int_0^{2\pi} \sum_{k=1}^n \frac{2(z_2 + z_1 cos\theta)}{G_\theta^k(z)} d\theta,$$

and

$$\widetilde{tr}(R^{-1}(z)\tau) = \frac{1}{4n\pi} \int_0^{2\pi} \sum_{k=1}^n \frac{2\omega_k(z_0 + \omega_k z_3)}{G_\theta^k(z)} d\theta.$$

By (9), summing all the results above, we obtain

$$\widetilde{tr}(\omega_R(z)) = d\left(\frac{1}{4n\pi} \int_0^{2\pi} \log(\prod_{k=1}^n G_\theta^k(z)) d\theta\right).$$

□

At the beginning of this section, we mentioned that a central linear functional ϕ will induce a non-trivial closed 1-form in the de Rham cohomology group of the projective resolvent set if $\phi(I) \neq 0$. In [16, Corollary 4.4], it is shown by the de Rham duality theorem that the $H^1_{de}(P^c(R_{\lambda_{D_\infty}}), \mathbb{C})$ is generated by $\frac{1}{2\pi} tr(\omega_{R_{\lambda_{D_\infty}}})$. So it is a natural question what the case is for D^n_∞. In fact, there are other central linear functionals that yield nontrivial elements in the cohomology group.

It is well known that for any discrete groups G and H

$$C^*(G \times H) = C^*(G) \otimes_{max} C^*(H),$$
$$C^*_r(G \times H) = C^*_r(G) \otimes_{min} C^*_r(H).$$

Moreover, if one of the groups, for example G, is amenable, then the group C^*-algebra $C^*(G)$ is nuclear and is isomorphic to $C^*_r(G)$. Back to our question, since both D_∞ and $\mathbb{Z}_n$ are amenable, $C^*(\mathbb{Z}_n \times D_\infty)$ is the closure of algebraic tensor product of $C^*(D_\infty)$ and $C^*(\mathbb{Z}_n)$ in $B(l^2(D^n_\infty))$. Therefore, a tensor product of two linear functionals, on the two respective C^*-algebras, extends to a linear functional on $C^*(D^n_\infty)$.

Apparently, $\dim C^*(\mathbb{Z}_n) = n$. For every element $x \in C^*(\mathbb{Z}_n)$, it takes the form

$$x = \sum_{\beta=0}^{n-1} b_\beta \lambda_{\mathbb{Z}_n}(\tau^\beta), b_\beta \in \mathbb{C}.$$

Since $\lambda_{\mathbb{Z}_n}(\tau)$ has the matrix representation

$$\lambda_{\mathbb{Z}_n}(\tau) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$C^*(\mathbb{Z}_n)$ can be identified as a subalgebra of $M_n(\mathbb{C})$ by the map φ defined by $\varphi(\lambda_{\mathbb{Z}_n}(\tau^\beta)) = \sum\limits_{i=n-\beta}^{n-1} E^{(n)}_{i(i-n+\beta)} + \sum\limits_{i=0}^{n-1-\beta} E^{(n)}_{i(i+\beta)}$. Here $\{E^{(n)}_{ij} : 0 \le i \le n-1, 0 \le j \le n-1\}$ is the generating set for $M_n(\mathbb{C})$, where $E^{(n)}_{ij}$ denotes the matrix with a 1 at the i-th row and j-th column and 0 elsewhere.

We first define n linear functionals ϕ_α, $0 \le \alpha \le n-1$, on $M_n(\mathbb{C})$ by

$$\phi_\alpha(E^{(n)}_{ij}) = \begin{cases} \frac{1}{n} & \text{if} \quad j - i = \alpha, \\ \frac{1}{n} & \text{if} \quad i - j = n - \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that every linear functional on $C^*(\mathbb{Z}_n)$ is a complex linear combination of $\{\phi_\alpha\}_{\alpha=0}^{n-1}$ by restriction.

In fact, $\phi_0 \otimes tr$ is just the trace on $C^*(D_\infty^n)$. Now define $\widetilde{\phi_\alpha} \triangleq \phi_\alpha - \phi_0$ for $1 \leq \alpha \leq n-1$, and $\widetilde{\phi_0} \triangleq \phi_0$. One can check that $\widetilde{\phi_\alpha}$ is not a trace for $1 \leq \alpha \leq n-1$, as it takes the value -1 at the identity. We first present the following observation.

Lemma 3.4. *The formula of* $(\phi_\alpha \otimes tr)((a_{ij})_{i,j=0}^{2n-1})$ *is*

$$(\phi_\alpha \otimes tr)((a_{ij})_{i,j=0}^{2n-1}) = \frac{1}{2n} tr(\sum_{i=2n-2\alpha}^{2n-1} a_{i(i-2n+2\alpha)} + \sum_{i=0}^{2n-2\alpha-1} a_{i(2\alpha+i)}),$$

$\forall 1 \leq \alpha \leq n-1$, $(a_{ij})_{i,j=0}^{2n-1} \in C^*(D_\infty^n)$.

Proof. Let $\{E_{ij} : 0 \leq i \leq 2n-1, 0 \leq j \leq 2n-1\}$ be the generating set for $M_{2n}(L(\mathbb{Z}))$, and $\{\widetilde{E}_{ij} : 0 \leq i \leq 1, 0 \leq j \leq 1\}$ for $M_2(L(\mathbb{Z}))$. Here

The following equation can be easily verified:

$$E_{ij} = E^{(n)}_{[\frac{i}{2}][\frac{j}{2}]} \otimes \widetilde{E}_{(2\{\frac{i}{2}\})(2\{\frac{j}{2}\})},$$

where the brackets $[x]$ denote the floor operation and $\{x\}$ denote the fractional part operation.

Thus,

$$\begin{aligned}(\phi_\alpha \otimes tr)(E_{ij}) &= \phi_\alpha(E^{(n)}_{[\frac{i}{2}][\frac{j}{2}]}) tr(\widetilde{E}_{(2\{\frac{i}{2}\})(2\{\frac{j}{2}\})}) \\ &= \frac{1}{n}\left(\delta_{\alpha([\frac{j}{2}]-[\frac{i}{2}])} + \delta_{(n-\alpha)([\frac{i}{2}]-[\frac{j}{2}])}\right)\frac{1}{2} tr(\Delta_{(2\{\frac{i}{2}\})(2\{\frac{j}{2}\})}) \\ &= \begin{cases} \frac{1}{2n} & \text{if} \quad j-i = 2\alpha, \\ \frac{1}{2n} & \text{if} \quad i-j = 2n-2\alpha, \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

where δ_{xy} is the kronecker symbol and

$$\Delta_{xy} = \begin{cases} I(L^2(\mathbb{Z})) & \text{if} \quad x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned}(\phi_\alpha \otimes tr)((a_{ij})_{i,j=0}^{2n-1}) &= \sum_{i,j=0}^{n-1} \phi_\alpha(E^{(n)}_{ij}) tr\left(\begin{pmatrix} a_{(2i)(2j)} & a_{(2i)(2j+1)} \\ a_{(2i+1)(2j)} & a_{(2i+1)(2j+1)} \end{pmatrix}\right) \\ &= \sum_{i=0}^{n-1-\alpha} \frac{1}{2n} tr(a_{(2i)(2i+2\alpha)} + a_{(2i+1)(2i+2\alpha+1)}) + \\ &\quad \sum_{i=n-\alpha}^{n-1} \frac{1}{2n} tr(a_{(2i)(2i+2\alpha-2n)} + a_{(2i+1)(2i+2\alpha-2n+1)})\end{aligned}$$

$$= \frac{1}{2n} tr(\sum_{i=2n-2\alpha}^{2n-1} a_{i(i-2n+2\alpha)} + \sum_{i=0}^{2n-2\alpha-1} a_{i(2\alpha+i)}).$$

□

Theorem 3.5. *$\{\widetilde{\phi_\alpha \otimes tr}\}_{\alpha=1}^{n-1}$ induces n-1 different 1-forms besides $Tr(\omega_R(z))$ in the cohomology group $H^1_{de}(P^c(R_{\lambda_{D^n_\infty}}), \mathbb{C})$, where Tr is the canonical trace on $C^*(D^n_\infty)$.*

Proof. Firstly, using the formula in Lemma 3.4, we get for $1 \leq \alpha \leq n-1$

$$(\phi_\alpha \otimes tr)(R(z)^{-1}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(-1)^{n-\alpha-1} a_{11}^{(n-\alpha-1)} z_3^{n+\alpha} d\theta}{\prod_{k=1}^{n} G_\theta^k(z)}$$
$$+ \frac{1}{2\pi} \int_0^{2\pi} \frac{(-1)^{\alpha}(a_{11}^{(n)} a_{11}^{(n-\alpha-1)} - a_{12}^{(n)} a_{12}^{(n-\alpha-1)} fg) z_3^{\alpha} d\theta}{\prod_{k=1}^{n} G_\theta^k(z)},$$

$$(\phi_\alpha \otimes tr)(R(z)^{-1}a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(-1)^{n-\alpha-1}(z_1 + z_2 cos\theta) a_{12}^{(n-\alpha-1)} z_3^{n+\alpha} d\theta}{\prod_{k=1}^{n} G_\theta^k(z)}$$
$$+ \frac{1}{2\pi} \int_0^{2\pi} \frac{(-1)^{\alpha}(z_1 + z_2 cos\theta)(a_{11}^{(n)} a_{12}^{(n-\alpha-1)} - a_{12}^{(n)} a_{11}^{(n-\alpha-1)}) z_3^{\alpha} d\theta}{\prod_{k=1}^{n} G_\theta^k(z)},$$

$$(\phi_\alpha \otimes tr)(R(z)^{-1}t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(-1)^{n-\alpha-1}(z_2 + z_1 cos\theta) a_{12}^{(n-\alpha-1)} z_3^{n+\alpha} d\theta}{\prod_{k=1}^{n} G_\theta^k(z)}$$
$$+ \frac{1}{2\pi} \int_0^{2\pi} \frac{(-1)^{\alpha}(z_2 + z_1 cos\theta)(a_{11}^{(n)} a_{12}^{(n-\alpha-1)} - a_{12}^{(n)} a_{11}^{(n-\alpha-1)}) z_3^{\alpha} d\theta}{\prod_{k=1}^{n} G_\theta^k(z)},$$

and

$$(\phi_\alpha \otimes tr)(R(z)^{-1}\tau) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(-1)^{n-\alpha} a_{11}^{(n-\alpha)} z_3^{n+\alpha-1} d\theta}{\prod_{k=1}^{n} G_\theta^k(z)}$$
$$+ \frac{1}{2\pi} \int_0^{2\pi} \frac{(-1)^{\alpha+1}(a_{11}^{(n)} a_{11}^{(n-\alpha)} - a_{12}^{(n)} a_{12}^{(n-\alpha)} fg) z_3^{\alpha-1} d\theta}{\prod_{k=1}^{n} G_\theta^k(z)}.$$

Fix $k \in \{0, 1, \cdots, n-1\}$ and choose a closed path

$$\gamma_k = \{((b+1)\omega_k, \omega_k c_{nk} e^{it} + \omega_k, 0, b) : 0 \le t \le 2\pi\},$$

where $c_{nk} = \frac{1}{2}\min\{\min_{j \ne k} b|\omega_k - \omega_j|, \min_j |(b+2)\omega_k - b\omega_j|\}$, and b is a non-negative constant. For every $-1 \le x \le 1$ and $j \in \{0, 1, \cdots, n-1\}$, we have that

$$\begin{aligned}&(z_0 + \omega_j z_3)^2 - z_1^2 - z_2^2 - 2z_1 z_2 x \\ &= (b\omega_k - b\omega_j - \omega_k c_{nk} e^{it})((b+2)\omega_k - b\omega_j + \omega_k c_{nk} e^{it}) \ne 0,\end{aligned}$$

This means $\gamma_k \in P^c(R_{\lambda_{D_\infty^n}})$ by Theorem 2.1.

On γ_k, for $0 \le \alpha \le n-1$, the path integral

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\gamma_k} (\phi_\alpha \otimes tr)(\omega_R(z)) \\
&= \frac{1}{2\pi i}\int_{\gamma_k} \frac{z_3^\alpha}{2\pi}\int_0^{2\pi} \frac{(-1)^{n-\alpha-1}(f+g)a_{12}^{n-\alpha-1} z_3^n}{\prod_{j=1}^n G_\theta^k(z)} d\theta dz_1 \\
&- \frac{1}{2\pi i}\int_{\gamma_k} \frac{z_3^\alpha}{2\pi}\int_0^{2\pi} \frac{(-1)^{\alpha+1}(f+g)(z_0^2 - z_1^2)^{n-\alpha-1} a_{12}^{\alpha+1}}{\prod_{j=1}^n G_\theta^k(z)} d\theta dz_1 \\
&= \frac{1}{4\pi}\int_0^{2\pi} \frac{b^{n+\alpha}\omega_k^{-\alpha-1}((b+2+c_{nk}e^{it})^{n-\alpha-1} - (b - c_{nk}e^{it})^{n-\alpha-1})}{\prod_{j=1}^n((b+2)\omega_k - b\omega_j + \omega_k c_{nk}e^{it}) \prod_{j\ne k}(b\omega_k - b\omega_j - \omega_k c_{nk} e^{it})} dt \\
&+ \frac{1}{4\pi}\int_{\gamma_k} \frac{b^\alpha \omega_k^{-\alpha-1}(b+2+c_{nk}e^{it})^{\alpha+1}((b+1)^2 - (c_{nk}e^{it}+1)^2)}{\prod_{j=1}^n((b+2)\omega_k - b\omega_j + \omega_k c_{nk}e^{it}) \prod_{j\ne k}(b\omega_k - b\omega_j - \omega_k c_{nk} e^{it})} dt \\
&- \frac{1}{4\pi}\int_{\gamma_k} \frac{b^\alpha \omega_k^{-\alpha-1}(b - c_{nk}e^{it})^{\alpha+1}((b+1)^2 - (c_{nk}e^{it}+1)^2)}{\prod_{j=1}^n((b+2)\omega_k - b\omega_j + \omega_k c_{nk}e^{it}) \prod_{j\ne k}(b\omega_k - b\omega_j - \omega_k c_{nk} e^{it})} dt \\
&= \frac{1}{4\pi i}\int_{\mathbb{T}} \frac{b^{n+\alpha}\omega_k^{-\alpha-1}((b+2+c_{nk}\omega)^{n-\alpha-1} - (b - c_{nk}\omega)^{n-\alpha-1})}{\omega \prod_{j=1}^n((b+2)\omega_k - b\omega_j + \omega_k c_{nk}\omega) \prod_{j\ne k}(b\omega_k - b\omega_j - \omega_k c_{nk}\omega)} d\omega \\
&+ \frac{1}{4\pi i}\int_{\mathbb{T}} \frac{b^\alpha \omega_k^{-\alpha-1}((b+2+c_{nk}\omega)^{\alpha+1})((b+1)^2 - (c_{nk}\omega+1)^2)}{\omega \prod_{j=1}^n((b+2)\omega_k - b\omega_j + \omega_k c_{nk}\omega) \prod_{j\ne k}(b\omega_k - b\omega_j - \omega_k c_{nk}\omega)} d\omega
\end{aligned}$$

$$-\frac{1}{4\pi i}\int_{\mathbb{T}}\frac{b^{\alpha}\omega_k^{-\alpha-1}(b-c_{nk}\omega)^{\alpha+1})((b+1)^2-(c_{nk}\omega+1)^2)}{\omega\prod_{j=1}^{n}((b+2)\omega_k-b\omega_j+\omega_k c_{nk}\omega)\prod_{j\neq k}(b\omega_k-b\omega_j-\omega_k c_{nk}\omega)}d\omega \tag{10}$$

Since $c_{nk}=\frac{1}{2}\min\{\min_{j\neq k}b|\omega_k-\omega_j|,\min_{j}|(b+2)\omega_k-b\omega_j|\}$, the only residue of the integral is the residue at $\omega=0$.

Thus,

$$\begin{aligned}(10)&=\frac{b^{n+\alpha}\omega_k^{n-\alpha-1}((b+2)^{n-\alpha-1}-b^{n-\alpha-1})}{2\prod_{j=1}^{n}((b+2)\omega_k-b\omega_j)\prod_{j\neq k}(b\omega_k-b\omega_j)}\\&\quad+\frac{b^{\alpha}\omega_k^{2n-\alpha-1}(b^2+2b)^{n-\alpha-1}((b+2)^{\alpha+1}-b^{\alpha+1})}{2\prod_{j=1}^{n}((b+2)\omega_k-b\omega_j)\prod_{j\neq k}(b\omega_k-b\omega_j)}\\&=\frac{\omega^{-\alpha-1}}{2}\frac{b^{n-1}[b^{\alpha+1}(b+2)^{n-\alpha-1}-b^n+(b+2)^n-(b+2)^{n-\alpha-1}b^{\alpha+1}]}{n\omega_k^{n-1}b^{n-1}((b+2)^n-b^n)}\\&=\frac{\omega_k^{-\alpha}}{2n}=:a_{k\alpha}\end{aligned}$$

Suppose that $(\widetilde{\phi_\alpha}\otimes tr)(\omega_R(z))$, $0\le\alpha\le n-1$, are linearly dependent. Then, $(\phi_\alpha\otimes tr)(\omega_R(z))$ are linearly dependent in $H^1_{de}(P^c(R_\lambda),\mathbb{C})$. Thus, $\exists\, b_\alpha\in\mathbb{C}, 0\le\alpha\le n-1$, *s.t.*

$$\sum_{\alpha=0}^{n-1}b_\alpha(\phi_\alpha\otimes tr)(\omega_R(z))=[0]. \tag{11}$$

Integrating (11) on γ_k, we have

$$\sum_{\alpha=0}^{n-1}b_\alpha a_{k\alpha}=0.$$

Choosing $k=0,1,\cdots,n-1$ in turn, we obtain a homogeneous linear system of equations with a coefficient matrix $(a_{k\alpha})_{k,\alpha=0}^{n-1}$. Since

$$\det((a_{k\alpha})_{k,\alpha=0}^{n-1})=(\frac{1}{2n})^n\prod_{0\le j<k\le n-1}(\frac{1}{\omega_k}-\frac{1}{\omega_j}),$$

the only solution to the homogeneous linear system of equations is the zero solution, i.e. $\forall 0\le\alpha\le n-1, b_\alpha=0$. Thus $(\widetilde{\phi_\alpha}\otimes tr)(R^{-1}(z))$ are n different elements in the cohomology group. □

Remark 1. Let $\gamma=\{z(t):0\le t\le 2\pi\}$ be a closed path in the resolvent set. By Theorem 2.1, $G^k_\theta(z)\neq 0$ on $P(R_\lambda)^c$. If we define the winding number $W(\gamma)$

of γ around the projective joint spectrum as the winding number of $\prod_{k=1}^{n} G_\theta^k(\gamma)$ around 0, then

$$W(\gamma) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{G_\theta^k(\gamma)} \frac{d\omega}{\omega} = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_0^{2\pi} \frac{d}{dt} \log(G_\theta^k(z(t)))dt.$$

Thus, the integral

$$\begin{aligned}\frac{1}{2\pi i} \int_\gamma (\phi_0 \otimes tr)(\omega_R(z)) &= \frac{1}{2\pi i} \int_\gamma d\Big(\frac{1}{4n\pi} \int_0^{2\pi} \log(\prod_{k=1}^{n} G_\theta^k(z))d\theta\Big) \\ &= \frac{1}{4n\pi} \int_0^{2\pi} \Big(\sum_{k=1}^{n} \frac{1}{2\pi i} \int_0^{2\pi} \frac{d}{dt} \log(G_\theta^k(z(t)))dt\Big)d\theta \\ &= \frac{W(\gamma)}{2n}.\end{aligned}$$

This provides the explanation for why a_{k0} is not an integer.

Remark 2. In the above theorem, we only provide n linear independent elements in the cohomology group. However, it still remains a question whether the cohomology group is n generated.

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