

Left braces of size p^2q^2

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ABSTRACT. We consider relatively prime integer numbers m and n such that each solvable group of order mn has a normal subgroup of order m . We prove that each brace of size mn is a semidirect product of a brace of size m and a brace of size n . We further give a method to classify braces of size mn from the classification of braces of sizes m and n . We apply this result to determine all braces of size p^2q^2 , for p and q odd primes satisfying some conditions which hold in particular for p a Germain prime and $q = 2p + 1$.

CONTENTS

1. Introduction	223
2. Left braces of size mn , for $\gcd(m, n) = 1$	224
3. Braces of size p^2 , for p an odd prime number	226
4. Groups of order p^2q^2	227
5. Left braces of size p^2q^2	228
Acknowledgments	237
References	237

1. Introduction

In [10] Rump introduced braces to study set-theoretic solutions of the Yang-Baxter equation. A (left) brace is a triple $(B, +, \cdot)$ where B is a set and $+$ and \cdot are binary operations such that $(B, +)$ is an abelian group, (B, \cdot) is a group and

$$a \cdot (b + c) + a = a \cdot b + a \cdot c,$$

for all $a, b, c \in B$. We call $(B, +)$ the additive group and (B, \cdot) the multiplicative group of the left brace. The cardinal of B is called the size of the brace. If $(B, +)$ is an abelian group, then $(B, +, +)$ is a brace, called trivial brace.

Let B_1 and B_2 be left braces. A map $f : B_1 \rightarrow B_2$ is said to be a brace morphism if $f(b + b') = f(b) + f(b')$ and $f(b \cdot b') = f(b) \cdot f(b')$ for all $b, b' \in B_1$. If f is bijective, we say that f is an isomorphism. In that case we say that the braces B_1 and B_2 are isomorphic.

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We recall the definition of direct and semidirect product of braces as defined in [5] and [11]. Let $(B_1, +, \cdot)$ and $(B_2, +, \cdot)$ be braces and $\tau : (B_2, \cdot) \rightarrow \text{Aut}(B_1, +, \cdot)$ be a group morphism. Define in $B_1 \times B_2$ operations $+$ and \cdot by

$$(a, b) + (a', b') = (a + a', b + b'), \quad (a, b) \cdot (a', b') = (a \cdot \tau(b)(a'), b \cdot b').$$

Then $(B_1 \times B_2, +, \cdot)$ is a brace which is called the semidirect product of the braces B_1 and B_2 via τ and will be denoted $B_1 \rtimes_{\tau} B_2$. If τ is the trivial morphism, then $(B_1 \times B_2, +, \cdot)$ is called the direct product of B_1 and B_2 .

We recall that, for a left brace $(B, +, \cdot)$ and each $a \in B$, we have a bijective map $\lambda_a : B \rightarrow B$ defined by $\lambda_a(b) = -a + a \cdot b$ which satisfies $\lambda_a(b + c) = \lambda_a(b) + \lambda_a(c)$, $a \cdot b = a + \lambda_a(b)$, $\lambda_{a \cdot b} = \lambda_a \circ \lambda_b$, for any a, b, c in B .

Left braces have been classified for sizes p^2, p^3 , for p a prime number ([3]); pq and p^2q , for p and q odd prime numbers ([1, 2, 4, 9]); $2p^2$, for p an odd prime number ([6]); $8p$, for p an odd prime number $\neq 3, 7$ ([7]) and for $12p$, for p an odd prime number ≥ 7 ([8]). In this paper we consider relatively prime integer numbers m and n such that each solvable group of order mn has a normal subgroup of order m . We prove that each brace of size mn is a semidirect product of a brace of size m and a brace of size n . We further give a method to classify braces of size mn from the classification of braces of sizes m and n . This is a generalization of the result obtained in [8] in the case in which m is prime. We apply our result to describe all braces of size p^2q^2 , for p and q odd primes satisfying $q > p, q \geq 5, p \mid q - 1, p \nmid q + 1, p^2 \nmid q - 1$. We note that these conditions hold in particular when p is an odd Germain prime and $q = 2p + 1$.

2. Left braces of size mn , for $\text{gcd}(m, n) = 1$

In this section we consider relatively prime integer numbers m and n and assume that each solvable group of order mn has a normal subgroup of order m . We prove that each brace of order mn is a semidirect product $B_1 \rtimes_{\tau} B_2$, where B_1 is a brace of size m , B_2 is a brace of size n and $\tau : (B_2, \cdot) \rightarrow \text{Aut}(B_1, +, \cdot)$ is a group morphism. Moreover, given such B_1 and B_2 , we determine when two group morphisms $\sigma, \tau : (B_2, \cdot) \rightarrow \text{Aut}(B_1, +, \cdot)$ provide isomorphic braces.

Theorem 2.1. *Let m and n be relatively prime integer numbers such that each solvable group of order mn has a normal subgroup of order m . Then each brace of size mn is a semidirect product of a brace of size m and a brace of size n .*

Proof. Let $(B, +, \cdot)$ be a brace of size mn . Let B_1 and B_2 be its unique additive subgroups of size m and n , respectively. In particular B_1 and B_2 are characteristic subgroups in $(B, +)$. Since, for each $a \in B$, λ_a is an automorphism of $(B, +)$, it leaves B_1 and B_2 setwise invariant. This implies that, for $a, b \in B_1$, we have $ab = a + \lambda_a(b) \in B_1$, as $\lambda_a(b) \in B_1$. Similarly, this can be applied to B_2 . So, B_1 and B_2 are subbraces of B and B_1 and B_2 are complements of one another. Let $a \in B_1$ and $b \in B_2$, then

$$ba = {}^b a b \Rightarrow b + \lambda_b(a) = {}^b a + \lambda_{b a}(b).$$

Since the multiplicative group of a brace is always solvable (see [5] Theorem 5.2), our hypothesis implies that (B_1, \cdot) is a normal subgroup of (B, \cdot) , hence ${}^b a \in B_1$. Using again that the λ -action leaves B_2 setwise invariant, we obtain $\lambda_{{}^b a}(b) \in B_2$. A comparison of the components shows ${}^b a = \lambda_b(a)$, i.e. under the λ -action, (B_2, \cdot) acts by automorphisms of $(B_1, +)$ and (B_1, \cdot) , that is, by brace automorphisms. Analogously

$$ab = ba^b \Rightarrow a + \lambda_a(b) = b + \lambda_b(a^b),$$

where $\lambda_a(b) \in B_2, \lambda_b(a^b) \in B_1$. Comparing components, we obtain $\lambda_a(b) = b$. Therefore $ab = a + \lambda_a(b) = a + b$ for $a \in B_1, b \in B_2$. Also, $ba = {}^b a + \lambda_{{}^b a}(b) = {}^b a + b = \tau_b(a) + b$ for an action $\tau : B_2 \rightarrow \text{Aut}(B_1)$.

Finally, for $a, a' \in B_1; b, b' \in B_2$, we have

$$\begin{aligned} (a + b)(a' + b') &= ab(a' + b') = a(ba' - b + bb') = a(\tau_b(a') + bb') \\ &= a\tau_b(a') - a + a(bb') = a\tau_b(a') + bb', \end{aligned}$$

where we have used the brace condition in the second and fourth equalities. Hence

$$B \rightarrow B_1 \rtimes_{\tau} B_2; a + b \mapsto (a, b)$$

is indeed a brace morphism. □

We want to see now when two semidirect products of braces B_1 and B_2 of coprime orders are isomorphic.

Proposition 2.2. *Let B_1, B_2 be braces with $\gcd(|B_1|, |B_2|) = 1$. Consider semidirect products $B_{\sigma} := B_1 \rtimes_{\sigma} B_2, B_{\tau} := B_1 \rtimes_{\tau} B_2$, for morphisms $\sigma, \tau : (B_2, \cdot) \rightarrow \text{Aut}(B_1, +, \cdot)$. An isomorphism $h : B_{\sigma} \rightarrow B_{\tau}$ is of the form (h_1, h_2) , where $h_i \in \text{Aut}(B_i), i = 1, 2$, and h_1 and h_2 satisfy*

$$\tau h_2 = h_1 \sigma.$$

Proof. The coprimality of $|B_1|$ and $|B_2|$ implies that the B_i are subbraces of B_{σ} and B_{τ} and furthermore, $(B_1, +)$ (respectively $(B_2, +)$) is the only subgroup of order m (respectively n) in $(B_{\sigma}, +)$ and $(B_{\tau}, +)$. Hence an isomorphism $h : B_{\sigma} \rightarrow B_{\tau}$ is of the form (h_1, h_2) , where $h_i \in \text{Aut}(B_i), i = 1, 2$. For $a, a' \in B_1, b, b' \in B_2$, we have

$$h((a, b) \cdot (a', b')) = h(a\sigma(b)(a'), bb') = (h_1(a\sigma(b)(a')), h_2(bb'))$$

and

$$\begin{aligned} h(a, b) \cdot h(a', b') &= (h_1(a), h_2(b)) \cdot (h_1(a'), h_2(b')) \\ &= (h_1(a)\tau(h_2(b))(h_1(a')), h_2(b)h_2(b')). \end{aligned}$$

We obtain

$$h_1(\sigma(b)(a') = \tau(h_2(b))(h_1(a')).$$

Replacing a' by $h_1^{-1}(a')$ results in the equation

$$h_1(\sigma(b)(h_1^{-1}(a')) = \tau(h_2(b))(a').$$

As a' and b are arbitrary, this implies

$$\tau h_2 = h_1 \sigma.$$

□

3. Braces of size p^2 , for p an odd prime number

In [3] Bachiller obtained the classification of braces of sizes p^2 and p^3 , up to isomorphism, for p a prime number. We recall it for braces $(B, +, \cdot)$ of size p^2 , for p odd. We note that in this case (B, \cdot) is isomorphic to $(B, +)$. For each brace, we give the group of brace automorphisms and an explicit isomorphism from (B, \cdot) to $(B, +)$.

3.1. $(B, +) \simeq \mathbf{Z}/(p^2)$. There are two braces, up to isomorphism, with additive group isomorphic to $\mathbf{Z}/(p^2)$, the trivial one and a brace with \cdot defined by

$$x_1 \cdot x_2 = x_1 + x_2 + px_1x_2.$$

In both cases, $(B, \cdot) \simeq \mathbf{Z}/(p^2)$. In the trivial case, we have

$$\text{Aut } B = \text{Aut}(\mathbf{Z}/(p^2)) \simeq (\mathbf{Z}/(p^2))^*.$$

In the nontrivial case, we have

$$\text{Aut } B = \{k \in (\mathbf{Z}/(p^2))^* : k \equiv 1 \pmod{p}\}$$

and an isomorphism from (B, \cdot) into $\mathbf{Z}/(p^2)$ is given by $n \mapsto n - pn(n-1)/2$.

3.2. $(B, +) \simeq \mathbf{Z}/(p) \times \mathbf{Z}/(p)$. We write the elements in $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ in vector form. There are two braces, up to isomorphism, with additive group isomorphic to $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$, the trivial one and a brace with \cdot defined by

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + y_1y_2 \\ y_1 + y_2 \end{pmatrix}.$$

In both cases, $(B, \cdot) \simeq \mathbf{Z}/(p) \times \mathbf{Z}/(p)$. In the trivial case, we have

$$\text{Aut } B = \text{Aut}(\mathbf{Z}/(p) \times \mathbf{Z}/(p)) \simeq \text{GL}(2, p).$$

In the nontrivial case, we have

$$\text{Aut } B = \left\{ \begin{pmatrix} d^2 & b \\ 0 & d \end{pmatrix} : b \in \mathbf{Z}/(p), d \in (\mathbf{Z}/(p))^* \right\}$$

and an isomorphism from (B, \cdot) into $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - y(y-1)/2 \\ y \end{pmatrix}.$$

4. Groups of order p^2q^2

We assume now that p and q are primes satisfying $p > 2$, $q > p$ and $q \geq 5$. These hypotheses imply that a group G of order p^2q^2 has a unique normal q -Sylow subgroup S_q of order q^2 . Indeed, the number n_q of q -Sylow subgroups of G satisfies $n_q \in \{1, p, p^2\}$ and $n_q \equiv 1 \pmod{q}$. Clearly $q \nmid p-1$ and $q \mid p^2-1$ implies $q \mid p-1$ or $q \mid p+1$ but, if $q > p$, the second condition holds only for $p=2$ and $q=3$. We obtain that a group of order p^2q^2 is the semidirect product of a normal subgroup S_q of order q^2 and a subgroup S_p of order p^2 . It is then determined by a group G_1 of order q^2 , a group G_2 of order p^2 and a morphism $\tau : G_2 \rightarrow \text{Aut}(G_1)$. We note that triples (G_1, G_2, τ) and (G'_1, G'_2, τ') provide isomorphic groups of order p^2q^2 if and only if there exist isomorphisms $f : G_1 \rightarrow G'_1, g : G_2 \rightarrow G'_2$ such that ${}^f\tau = \tau'g$. The groups of order p^2q^2 may then be described by determining the equivalence classes of morphisms $\tau : G_2 \rightarrow \text{Aut}(G_1)$ under the relation

$$\tau \sim \tau' \Leftrightarrow \exists (f, g) \in \text{Aut } G_1 \times \text{Aut } G_2 : {}^f\tau = \tau'g.$$

Let us further assume that p and q satisfy $p \mid q-1, p \nmid q+1$ and $p^2 \nmid q-1$. If $G_1 \simeq \mathbf{Z}/(q^2)$ then $\text{Aut } G_1 \simeq (\mathbf{Z}/(q^2))^* \simeq \mathbf{Z}/q(q-1)$. The assumptions $p \mid q-1$ and $p^2 \nmid q-1$ imply that $\text{Aut } G_1$ contains a unique subgroup of order p but no subgroup of order p^2 . If $G_1 \simeq \mathbf{Z}/(q) \times \mathbf{Z}/(q)$, then $\text{Aut } G_1 \simeq \text{GL}(2, q)$ and $|\text{GL}(2, q)| = (q+1)q(q-1)^2$. The assumptions $p \mid q-1, p \nmid q+1$ and $p^2 \nmid q-1$ imply that $\text{Aut } G_1$ contains elements of order p but no element of order p^2 .

Since τ and ${}^f\tau$, for $f \in \text{GL}(2, q)$, give isomorphic groups of order p^2q^2 , we need to determine the subgroups of order p of $\text{GL}(2, q)$, up to conjugation. This is done in the following lemma which is easy to prove.

Lemma 4.1. *For λ a fixed generator of the unique subgroup of order p of $(\mathbf{Z}/(q))^*$, a system of representatives of the conjugation classes of subgroups of order p of $\text{GL}(2, q)$ is*

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^k \end{pmatrix} \right\rangle, \quad (4.1)$$

for k running over a system of representatives of elements of $(\mathbf{Z}/(p))^*$, different from 1 and -1 , under the relation $k \sim \ell$ if and only if $k\ell \equiv 1 \pmod{p}$.

The number of subgroups of order p of $\text{GL}(2, q)$ up to conjugation is then $(p+3)/2$.

We may now describe the groups of order p^2q^2 for primes p and q satisfying the following conditions.

$$q > p, p > 2, q \geq 5, p \mid q-1, p \nmid q+1, p^2 \nmid q-1. \quad (4.2)$$

Lemma 4.2. *Let p and q satisfying (4.2). Let G be a group of order p^2q^2 and let us denote by S_q the unique q -Sylow subgroup of G .*

1) *Assume $S_q \simeq \mathbf{Z}/(q^2)$ and let α denote a fixed generator of the unique subgroup of order p of $(\mathbf{Z}/(q^2))^*$. In this case, G is isomorphic to one of the following groups.*

1.1) $\mathbf{Z}/(p^2q^2)$;

1.2) $\mathbf{Z}/(q^2) \rtimes \mathbf{Z}/(p^2)$ with product given by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 + \alpha^{y_1}x_2, y_1 + y_2);$$

1.3) $\mathbf{Z}/(pq^2) \times \mathbf{Z}/(p)$;

1.4) $\mathbf{Z}/(q^2) \rtimes (\mathbf{Z}/(p) \times \mathbf{Z}/(p))$ with product given by

$$(x_1, \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}) \cdot (x_2, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}) = (x_1 + \alpha^{y_1}x_2, \begin{pmatrix} y_1+y_2 \\ z_1+z_2 \end{pmatrix}).$$

2) *Assume $S_q \simeq \mathbf{Z}/(q) \times \mathbf{Z}/(q)$ and let λ denote a fixed generator of the unique subgroup of order p of $(\mathbf{Z}/(q))^*$. In this case, G is isomorphic to one of the following groups.*

2.1) $\mathbf{Z}/(p^2q) \times \mathbf{Z}/(q)$;

2.2) *one of the $(p+3)/2$ groups $(\mathbf{Z}/(q) \times \mathbf{Z}/(q)) \rtimes_M \mathbf{Z}/(p^2)$ with product given by*

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, z_1 \right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, z_2 \right) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + M^{z_1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, z_1 + z_2 \right),$$

where M denotes one of the matrices in (4.1).

2.3) $\mathbf{Z}/(pq) \times \mathbf{Z}/(pq)$;

2.4) *one of the $(p+3)/2$ groups $(\mathbf{Z}/(q) \times \mathbf{Z}/(q)) \rtimes_M (\mathbf{Z}/(p) \times \mathbf{Z}/(p))$, with product given by*

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix} \right) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + M^{z_1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_1+z_2 \\ t_1+t_2 \end{pmatrix} \right),$$

where M denotes one of the matrices in (4.1);

2.5) $(\mathbf{Z}/(q) \times \mathbf{Z}/(q)) \rtimes_\lambda (\mathbf{Z}/(p) \times \mathbf{Z}/(p))$ with product given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix} \right) = \left(\begin{pmatrix} x_1 + \lambda^{t_1}x_2 \\ y_1 + \lambda^{z_1+t_1}y_2 \end{pmatrix}, \begin{pmatrix} z_1+z_2 \\ t_1+t_2 \end{pmatrix} \right).$$

5. Left braces of size p^2q^2

In this section we consider primes p and q satisfying the conditions in (4.2). At the beginning of Section 4, we have seen that, under these assumptions, $m = q^2$ and $n = p^2$ satisfy the conditions in Theorem 2.1. Hence, every brace of size p^2q^2 is the semidirect product of a brace B_1 of size q^2 and a brace B_2 of size p^2 . We use the description of braces of order p^2 recalled in Section 3 and Proposition 2.2 to determine all braces of size p^2q^2 , for p and q satisfying the conditions (4.2). We note that, in particular, these conditions are satisfied when p is an odd Germain prime and $q = 2p + 1$.

For the description of the multiplicative groups of the braces of size p^2q^2 given below we shall use the explicit isomorphism from (B_2, \cdot) to $(B_2, +)$ given in Sections 3.1 and 3.2, respectively. Using these isomorphisms, one may prove that the description of the action of $\text{Aut } B_2$ on (B_2, \cdot) looks the same as its action on $(B_2, +)$ (see [9] Lemma 7).

5.1. $(B_1, +) = \mathbf{Z}/(q^2)$ and $(B_2, +) = \mathbf{Z}/(p^2)$. In this section we describe braces of size p^2q^2 whose additive law is given by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \quad (5.1)$$

for $x_1, y_1 \in B_1; x_2, y_2 \in B_2$.

5.1.1. B_1 trivial brace. In this case, $\text{Aut } B_1 = (\mathbf{Z}/(q^2))^*$. Since $\text{Aut } B_1$ is abelian, $h_1\tau = \tau$, for every morphism τ from (B_2, \cdot) to $\text{Aut } B_1$.

The morphisms from $\mathbf{Z}/(p^2)$ to $\text{Aut } B_1$ are τ_i defined by $1 \mapsto \alpha^i$, for α a fixed generator of the unique subgroup of order p of $\text{Aut } B_1$, $0 \leq i \leq p-1$, where $i=0$ corresponds to the trivial morphism.

If B_2 is trivial, for $h_2 \in \text{Aut } B_2$ defined by $h_2(1) = i$, with $p \nmid i$, we have $\tau_i = \tau_1 h_2$. We obtain then two braces, the first one is the direct product of B_1 and B_2 , with multiplicative law given by

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \quad (5.2)$$

and the second one has multiplicative law given by

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 + \alpha^{x_2} y_1, x_2 + y_2), \quad (5.3)$$

for $x_1, y_1 \in B_1; x_2, y_2 \in B_2; \alpha$ a fixed element of order p of $(\mathbf{Z}/(q^2))^*$.

If B_2 is nontrivial, $\text{Aut } B_2 = \{k \in (\mathbf{Z}/(p^2))^* : k \equiv 1 \pmod{p}\}$ and, for the morphisms τ_i defined above we have $\tau_i h_2 = \tau_i$, for each $h_2 \in \text{Aut } B_2$. We obtain p braces, including the direct product one. Taking into account the isomorphism from (B_2, \cdot) into $\mathbf{Z}/(p^2)$ given in Section 3.1 and that α has order p , their multiplicative laws are given by

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 + \alpha^{ix_2} y_1, x_2 + y_2 + px_2 y_2), \quad (5.4)$$

for $x_1, y_1 \in B_1; x_2, y_2 \in B_2; i = 0, \dots, p-1; \alpha$ a fixed element of order p of $(\mathbf{Z}/(q^2))^*$.

5.1.2. B_1 nontrivial brace. In this case, $\text{Aut } B_1 = \{k \in (\mathbf{Z}/(q^2))^* : k \equiv 1 \pmod{q}\} \simeq \mathbf{Z}/(q)$. Then the unique morphism τ from $(B_2, \cdot) \simeq \mathbf{Z}/(p^2)$ to $\text{Aut } B_1$ is the trivial one. We obtain two braces which are direct products of B_1 and B_2 , where B_2 is either trivial or nontrivial. Their multiplicative laws are given by

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 + y_1 + qx_1 y_1, x_2 + y_2), \quad (5.5)$$

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 + y_1 + qx_1 y_1, x_2 + y_2 + px_2 y_2), \quad (5.6)$$

for $x_1, y_1 \in B_1; x_2, y_2 \in B_2$.

Summing up, we have obtained the following result.

Theorem 5.1. *Let p and q be primes satisfying $q > p, q \geq 5, p \mid q - 1, p \nmid q + 1$ and $p^2 \nmid q - 1$. There are $p + 4$ braces with additive group $\mathbf{Z}/(p^2q^2)$. Four of them have multiplicative group $\mathbf{Z}/(p^2q^2)$ and the remaining p have multiplicative group $\mathbf{Z}/(q^2) \rtimes \mathbf{Z}/(p^2)$.*

5.2. $(B_1, +) = \mathbf{Z}/(q^2)$ and $(B_2, +) = \mathbf{Z}/(p) \times \mathbf{Z}/(p)$. In this section we describe braces of size p^2q^2 whose additive law is given by

$$(x_1, \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}) + (x_2, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}) = (x_1 + x_2, \begin{pmatrix} y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}), \quad (5.7)$$

for $x_1, x_2 \in B_1; \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \in B_2$.

5.2.1. B_1 trivial brace. In this case, $\text{Aut } B_1 \simeq (\mathbf{Z}/(q^2))^*$. Since $\text{Aut } B_1$ is abelian, we have $h_1\tau = \tau$, for every morphism τ from G_2 to $\text{Aut } B_1$ and $h_1 \in \text{Aut } B_1$.

If B_2 is trivial, every nontrivial morphism $\tau : \mathbf{Z}/(p) \times \mathbf{Z}/(p) \rightarrow (\mathbf{Z}/(q^2))^*$ is equal to $\tau_0 h_2$, for $h_2 \in \text{Aut } B_2 \simeq \text{GL}(2, p)$ and τ_0 defined by $\tau_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha, \tau_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$, for α a fixed element of order p in $(\mathbf{Z}/(q^2))^*$. We obtain one brace whose multiplicative law is given by

$$(x_1, \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}) \cdot (x_2, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}) = (x_1 + \alpha^{y_1} x_2, \begin{pmatrix} y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}), \quad (5.8)$$

where α is an element of order p in $\text{Aut } B_1$. Besides, we have the direct product of B_1 and B_2 with multiplicative law given by

$$(x_1, \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}) \cdot (x_2, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}) = (x_1 + x_2, \begin{pmatrix} y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}), \quad (5.9)$$

If B_2 is nontrivial, $\text{Aut } B_2 = \{ \begin{pmatrix} d^2 & b \\ 0 & d \end{pmatrix} : d \in (\mathbf{Z}/(p))^*, b \in \mathbf{Z}/(p) \}$. Every nontrivial morphism τ from $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ to $\text{Aut } B_1$ is equal to $\tau_0 g$, for $g \in \text{GL}(2, p)$ and τ_0 defined by $\tau_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha, \tau_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$, for α a fixed element of order p in $\text{Aut } B_1$. By computation, we obtain that, for $g_1, g_2 \in \text{GL}(2, p)$, we have $\tau_0 g_1 = \tau_0 g_2$ if and only if the first rows of g_1 and g_2 are equal. We obtain then that the set of nontrivial morphisms τ from $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ to $\text{Aut } B_1$ is precisely $\{ \tau_0 \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in (\mathbf{Z}/(p))^*, b \in \mathbf{Z}/(p) \} \cup \{ \tau_0 \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix} : b \in (\mathbf{Z}/(p))^* \}$. Now, for $\tau := \tau_0 \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \tau' := \tau_0 \begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix}$, there exists $h_2 \in \text{Aut } B_2$ such that $\tau' h_2 = \tau$ if and only if a'/a is a square; for $\tau := \tau_0 \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}, \tau' := \tau_0 \begin{pmatrix} 0 & b' \\ 1 & 0 \end{pmatrix}$, there always exists $h_2 \in \text{Aut } B_2$ such that $\tau' h_2 = \tau$; for $\tau := \tau_0 \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \tau' := \tau_0 \begin{pmatrix} 0 & b' \\ 1 & 0 \end{pmatrix}$, there exists no $h_2 \in \text{Aut } B_2$ such that $\tau' h_2 = \tau$. We obtain then three braces. By considering the isomorphism from (B_2, \cdot) into $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ given in Section 3.2, their multiplicative laws are given by

$$(x_1, \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}) \cdot (x_2, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}) = (x_1 + \alpha^{y_1 - z_1(z_1 - 1)/2} x_2, \begin{pmatrix} y_1 + y_2 + z_1 z_2 \\ z_1 + z_2 \end{pmatrix}), \quad (5.10)$$

$$(x_1, \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}) \cdot (x_2, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}) = (x_1 + \alpha^{a(y_1 - z_1(z_1 - 1)/2)} x_2, \begin{pmatrix} y_1 + y_2 + z_1 z_2 \\ z_1 + z_2 \end{pmatrix}), \quad (5.11)$$

and

$$(x_1, \binom{y_1}{z_1}) \cdot (x_2, \binom{y_2}{z_2}) = (x_1 + \alpha^{z_1} x_2, \binom{y_1 + y_2 + z_1 z_2}{z_1 + z_2}), \quad (5.12)$$

respectively, where α is a fixed element of order p in $\text{Aut } B_1$ and a is a fixed quadratic nonresidue modulo p . Besides, we have the direct product of B_1 and B_2 with multiplicative law given by

$$(x_1, \binom{y_1}{z_1}) \cdot (x_2, \binom{y_2}{z_2}) = (x_1 + x_2, \binom{y_1 + y_2 + z_1 z_2}{z_1 + z_2}), \quad (5.13)$$

5.2.2. B_1 nontrivial brace. In this case, $\text{Aut } B_1 = \{k \in (\mathbf{Z}/(q^2))^* : k \equiv 1 \pmod{q}\} \simeq \mathbf{Z}/(q)$. Then the unique morphism τ from $G_2 \simeq \mathbf{Z}/(p) \times \mathbf{Z}/(p)$ to $\text{Aut } B_1$ is the trivial one. We obtain then just two braces which are the direct product of B_1 and B_2 , corresponding to B_2 trivial and B_2 nontrivial. Their multiplicative laws are given by

$$(x_1, \binom{y_1}{z_1}) \cdot (x_2, \binom{y_2}{z_2}) = (x_1 + x_2 + qx_1x_2, \binom{y_1 + y_2}{z_1 + z_2}), \quad (5.14)$$

$$(x_1, \binom{y_1}{z_1}) \cdot (x_2, \binom{y_2}{z_2}) = (x_1 + x_2 + qx_1x_2, \binom{y_1 + y_2 + z_1 z_2}{z_1 + z_2}), \quad (5.15)$$

Summing up, we have obtained the following result.

Theorem 5.2. *Let p and q be primes satisfying $q > p$, $q \geq 5$, $p \mid q - 1$, $p \nmid q + 1$ and $p^2 \nmid q - 1$. There are eight braces with additive group $\mathbf{Z}/(pq^2) \times \mathbf{Z}/(p)$. Four of them have multiplicative group $\mathbf{Z}/(pq^2) \times \mathbf{Z}/(p)$ and the remaining four have multiplicative group $\mathbf{Z}/(q^2) \rtimes (\mathbf{Z}/(p) \times \mathbf{Z}/(p))$.*

5.3. $(B_1, +) = \mathbf{Z}/(q) \times \mathbf{Z}/(q)$ and $(B_2, +) = \mathbf{Z}/(p^2)$. In this section we describe braces of size p^2q^2 whose additive law is given by

$$\left(\binom{x_1}{y_1}, z_1\right) + \left(\binom{x_2}{y_2}, z_2\right) = \left(\binom{x_1 + x_2}{y_1 + y_2}, z_1 + z_2\right), \quad (5.16)$$

for $\left(\binom{x_1}{y_1}, z_1\right), \left(\binom{x_2}{y_2}, z_2\right) \in B_1, z_1, z_2 \in B_2$.

5.3.1. B_1 trivial brace. In this case, $\text{Aut } B_1 = \text{GL}(2, q)$. Every morphism from $\mathbf{Z}/(p^2)$ to $\text{Aut } B_1 = \text{GL}(2, q)$ is equal to $h_1 \tau$ for some $h_1 \in \text{Aut } B_1$ and τ defined by $\tau(1) = M^\ell$ for M one of the matrices in (4.1) and $1 \leq \ell \leq p - 1$.

If B_2 is trivial, $\text{Aut } B_2 = \text{Aut } \mathbf{Z}/(p^2)$. For $\tau : \mathbf{Z}/(p^2) \rightarrow \text{Aut } B_1$ defined by $\tau(1) = M$ and $h_2 \in \text{Aut } \mathbf{Z}/(p^2)$, we have $\tau h_2(1) = M^{h_2(1)}$. Hence for morphisms τ, τ' with $\tau(1) = M$ and $\tau'(1) = M^\ell$, one has $\tau \sim \tau'$. We have then one brace for each conjugation class of subgroups of order p in $\text{GL}(2, q)$. We obtain $(p + 3)/2$ braces, whose multiplicative laws are given by

$$\left(\binom{x_1}{y_1}, z_1\right) \cdot \left(\binom{x_2}{y_2}, z_2\right) = \left(\binom{x_1}{y_1} + M^{z_1} \binom{x_2}{y_2}, z_1 + z_2\right), \quad (5.17)$$

for M one of the matrices in (4.1). Besides, we obtain the direct product of B_1 and B_2 whose multiplicative law is given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, z_1\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, z_2\right) = \left(\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}, z_1+z_2\right), \quad (5.18)$$

If B_2 is nontrivial, we have $\text{Aut } B_2 = \{k \in (\mathbf{Z}/(p^2))^* : k \equiv 1 \pmod{p}\}$. Since a nontrivial morphism τ from (B_2, \cdot) to $\text{Aut } B_1$ sends 1 to an element of order p , we have $\tau h_2 = \tau$ for $h_2 \in \text{Aut } B_2$. As noted above, a nontrivial morphism τ from $\mathbf{Z}/(p^2)$ to $\text{Aut } B_1$ is equal to ${}^{h_1}\tau$ for some $h_1 \in \text{Aut } B_1$ and τ defined by $\tau(1) = M^\ell$ for M one of the matrices in (4.1) and $1 \leq \ell \leq p-1$. Let us see if for some $\ell \in \{2, \dots, p-1\}$ and some matrix M in (4.1), the matrices M and M^ℓ are conjugate by some element in $\text{GL}(2, q)$. This is so only for $M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $\ell = p-1$. In this case, there are $p-1$ braces for each matrix M different from $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $(p-1)/2$ for this last one. By considering the isomorphism from (B_2, \cdot) into $\mathbf{Z}/(p^2)$ given in Section 3.1 and taking into account that M denotes a matrix of order p , we obtain $\frac{p+1}{2}(p-1) + \frac{p-1}{2} = \frac{(p-1)(p+2)}{2}$ braces whose multiplicative laws are given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, z_1\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, z_2\right) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + M^{\ell z_1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, z_1+z_2+pz_1z_2\right), \quad (5.19)$$

for M one of the matrices in (4.1) and with $1 \leq \ell \leq p-1$, for $M \neq \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$; $1 \leq \ell \leq (p-1)/2$, for $M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

Besides, we obtain the direct product of B_1 and B_2 whose multiplicative law is given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, z_1\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, z_2\right) = \left(\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}, z_1+z_2+pz_1z_2\right), \quad (5.20)$$

5.3.2. B_1 nontrivial brace. If B_1 is nontrivial,

$$\text{Aut } B_1 = \left\{ \left(\begin{pmatrix} d^2 & b \\ 0 & d \end{pmatrix} : b \in \mathbf{Z}/(q), d \in (\mathbf{Z}/(q^2))^* \right) \right\}.$$

The matrices of order p in $\text{Aut } B_1$ are conjugate to some diagonal matrix of the form $\begin{pmatrix} d^2 & 0 \\ 0 & d \end{pmatrix}$ with d an element of order p in $(\mathbf{Z}/(q))^*$. For λ a chosen element of order p in $(\mathbf{Z}/(q))^*$, the morphisms τ from $\mathbf{Z}/(p^2)$ to $\text{Aut } B_1$ are given by $\tau(1) = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix}^\ell$, for $1 \leq \ell \leq p-1$. We note that $\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^k \end{pmatrix}^2$, with $k = (p+1)/2$.

If B_2 is trivial, for $\tau : \mathbf{Z}/(p^2) \rightarrow \text{Aut } B_1$ defined by $\tau(1) = M$, we have $\tau h_2(1) = M^{h_2(1)}$. Hence for morphisms τ, τ' with $\tau(1) = M$ and $\tau'(1) = M^\ell$, one has $\tau \sim \tau'$. We may then reduce to the case where $\tau(1) = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix}$ and we obtain one brace whose multiplicative law is given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, z_1\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, z_2\right) = \left(\begin{pmatrix} x_1+\lambda^{2z_1}x_2+\lambda^{2z_1}x_1x_2 \\ y_1+\lambda^{z_1}y_2 \end{pmatrix}, z_1+z_2\right). \quad (5.21)$$

Besides, we have the direct product whose multiplicative law is given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, z_1\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, z_2\right) = \left(\begin{pmatrix} x_1+x_2+x_1x_2 \\ y_1+y_2 \end{pmatrix}, z_1+z_2\right). \quad (5.22)$$

If B_2 is nontrivial, we have $\text{Aut } B_2 = \{k \in (\mathbf{Z}/(p^2))^* : k \equiv 1 \pmod{p}\}$, as above. For $h_2 \in \text{Aut } B_2$ and $\tau : (B_2, \cdot) \rightarrow \text{Aut } B_1$, we have $\tau h_2 = \tau$. We

obtain then $p - 1$ braces. By considering again the isomorphism from (B_2, \cdot) into $\mathbf{Z}/(p^2)$ given in Section 3.1 and taking into account that $\tau(1)$ is a matrix of order p , their multiplicative laws are given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, z_1 \right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, z_2 \right) = \left(\begin{pmatrix} x_1 + \lambda^{2\ell z_1} x_2 + \lambda^{2\ell z_1} x_1 x_2 \\ y_1 + \lambda^{\ell z_1} y_2 \end{pmatrix}, z_1 + z_2 + pz_1 z_2 \right), \quad (5.23)$$

where λ is a fixed element of order p in $(\mathbf{Z}/(q))^*$ and $1 \leq \ell \leq p - 1$. Besides, we have the direct product whose multiplicative law is given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, z_1 \right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, z_2 \right) = \left(\begin{pmatrix} x_1 + x_2 + x_1 x_2 \\ y_1 + y_2 \end{pmatrix}, z_1 + z_2 + pz_1 z_2 \right). \quad (5.24)$$

Summing up, we have obtained the following result.

Theorem 5.3. *Let p and q be primes satisfying $q > p, q \geq 5, p \mid q - 1, p \nmid q + 1$ and $p^2 \nmid q - 1$. There are $(p^2 + 4p + 9)/2$ braces with additive group $\mathbf{Z}/(p^2q) \times \mathbf{Z}/(q)$.*

- There are four such braces with multiplicative group $\mathbf{Z}/(p^2q) \times \mathbf{Z}/(q)$;
- for each of the matrices M in (4.1) different from $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{(p+1)/2} \end{pmatrix}$, there are p such braces with multiplicative group $(\mathbf{Z}/(q) \times \mathbf{Z}/(q)) \rtimes_M \mathbf{Z}/(p^2)$;
- for $M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, there are $(p + 1)/2$ such braces with multiplicative group $(\mathbf{Z}/(q) \times \mathbf{Z}/(q)) \rtimes_M \mathbf{Z}/(p^2)$;
- for $M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{(p+1)/2} \end{pmatrix}$, there are $2p$ such braces with multiplicative group $(\mathbf{Z}/(q) \times \mathbf{Z}/(q)) \rtimes_M \mathbf{Z}/(p^2)$.

5.4. $(B_1, +) = \mathbf{Z}/(q) \times \mathbf{Z}/(q)$ and $(B_2, +) = \mathbf{Z}/(p) \times \mathbf{Z}/(p)$. In this section we describe braces of size p^2q^2 whose additive law is given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix} \right) + \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix} \right) = \left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}, \begin{pmatrix} z_1 + z_2 \\ t_1 + t_2 \end{pmatrix} \right), \quad (5.25)$$

for $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in B_1; \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix} \in B_2$.

5.4.1. B_1 trivial brace. In this case, $\text{Aut } B_1 = \text{GL}(2, q)$. A nontrivial morphism τ from $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ to $\text{Aut } B_1$ either has an order p kernel or is injective. In the first case, it is equal to $h_1 \tau$ for some $h_1 \in \text{Aut } B_1$ and τ defined by $\tau(u) = M, \tau(v) = \text{Id}$, for some $\mathbf{Z}/(p)$ -basis (u, v) of $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$, where M is one of the matrices in (4.1). In the second case, it is equal to $h_1 \tau$ for some $h_1 \in \text{Aut } B_1$ and τ defined by $\tau(u) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \tau(v) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, for an element λ of order p in $(\mathbf{Z}/(q))^*$ and some basis (u, v) of $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$. Indeed, all subgroups of order p^2 of $\text{GL}(2, q)$ are conjugate, as they are the p -Sylow subgroups of $\text{GL}(2, q)$, and $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ and $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ are a basis of the subgroup of order p^2 whose elements are diagonal matrices.

If B_2 is trivial, we have $\text{Aut } B_2 = \text{GL}(2, p)$. For τ defined by $\tau(u) = M, \tau(v) = \text{Id}$, for some $\mathbf{Z}/(p)$ -basis (u, v) of $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$, we have $\tau = \tau_0 h_2$, for h_2 defined by $h_2(u) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, h_2(v) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and τ_0 defined by $\tau_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = M, \tau_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \text{Id}$. We obtain then $(p + 3)/2$ braces whose multiplicative laws are given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix}\right) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + M^{z_1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_1+z_2 \\ t_1+t_2 \end{pmatrix}\right), \quad (5.26)$$

for M one of the matrices in (4.1). In the case when τ is injective, for an adequate h_2 , we have $\tau = \tau_0 h_2$, for τ_0 defined by $\tau_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, $\tau_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, where λ is a fixed element of order p in $(\mathbf{Z}/(q))^*$. We obtain then one brace whose multiplicative law is given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix}\right) = \left(\begin{pmatrix} x_1 + \lambda^{t_1} x_2 \\ y_1 + \lambda^{z_1 + t_1} y_2 \end{pmatrix}, \begin{pmatrix} z_1 + z_2 \\ t_1 + t_2 \end{pmatrix}\right), \quad (5.27)$$

for λ a fixed element of order p in $(\mathbf{Z}/(q))^*$. Besides, we have the direct product, whose multiplicative law is given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix}\right) = \left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}, \begin{pmatrix} z_1 + z_2 \\ t_1 + t_2 \end{pmatrix}\right). \quad (5.28)$$

If B_2 is nontrivial, we have $\text{Aut } B_2 = \left\{ \begin{pmatrix} d & b \\ 0 & d \end{pmatrix} : d \in (\mathbf{Z}/(p))^*, b \in \mathbf{Z}/(p) \right\}$, as in Section 5.2.1. Now every morphism τ from $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ to $\text{Aut } B_1$ with an order p kernel is equal to $\tau_0 g$, for $g \in \text{GL}(2, p)$ and τ_0 defined by $\tau_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = M$, $\tau_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \text{Id}$, for M one of the matrices in (4.1). Similarly as in Section 5.2.1, we obtain that the set of nontrivial morphisms τ from $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ to $\text{Aut } B_1$ is precisely $\left\{ \tau_0 \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in (\mathbf{Z}/(p))^*, b \in \mathbf{Z}/(p) \right\} \cup \left\{ \tau_0 \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix} : b \in (\mathbf{Z}/(p))^* \right\}$. Moreover, again as in Section 5.2.1, under the relation

$$\tau \sim \tau' \Leftrightarrow \exists h_2 \in \text{Aut } B_2 : \tau' h_2 = \tau,$$

we are left with τ_0 , $\tau_0 \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, for $a \in (\mathbf{Z}/(p))^*$ a non-square element, and $\tau_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Now, if $M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, the matrices M and M^{-1} are conjugate by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{GL}(2, q) = \text{Aut } B_1$. Hence, for $h_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have $h_1 \tau_0 = \tau_0 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ which implies that, if -1 is not a square in $\mathbf{Z}/(p)$, then the orbits corresponding to τ_0 and $\tau_0 \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ coincide. We obtain then two braces corresponding to $M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and three corresponding to the other matrices. Summing up, there are $(3/2)(p+3)$ braces if $p \equiv 1 \pmod{4}$ and $(3/2)(p+3) - 1$ braces if $p \equiv 3 \pmod{4}$. Taking into account the isomorphism from (B_2, \cdot) into $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ given in Section 3.2, the corresponding multiplicative laws are given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix}\right) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + M^{z_1 - t_1(t_1-1)/2} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_1 + z_2 + t_1 t_2 \\ t_1 + t_2 \end{pmatrix}\right), \quad (5.29)$$

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix}\right) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + M^{a(z_1 - t_1(t_1-1)/2)} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_1 + z_2 + t_1 t_2 \\ t_1 + t_2 \end{pmatrix}\right), \quad (5.30)$$

and

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix}\right) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + M^{t_1} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_1 + z_2 + t_1 t_2 \\ t_1 + t_2 \end{pmatrix}\right), \quad (5.31)$$

respectively, where M is one of the matrices in (4.1) and a is a fixed quadratic nonresidue modulo p with the exception that, for $p \equiv 3 \pmod{4}$ and $M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, the braces with multiplicative laws (5.29) and (5.30) are isomorphic.

As established above, an injective morphism τ from $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ to $\text{Aut } B_1$ is equal to $h_1\tau$ for some $h_1 \in \text{GL}(2, q)$ and τ defined by $\tau(u) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, $\tau(v) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, for an element λ of order p in $(\mathbf{Z}/(q))^*$ and some $\mathbf{Z}/(p)$ -basis (u, v) of $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$. A transversal of $\text{Aut } B_2$ in $\text{GL}(2, p)$ is

$$\left\{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} : a \in (\mathbf{Z}/(p))^*, c \in \mathbf{Z}/(p) \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ 1 & 0 \end{pmatrix} : c \in (\mathbf{Z}/(p))^* \right\},$$

hence any injective morphism τ from $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ to $\text{Aut } B_1$ is equivalent under the relation in Proposition 2.2 either to $\tau_{a,c} = \tau_0 h_2$ for $h_2 = \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix}$ for some $a \in (\mathbf{Z}/(p))^*, c \in \mathbf{Z}/(p)$ or to $\tau_c = \tau_0 h_2$ for $h_2 = \begin{pmatrix} 0 & c \\ 1 & 0 \end{pmatrix}$ for some $c \in (\mathbf{Z}/(p))^*$, where τ_0 is defined by $\tau_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, $\tau_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. Now the normalizer of $\langle \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \rangle$ in $\text{GL}(2, q)$ consists of diagonal and anti-diagonal matrices. Conjugation by a diagonal matrix leaves diagonal matrices fixed and for an anti-diagonal h_1 we have $h_1 \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, $h_1 \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. We obtain then $h_1 \tau_{a,c} = \tau_{-a, a+c}$, for $h_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and no further equivalences. This gives $(p(p-1)/2) + p - 1 = (p^2 + p - 2)/2$ braces. With λ an element of order p in $(\mathbf{Z}/(q))^*$, and taking into account the isomorphism from (B_2, \cdot) into $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ given in Section 3.2, their multiplicative laws are given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix} \right) = \left(\begin{pmatrix} x_1 + \lambda^{(z_1 - t_1(t_1 - 1)/2)(a+c) + t_1} x_2 \\ y_1 + \lambda^{(z_1 - t_1(t_1 - 1)/2)c + t_1} y_2 \end{pmatrix}, \begin{pmatrix} z_1 + z_2 + t_1 t_2 \\ t_1 + t_2 \end{pmatrix} \right), \quad (5.32)$$

for some $(a, c) \in (\mathbf{Z}/(p))^* \times \mathbf{Z}/(p)$ where the braces corresponding to (a, c) and $(-a, a+c)$ are isomorphic, and

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix} \right) = \left(\begin{pmatrix} x_1 + \lambda^{z_1 - t_1(t_1 - 1)/2} x_2 \\ y_1 + \lambda^{z_1 - t_1(t_1 - 1)/2 + ct_1} y_2 \end{pmatrix}, \begin{pmatrix} z_1 + z_2 + t_1 t_2 \\ t_1 + t_2 \end{pmatrix} \right), \quad (5.33)$$

for some $c \in (\mathbf{Z}/(p))^*$. Besides, we have the direct product of B_1 and B_2 with multiplicative law given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix} \right) = \left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}, \begin{pmatrix} z_1 + z_2 + t_1 t_2 \\ t_1 + t_2 \end{pmatrix} \right), \quad (5.34)$$

5.4.2. B_1 nontrivial brace. In this case, $\text{Aut } B_1 = \left\{ \begin{pmatrix} d^2 & b \\ 0 & d \end{pmatrix} : d \in (\mathbf{Z}/(q))^*, b \in \mathbf{Z}/(q) \right\} \subset \text{GL}(2, q)$. Since the only subgroup of order p of $\text{Aut } B_1$ is $\langle \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix} \rangle$, for $\lambda \in (\mathbf{Z}/(q))^*$ of order p , a nontrivial morphism τ from $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ to $\text{Aut } B_1$ has an order p kernel and is defined by $\tau(u) = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix}$, $\tau(v) = \text{Id}$, for some $\mathbf{Z}/(p)$ -basis (u, v) of $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$.

If B_2 is trivial, $\text{Aut } B_2 = \text{GL}(2, p)$. For τ defined by $\tau(u) = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix}$, $\tau(v) = \text{Id}$, for some basis (u, v) of $(B_2, \cdot) = \mathbf{Z}/(p) \times \mathbf{Z}/(p)$, we have $\tau = \tau_0 h_2$, for h_2 defined by $h_2(u) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $h_2(v) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and τ_0 defined by $\tau_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix}$, $\tau_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \text{Id}$. We obtain then one brace, whose multiplicative law is given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix} \right) = \left(\begin{pmatrix} x_1 + \lambda^{2z_1} x_2 + \lambda^{z_1} y_1 y_2 \\ y_1 + \lambda^{z_1} y_2 \end{pmatrix}, \begin{pmatrix} z_1 + z_2 \\ t_1 + t_2 \end{pmatrix} \right), \quad (5.35)$$

for λ a fixed element of order p in $(\mathbf{Z}/(q))^*$. Besides, we have the direct product whose multiplicative law is given by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix}\right) = \left(\begin{pmatrix} x_1+x_2+y_1y_2 \\ y_1+y_2 \end{pmatrix}, \begin{pmatrix} z_1+z_2 \\ t_1+t_2 \end{pmatrix}\right). \quad (5.36)$$

If B_2 is nontrivial, $\text{Aut } B_2 = \left\{ \begin{pmatrix} d^2 & b \\ 0 & d \end{pmatrix} : d \in (\mathbf{Z}/(p))^*, b \in \mathbf{Z}/(p) \right\} \subset \text{GL}(2, p)$. As in Section 5.2.1, we obtain that the set of nontrivial morphisms τ from $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ to $\text{Aut } B_1$ is precisely

$$\{\tau_0 \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in (\mathbf{Z}/(p))^*, b \in \mathbf{Z}/(p)\} \cup \{\tau_0 \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix} : b \in (\mathbf{Z}/(p))^*\},$$

for τ_0 defined by $\tau_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix}$, $\tau_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \text{Id}$. Again, under the relation in Proposition 2.2, we have three orbits corresponding to the matrices Id , $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, for a non-square, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We obtain then three braces. Taking into account the isomorphism from (B_2, \cdot) into $\mathbf{Z}/(p) \times \mathbf{Z}/(p)$ given in Section 3.2, their multiplicative laws are given by

$$\begin{aligned} & \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix}\right) \\ &= \left(\begin{pmatrix} x_1 + \lambda^{2(z_1-t_1(t_1-1)/2)} x_2 + \lambda^{z_1-t_1(t_1-1)/2} y_1 y_2 \\ y_1 + \lambda^{z_1-t_1(t_1-1)/2} y_2 \end{pmatrix}, \begin{pmatrix} z_1+z_2+t_1 t_2 \\ t_1+t_2 \end{pmatrix}\right), \end{aligned} \quad (5.37)$$

$$\begin{aligned} & \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix}\right) \\ &= \left(\begin{pmatrix} x_1 + (a\lambda^2)^{(z_1-t_1(t_1-1)/2)} x_2 + \lambda^{z_1-t_1(t_1-1)/2} y_1 y_2 \\ y_1 + \lambda^{z_1-t_1(t_1-1)/2} y_2 \end{pmatrix}, \begin{pmatrix} z_1+z_2+t_1 t_2 \\ t_1+t_2 \end{pmatrix}\right), \end{aligned} \quad (5.38)$$

for a fixed quadratic nonresidue a modulo p , and

$$\begin{aligned} & \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix}\right) \\ &= \left(\begin{pmatrix} x_1 + \lambda^{(z_1-t_1(t_1-1)/2)} x_2 + \lambda^{2(z_1-t_1(t_1-1)/2)} y_1 y_2 \\ y_1 + \lambda^{2(z_1-t_1(t_1-1)/2)} y_2 \end{pmatrix}, \begin{pmatrix} z_1+z_2+t_1 t_2 \\ t_1+t_2 \end{pmatrix}\right). \end{aligned} \quad (5.39)$$

Besides, we have the direct product with multiplicative law defined by

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_2 \\ t_2 \end{pmatrix}\right) = \left(\begin{pmatrix} x_1+x_2+y_1y_2 \\ y_1+y_2 \end{pmatrix}, \begin{pmatrix} z_1+z_2+t_1 t_2 \\ t_1+t_2 \end{pmatrix}\right). \quad (5.40)$$

Summing up, we have obtained the following result.

Theorem 5.4. *Let p and q be primes satisfying $q > p$, $q \geq 5$, $p \mid q-1$, $p \nmid q+1$ and $p^2 \nmid q-1$. There are $\frac{p^2+5p}{2}+14$ (resp. $\frac{p^2+5p}{2}+13$) braces with additive group $\mathbf{Z}/(pq) \times \mathbf{Z}/(pq)$ if $p \equiv 1 \pmod{4}$ (resp. if $p \equiv 3 \pmod{4}$).*

- There are four of them with multiplicative group $\mathbf{Z}/(pq) \times \mathbf{Z}/(pq)$;
- for each of the matrices M in (4.1) different from $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{(p+1)/2} \end{pmatrix}$, there are four of them with multiplicative group $(\mathbf{Z}/(q) \times \mathbf{Z}/(q)) \rtimes_M (\mathbf{Z}/(p) \times \mathbf{Z}/(p))$;

- c) for $M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, there are four (resp. three) such braces with multiplicative group $(\mathbf{Z}/(q) \times \mathbf{Z}/(q)) \rtimes_M (\mathbf{Z}/(p) \times \mathbf{Z}/(p))$, if $p \equiv 1 \pmod{4}$ (resp. if $p \equiv 3 \pmod{4}$);
- d) for $M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{(p+1)/2} \end{pmatrix}$, there are eight of them with multiplicative group $(\mathbf{Z}/(q) \times \mathbf{Z}/(q)) \rtimes_M (\mathbf{Z}/(p) \times \mathbf{Z}/(p))$;
- e) there are $(p^2 + p)/2$ of them with multiplicative group $(\mathbf{Z}/(q) \times \mathbf{Z}/(q)) \rtimes_\lambda (\mathbf{Z}/(p) \times \mathbf{Z}/(p))$.

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