New York Journal of Mathematics

New York J. Math. **31** (2025) 321–367.

Overfare of harmonic functions on Riemann surfaces

Eric Schippers and **Wolfgang Staubach**

ABSTRACT. This is the first in a series of four papers developing a scattering theory for harmonic functions/one-forms on Riemann surfaces. In this paper we prove the following. Let $\mathscr{R}$ be a compact Riemann surface split into two surfaces Σ_1 and Σ_2 by a complex of quasicircles. Given a harmonic function with L^2 derivatives on one of the pieces Σ_1, there is a unique harmonic function with L^2 derivatives on the other piece Σ_2 with the same boundary values as the original function in a certain conformally invariant non-tangential sense. We call the new harmonic function the overfare of the original function. This overfare map is well-defined and bounded with respect to Dirichlet semi-norm provided that Σ_1 is connected. For Weil-Petersson quasicircles, it is bounded with respect to the Sobolev H^1-norm.

1. Introduction

1.1. Statement of results and literature.

This paper is the first part of a longer work [29] establishing a scattering theory of one-forms on Riemann surfaces, which we have divided into four parts. A non-technical exposition of some aspects of this scattering theory can be found in [33]. The scattering process starts by dividing a Riemann surface into two pieces (which themselves may not be connected) by a collection of Jordan curves. Alternatively, we can think of a compact surface obtained by sewing together several surfaces, and the Jordan curves are the seams. Then one considers functions or one-forms which are separately harmonic on the pieces, and share boundary values on the seams. The function (or one-form) on one side

Received December 12, 2023.

2010 *Mathematics Subject Classification.* 14F40, 30F15, 30F30, 35P99, 51M15.

Key words and phrases. Overfare operator, scattering, bordered surfaces, quasicircles, bounded zero mode quasicircles, conformally nontangential limits, conformal Sobolev spaces.

ISSN 1076-9803/2025

of the surface is obtained from the function (or one-form) on the other side using a mapping which we refer to as "overfare", which is the backbone of this particular kind of scattering theory. In order that the results be applicable to Teichmüller theory and conformal field theory, it is necessary that these seams can be quasicircles. Furthermore, much evidence exists that quasicircles are analytically natural for the scattering theory [28].

This first paper develops the analytic theory of harmonic functions which share boundary values on the seams. In the second paper [30], we develop the theory of shared boundary values for one-forms, dealing with analytic and cohomological issues which arise. These results are in turn applied in two papers developing the global analysis and geometry of the scattering process in terms of integral operators of Schiffer [31, 32]. We prove index theorems for these operators and it is shown that this scattering process is unitary. Also, deep connections with both classical period matrices in geometry, and the Plemelj-Sokhotski jump formula in function theory are illustrated.

Returning to the paper at hand, let $\mathscr{R}$ be a Riemann surface split into two pieces Σ_1 and Σ_2 by a Jordan curve or complex of curves. Given a harmonic function with L^2 derivatives on one of the pieces Σ_1 (a Dirichlet harmonic function), we find its boundary values. The "overfare" is the harmonic function on the other piece Σ_2 with the same boundary values as the original function. This is well-defined and bounded provided that the curves in the complex are quasicircles. Because the seams are quasicircles, which in general can be highly irregular, there are many obstacles to overcome, even in the formulation of the statement. Nevertheless, in the end one obtains a clear and clean picture.

The **main results** are that the overfare process is well-defined (Theorem 3.11) and is bounded with respect to the Dirichlet semi-norm (Theorem 3.43) — that is, with respect to the L^2 norm on derivatives. We also establish boundedness with control of constants in the case of Weil-Petersson class quasicircles (Theorem 3.42).

The results of this paper extend those in the case of the plane and Riemann surfaces split by a single curve. These results have had applications to approximation theory in the plane [24] and on Riemann surfaces [35], conformal field theory [22, 21], and global analysis of integral operators on Riemann surfaces [27]. For a survey in the plane see [28].

For further applications to Teichmüller theory and conformal field theory, it was necessary to extend the overfare results to more general configurations of sewn surfaces which arise there – one requires for example surfaces with many boundary curves and disks sewn on; self-sewn surfaces; and surfaces sewn along many curves. The purpose of this paper is to prove overfare results sufficient to apply to these general cases. In [31, 32], the results of the present paper together with [30] are applied to derive a unitary scattering theory; characterize solutions to the holomorphic boundary value problems for one-forms; generalize the Grunsky inequalities to collections of maps into compact Riemann surfaces of genus g; and derive a generalized period map for bordered Riemann surfaces which unifies the classical period map for compact surfaces with the Kirillov–Yuri'ev–Nag–Sullivan period mapping of universal Teichmüller space. The results of these sequels to this paper have also been used in approximation theory of holomorphic one-forms on Riemann surfaces [34].

The method of proof involves a surgery technique, the core of which is that a certain "bounce operator" is bounded (Theorem 3.24). This operator takes harmonic functions on collars of boundary curves of a surface, to harmonic functions on the entire surface with the same boundary values. This allows us to use the boundedness of overfare in the plane to prove boundedness on Riemann surfaces through "cutting and pasting". Indeed this technique is itself of interest and should have further applications.

To obtain these results, there are two analytic problems to be resolved. The first is to define the boundary values in preparation for overfare, and the second is to show the existence and continuous dependence of the overfare. The first problem is in a certain sense independent of the boundary regularity, while the second problem is more delicate and sensitive to the regularity of the curve.

In defining the boundary values, the nature of the approach to the boundary can be defined either extrinsically in terms of the geometry of the ambient space containing the curve, or in terms of the intrinsic geometry of the region on which the function is defined. For example, since harmonic functions with L^2 derivatives are in the Sobolev space H^1 for a wide class of curves, one could consider the Sobolev trace to the boundary; in this case, one would need to take into account the regularity of the boundary for this to be defined. The possibility of dealing with boundaries that may not be rectifiable would add additional difficulties that brings one into the realm of geometric measure theory see [13], [14]. Instead, our approach to boundary values proceeds intrinsically, viewing the boundary as the ideal boundary of Σ_1, which does not depend on the geometry of the boundary in $\mathscr{R}$. Indeed, it can be regarded as an analytic Jordan curve in the double of Σ_1.

Our intrinsic approach to boundary values in some sense originates with H. Osborn [17], who considered the boundary values of harmonic Dirichlet functions in planar domains Σ_1 along orthogonal trajectories of Green's function of that domain. This is conformally invariant and hence intrinsic, and can be formulated in terms of the ideal boundary. We improve this "radial" approach by defining a kind of conformally non-tangential boundary value (referred to as CNT boundary values), in which non-tangential cones are defined in terms of "collar charts" taking collar neighbourhoods of the boundary to annuli. Then, a classical theorem of A. Beurling applies to show that the boundary values exist except on a Borel set of logarithmic capacity zero in the circle under the chart (we call this a null set). We show that this notion of boundary value is independent of the choice of collar chart; this is essentially because the angle of approach to the ideal boundary is a well-defined conformal invariant. Thus we show that the boundary values are defined not just along orthogonal trajectories of Green's function but along any non-tangentially approaching curve. The independence of the boundary values on the choice of collar chart is a key tool in the application of the cutting and sewing approach to boundary value problems which we have developed in this and other papers [26], [27].

The overfare process, on the other hand, is extrinsic: the regularity of the boundary curve is a crucial issue. We work with quasicircles; there are several reasons for this choice. The first is geometric: at a foundational level, Teichmüller theory of bordered surfaces involves viewing these surfaces as subsets of compact surfaces bounded by quasicircles. Classically, this is seen in the quasi-Fuschsian model of Teichmüller space [15]; for example, the universal Teichmüller space can be viewed as the set of (normalized) planar domains bounded by quasicircles. The first author's work with D. Radnell [19], [20] also shows that the Teichmüller space can be modelled via a set of punctured compact surfaces capped by domains bounded by quasicircles, and that this leads to a natural fibre structure on Teichmüller space. This model and fibre structure has its roots in ideas of conformal field theory. Thus, in this work, we choose quasicircles in order to have sufficient generality in order to provide the groundwork for applying our results to Teichmüller theory and conformal field theory.

The second reason for choosing quasicircles is analytic. The authors showed in [25] that in the Riemann sphere, the overfare exists and is bounded precisely for quasicircles. This follows from a theorem of Nag-Sullivan/Vodopy'anov that shows that quasisymmetries are precisely the bounded composition operators on the homogeneous Sobolev space $\dot{H}^{1/2}$ on the circle. As we will see in the third paper, this also relates to several characterizations of quasicircles in terms of the Cauchy-type and Schiffer integral operators which play the main role in this paper. A survey of such results in the Riemann sphere can be found in [28].

It should also be noted that the Sobolev theory techniques by themselves are not sufficient in dealing with all aspects of the boundary value problems that are involved in this paper, since Sobolev spaces involve functions defined up to sets of Lebesgue measure zero. In fact, one needs to establish that boundary values exist up to a set which maps under a collar chart to a Borel set of logarithmic capacity zero in the unit circle. We call such sets null sets. By our earlier results, for quasicircles, a set which is null with respect to a collar chart on one side of the curve must be null with respect to a collar chart on the other side. This fact is central to establishing a well-defined overfare of harmonic functions. However, the claim fails if in the discussion above one replaces capacity zero with Lebesgue measure zero on the circles. Thus Sobolev theory on its own is not sufficient.

As described above, in this paper we extend our previous overfare results to Riemann surfaces divided by many curves, rather than just a single curve. There is an obstacle to doing so. If the region Σ_2 is bounded by several curves, but Σ_1 is not connected, then the Dirichlet seminorm is not controlled by the Dirichlet norm of the input. This is because one may add different constants to different connected components of Σ_1, driving up the seminorm of the overfare, while the Dirichlet norm on the originating surface is unchanged. If the originating surface is connected, this issue does not arise, and we are able to prove boundedness of overfare with respect to the Dirichlet seminorm.

One can also obtain boundedness with respect to a genuine norm if more regularity is assumed. We introduce a conformally invariant Sobolev space (Definition 3.20) with a norm in which, rather than adding the L^2 norm of the function in the case of classical Sobolev spaces, we add an integral of the function around a boundary curve. With no connectivity assumptions, we obtain boundedness of overfare with respect to this conformally invariant norm, for curves with greater regularity. It suffices that the quasicircles are so-called Weil-Petersson quasicircles. For both of these results, in this paper we use a more flexible method of proof than in [26], which makes systematic use of boundedness of the so-called bounce operator (see Definition 3.23).

Acknowledgements. The first author was partially supported by the National Sciences and Engineering Research Council of Canada. The second author is grateful to Andreas Strömbergsson for partial financial support through a grant from Knut and Alice Wallenberg Foundation. Finally we would like to thank the referee for suggestions which improved the presentation of the paper.

1.2. Outline of the paper. In Section 2 we gather the preliminary material about Riemann surfaces, their boundaries, and spaces of harmonic and holomorphic functions and forms. Section 3 defines the conformally non-tangential boundary values of Dirichlet bounded harmonic functions, and proves the existence and boundedness of the overfare map. Conformally non-tangential boundary values are defined in Subsection 3.2. The so-called anchor lemmas (certain boundary integrals are independent of the collar charts) can be found as Lemmas 3.14 and 3.15. The boundedness of the "bounce operator" is dealt with in Subsection 3.5. The boundedness of composition operators associated to WP-class quasisymmetries in Sobolev $H^{1/2}$ space of the unit circle, is proven in Subsection 3.6. Finally the boundedness of the the overfare operator for general quasicircles in proven in Subsection 3.6 as well.

2. Riemann surfaces, harmonic measures and Green's function

2.1. About this section. This section gathers the definitions and basic results used throughout the paper. This includes Dirichlet spaces of functions and Bergman spaces of forms; Riemann surfaces, their boundaries and specialized charts called collar charts; sewing; Green's functions on compact surfaces and surfaces with boundary; Sobolev spaces; and harmonic measures and boundary period matrices.

2.2. Bordered surfaces. We briefly recall the definition of a bordered surface in order to remove any ambiguity. See for example [1] for a complete treatment.

In what follows we denote by $\mathbb{A}_{a,b}$ the annulus $\{z \,:\, a < |z| < b\}$. Let $\mathbb{C}$ denote the complex plane, and $\bar{\mathbb{C}}$ denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Let $\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\}$ denote the open unit disk, and let $\mathbb{H} = \{z \in \mathbb{C} \,:\, \operatorname{Im} z > 0\}$ denote the upper half plane. Let $\operatorname{cl}(\mathbb{H})$ denote its closure. In general we will let cl denote closure throughout. Let $\mathbb{S}^1 = \{z \in \mathbb{C} \,:\, |z| = 1\}$ denote the unit circle.

Definition 2.1. We say that a connected Hausdorff topological space $\hat{\Sigma}$ is a *bordered Riemann surface* if there is an atlas of charts $\phi \,:\, U \to \operatorname{cl}(\mathbb{H})$ with the following properties.

(1) Each chart is a local homeomorphism with respect to the relative topology;
(2) Every point in $\hat{\Sigma}$ is contained in the domain of some chart;
(3) Given any pair of charts $\phi_k \,:\, U_k \to \operatorname{cl}(\mathbb{H})$, $k = 1, 2$, if $U_1 \cap U_2$ is non-empty, then $\phi_1 \circ \phi_2^{-1}$ is a biholomorphism on $U_1 \cap U_2 \cap \mathbb{H}$.

This defines a distinction between interior and border points (see e.g. [1, pp 23–24]). That is, we say p is on the border if there is a chart in the atlas such that $\phi(p)$ is on the real axis, and p is in the interior if there a chart mapping p to a point in $\mathbb{H}$. In either case, if the claim holds for one chart, it holds for all of them. We will denote the set of interior points by Σ and the set of border points by $\partial\Sigma$. We call $\partial\Sigma$ the border, and note that the border is also the topological boundary of Σ in $\hat{\Sigma}$. Observe that Σ is a Riemann surface in the standard sense.

We will call a chart ϕ which contains a boundary point in its domain a "boundary chart" or "border chart". Let Σ^d be the double of Σ, which as a set consists of $\hat{\Sigma}$ together with a set of conjugate points $\tilde{z}$ for each $z \in \Sigma$; for points on the border, $\tilde{z} = z$. An atlas is constructed as follows. Assume that $\phi_k \,:\, U_k \to \operatorname{cl}(\mathbb{H})$, $k = 1, 2$, are charts such that $U_1 \cap U_2 \cap \partial\Sigma$ is non-empty. Then by the Schwarz reflection principle, $\phi_1 \circ \phi_2^{-1}$ extends to a biholomorphism of an open set containing $\phi_2(U_1 \cap U_2)$. This open set can be taken to be the union of $\phi_2(U_1 \cap U_2)$ with its reflection in the real axis. In the usual construction of the double, any chart ϕ which contains border points can be extended to a chart in the double by reflection. By the above argument, the overlap map $\phi_1 \circ \phi_2^{-1}$ is a biholomorphism for any pair ϕ_1, ϕ_2 of such extensions. Charts which do not contain boundary points in their domain have conjugate charts with domains in $\tilde{\Sigma}$. This defines the atlas on the double of Σ^d of Σ.

Remark 2.2. Once the border structure is established as above, for convenience we will allow interior charts to have image in $\mathbb{C}$ and not necessarily in $\mathbb{H}$. In particular, we will also consider border charts which map into the closure of the disk $\mathbb{D} = \{z \in \bar{\mathbb{C}} : |z| < 1\}$, with border points mapping to $|z| = 1$. Every such chart is a border chart in the original sense after composition by a Möbius transformation.

We consider bordered Riemann surfaces with n simple closed borders and g handles. The precise definition is the following.

Definition 2.3. We say that Σ is a *bordered Riemann surface of type* (g, n), if it is bordered (in the sense Definition 2.1), the border has n connected components, each of which is homeomorphic to $\mathbb{S}^1$, and its double Σ^d is a compact surface of genus $2g + n - 1$.

Visually, a bordered surface of type (g, n) is a g-handled surface bounded by n simple closed curves. We order the borders and label them accordingly, so that $\partial\Sigma = \partial_1\Sigma \cup \cdots \cup \partial_n\Sigma$. The borders can be identified with analytic curves in the double Σ^d, and we can view the union $\hat{\Sigma} = \Sigma \cup \partial\Sigma$ as $\mathrm{cl}(\Sigma)$ where the closure is taken in the double Σ^d.

Finally, we observe that borders are conformally invariant. That is, if Σ_1 and Σ_2 are bordered surfaces, then any biholomorphism $f : \Sigma_1 \to \Sigma_2$ extends to a homeomorphism of the borders. In fact, f extends to a biholomorphism between the doubles Σ_1^d and Σ_2^d which takes $\partial\Sigma_1$ to $\partial\Sigma_2$. Finally, if only one of the two surfaces has a border, say Σ_1, then one can endow Σ_2 with a border using f. In particular, there is a unique maximal border structure.

Remark 2.4. Note that if Σ has type (g, n), the border structure is maximal, since Σ^d is a compact surface.

Definition 2.5. We say that a homeomorphic image Γ of $\mathbb{S}^1$ is a *strip-cutting Jordan curve* if it is contained in an open set U and there is a biholomorphism $\phi : U \to \mathbb{A}_{r,R}$ for some $r < 1 < R$, in such a way that $\phi(\Gamma)$ is isotopic to $\mathbb{S}^1$. We call U a doubly-connected neighbouhood of Γ and ϕ a doubly-connected chart.

Remark 2.6. If Γ is a strip-cutting curve, by shrinking $\mathbb{A}_{r,R}$, we can assume that (1) ϕ extends biholomorphically to an open neighourhood of $\mathrm{cl}\,(U)$, (2), that the boundary curves of U are themselves strip cutting (in fact analytic Jordan curves), and (3) that Γ is isotopic to each of the boundary curves (using ϕ^{-1} to provide the isotopy).

Remark 2.7. An analytic Jordan curve is by definition strip-cutting.

Throughout the paper we consider *nested Riemann surfaces*. That is, we are given a type (g, n) bordered surface Σ, another Riemann surface $\mathscr{R}$ which is compact, and a holomorphic inclusion map $\iota(\Sigma) \subset \mathscr{R}$. Assume that the closure of Σ is compact in $\mathscr{R}$, and furthermore the boundary consists of n closed strip-cutting Jordan curves, which do not intersect. In that case, the inclusion map ι extends homeomorphically to a map from the border to the strip-cutting Jordan curves. Thus $\partial\Sigma$ is in one-to-one correspondence with its image under the homeomorphic extension of ι, and the image is the boundary of $\iota(\Sigma)$ in the ordinary topological sense. For this reason, we will not notationally distinguish Σ from $\iota(\Sigma)$, and use the notation $\partial\Sigma$ for both the boundary of $\iota(\Sigma)$ in $\mathscr{R}$ and the abstract border of Σ.

In fact, if the boundary of Σ is a strip-cutting Jordan curve, it is also a border.

Theorem 2.8. *Let Σ be an open connected subset of a Riemann surface $\mathscr{R}$. Assume that the topological boundary of Σ in $\mathscr{R}$ is a finite collection $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$ of strip-cutting Jordan curves. Furthermore suppose that there are doubly-connected charts $\phi_k : U_k \to \mathbb{A}_k$ of Γ_k for $k = 1, \ldots, n$ (where $\mathbb{A}_k$'s are annuli) such that the closures of U_k are mutually disjoint, and $U_k \backslash \Gamma$ consists of two connected components, one of which is entirely contained in Σ and one which is in $\mathscr{R} \backslash \Sigma$. Then Σ is a bordered surface and the inclusion map is a homeomorphism.*

Proof. First, observe that Σ has a unique complex structure compatible with $\mathscr{R}$, so we let $\mathscr{A}$ be an atlas compatible with this structure.

Let U_k^+ denote the component of $U_k \backslash \Gamma$ in Σ. Then $\phi_k(U_k^+)$ is an open subset of $\mathbb{C}$ bounded by two Jordan curves, one of which is a boundary γ of $\mathbb{A}_k$ and one of which is the Jordan curve $\phi_k(\Gamma)$. By [7, Theorems 3.3, 3.4 Sect 15.3], there is a biholomorphism $\psi_k : \phi_k(U_k^+) \to \mathbb{A}_{r,1}$ which extends to a homeomorphism of the boundaries, taking γ to $|z| = r$ and $\phi_k(\Gamma)$ to $\mathbb{S}^1$. Adjoining the points in Γ_k to Σ, Then

$$\mathscr{A} \cup \left\{ \psi_1 \circ \phi_1|_{U_1^+}, \ldots, \psi_n \circ \phi_n|_{U_n^+} \right\}$$

is an atlas making $\Sigma \cup \partial\Sigma$ into a bordered surface. □

Remark 2.9. The embedding of the border $\partial\Sigma$ in $\mathscr{R}$ need not be regular. That is, the inclusion map does not extend to a smooth or analytic map from $\partial\Sigma$ onto its image under inclusion ι, unless the image consists of smooth or analytic curves.

By another application of [7, Theorems 3.3, 3.4 Sect 15.3], it is easily shown that if $\Sigma_1 \subset \mathscr{R}_1$ and $\Sigma_2 \subset \mathscr{R}_2$ satisfy the conditions above, and $f : \Sigma_1 \to \Sigma_2$ is a biholomorphism, then f extends continuously to a map taking each Jordan curve in $\partial\Sigma_1$ homeomorphically to one of the Jordan curves of $\partial\Sigma_2$.

It is helpful to have the following distinction in mind throughout the paper: certain statements are "intrinsic" while others are "extrinsic". Intrinsic statements about a Riemann surface Σ are those which depend only on the surface itself and are unchanged under a biholomorphism. For example, the border is intrinsic, and the harmonic function which is one on $\partial_k\Sigma$ and 0 on other curves is intrinsic. Extrinsic statements about a Riemann surfaces Σ nested in another surface $\mathscr{R}$, are those which make reference to $\mathscr{R}$. For example, "strip-cutting" is an extrinsic property, as is the regularity of $\iota(\partial\Sigma)$. An example of an extrinsic object is the restriction of Green's function of $\mathscr{R}$ to Σ (see the next subsection for the definition of Green's functions).

When dealing with intrinsically phrased boundary value problems, regularity of the boundary is not an issue, since we can treat the boundary as a border and thus we have its analytic structure at our disposal. Examples of this are the

Dirichlet problem (here with less regular data) and CNT boundary values of Dirichlet bounded harmonic functions on Σ in Section 3.2. On the other hand, when dealing with extrinsically phrased boundary value problems, regularity of the boundary is a major concern. Overfare, in which the boundary values of a harmonic function on Σ become data for the Dirichlet problem on $\mathscr{R}\backslash\Sigma$, is of this nature.

2.3. Collar charts. We define a kind of chart on bordered surfaces near the boundary, which we call a collar chart.

Definition 2.10. Let Σ be a bordered Riemann surface of type (g,n). A biholomorphism $\phi : U \to \mathbb{A}_{r,1}$ is called a *collar chart* of $\partial_k\Sigma$ (for some fixed k) if U is an open set in Σ bounded by two Jordan curves $\partial\Sigma$ and Γ, such that Γ is isotopic to $\partial_k\Sigma$ within the closure of U. A domain U is a collar neighbourhood of $\partial_k\Sigma$ if it is the domain of some collar chart.

Proposition 2.11. *Let Σ be a type (g,n) surface. Then every boundary curve $\partial_k\Sigma$ has a collar chart.*

Proof. Let Σ^d be the double of Σ, so that each boundary $\partial_k\Sigma$ is an analytic Jordan curve and hence strip-cutting. Let U_k, U_k^+, ϕ_k and ψ_k be as in the proof of Theorem 2.8. Then $\psi_k \circ \phi_k|_{U_k^+}$ is a collar chart. □

Furthermore, we have the following consequence of Carathéodory's theorem.

Theorem 2.12. *Let Σ be a bordered surface and Γ be a component of the border which is homeomorphic to $\mathbb{S}^1$. If $\phi : U \to \mathbb{A}$ is a collar chart, then ϕ extends continuously to $\partial_k\Sigma$. The extension is a homeomorphism of $\partial_k\Sigma$ onto $\mathbb{S}^1$.*

Proof. Γ is an analytic Jordan curve in the double, and hence strip-cutting. Let $\psi : V \to \mathbb{A}$ be a doubly-connected chart for Γ. By shrinking V we may assume that the boundaries of $\psi(V)$ are Jordan curves. Then $\psi \circ \phi^{-1}$ maps $\mathbb{A}$ onto a doubly-connected region bounded by Jordan curves, so the claim follows from [7, Theorems 3.4 Sect 15.3]. □

To keep the notation simple, we will also denote the continuous extension by ϕ.

Remark 2.13 (Isotopy and extension). By shrinking r, for any collar chart $\phi : U \to \mathbb{A}_{1,r}$ we can always assume that the inner boundary is an analytic curve and ϕ has an analytic extension to this curve. Furthermore, $H(t,\theta) = \phi^{-1}(e^t e^{i\theta})$ defines an isotopy between the level curve $|\phi| = r$ and $\partial_k\Sigma$, running through the level curves of $|\phi|$.

The homeomorphic extension is analytic on the border. This can be phrased in various ways, one of which is as follows. Treat Σ as a subset of its double Σ^d with involution $z \mapsto \tilde{z}$. For a collar neighbourhood U of $\partial_k\Sigma$, let $U^d = U \cup \tilde{U} \cup \partial_k\Sigma$. We then have

Proposition 2.14. *Let $\phi : U \to \mathbb{A}_{r,1}$ be a collar chart. Let $U^d = U \cup \tilde{U} \cup \partial_k\Sigma$ be the double of U. If Σ is included in its double Σ^d, then ϕ extends to a doubly-connected chart ϕ^d of $\partial_k\Sigma$ mapping U^d onto the annulus $\mathbb{A}_{r,1/r}$ satisfying $\phi^d(\tilde{z}) = 1/\overline{\phi^d(z)}$.*

Remark 2.15. In particular, the border charts give a well-defined meaning to continuous, C^k, and analytic functions on $\partial_k\Sigma$ for $k = 1, \dots, n$. Similarly it gives a meaning to continuous, C^k, and analytic vector fields or one-forms on the boundary. For example, a one-form α on $\partial_k\Sigma$ is continuous, C^k, or analytic if its expression in a boundary chart $\psi : U \to \mathbb{H}$ near p is $h(x)\,dx$ where h is continuous, C^k or analytic respectively, and this holds for all $p \in \partial_k\Sigma$. If the property holds for any collection of boundary charts covering $\partial_k\Sigma$ then it holds for all boundary charts. In particular, it is enough that the property in question holds for one collar chart; that is, α is continuous, C^k, or analytic if and only if in the local coordinates defined using $\phi|_{\partial_k\Sigma}$ for a collar chart ϕ, α is given by $h(e^{i\theta})\,d\theta$ where h is respectively continuous, C^k or analytic on $\mathbb{S}^1$.

Finally, we have the following useful fact.

Proposition 2.16. *Let Σ be a Riemann surface with border Γ homeomorphic to $\mathbb{S}^1$, and let U and V be collar neighbourhoods of a boundary curve $\partial_k\Sigma$. There is a collar chart $\phi : W \to \mathbb{A}_{r,1}$ such that $W \subseteq U \cap V$. Moreover r can be chosen so that the inner boundary of W is contained in $U \cap V$.*

Proof. By Remark 2.13 we can choose collar neighbourhoods U' and V' whose inner boundaries are analytic curves γ_1 and γ_2 contained in U and V, with corresponding collar charts $\psi_{U'}$ and $\psi_{V'}$ extending analytically to γ_1 and γ_2. By composing with $\psi_{U'}$, we can assume that $\Gamma = \mathbb{S}^1$, $\psi_{U'}(z) = z$, $U' = \mathbb{A}_{r,1}$ for some r, and $\gamma_1 = \{z : |z| = r\}$.

Now let M be the maximum value of $|\psi_{V'}(z)|$ on γ_2, which exists because γ_2 is compact. In that case $\mathrm{cl}\,(\mathbb{A}_{s,1}) \subseteq V' \cap U'$ for $s = (1 + M)/2$. We may now choose $W = \mathbb{A}_{s,1}$ and $\phi(z) = z$ to prove the claim. □

Proposition 2.17. *Let Γ be a strip-cutting Jordan curve in $\mathscr{R}$, and let $\phi : U \to \mathbb{A}_{r,R}$ be a doubly-connected chart. There are collar charts $\psi_k : U_k \to \mathbb{A}$ with $U_k \subseteq U \cap \Sigma_k$ for $k = 1, 2$. U_k may be chosen so that their inner boundaries are analytic curves contained in U.*

Proof. Applying the proof of Theorem 2.8 to each side of Γ we obtain the desired ψ_k. □

2.4. Function spaces and holomorphic and harmonic forms. In this paper, we will denote by C positive constants in the inequalities whose value is not crucial to the problem at hand. The value of C may differ from line to line, but in each instance could be estimated if necessary. Moreover, when the values of constants in our estimates are of no significance for our main purpose, then we use the notation $a \lesssim b$ as a shorthand for $a \leq Cb$. If $a \lesssim b$ and $b \lesssim a$ then we write $a \approx b$.

On any Riemann surface, define the dual of the almost complex structure, $*$ in local coordinates $z = x + iy$, by

$$* (a\,dx + b\,dy) = a\,dy - b\,dx.$$

This is independent of the choice of coordinates. It can also be computed in coordinates that for any complex function h

$$2\partial h = dh + i * dh. \tag{2.1}$$

Definition 2.18. We say a complex-valued function f on an open set U is *harmonic* if it is C^2 on U and $d * df = 0$. We say that a complex one-form α is harmonic if it is C^1 and satisfies both $d\alpha = 0$ and $d * \alpha = 0$.

Equivalently, harmonic one-forms are those which can be expressed locally as df for some harmonic function f. Harmonic one-forms and functions must of course be C^∞.

Denote complex conjugation of functions and forms with a bar, e.g. $\overline{\alpha}$. A holomorphic one-form is one which can be written in coordinates as $h(z)\,dz$ for a holomorphic function h, while an anti-holomorphic one-form is one which can be locally written $\overline{h(z)}\,d\bar{z}$ for a holomorphic function h.

Denote by $L^2(U)$ the set of one-forms ω on an open set U which satisfy

$$\iint_U \omega \wedge * \overline{\omega} < \infty$$

(observe that the integrand is positive at every point, as can be seen by writing the expression in local coordinates). This is a Hilbert space with respect to the inner product

$$(\omega_1, \omega_2) = \iint_U \omega_1 \wedge * \overline{\omega_2}. \tag{2.2}$$

Definition 2.19. The *Bergman space of holomorphic one forms* is

$$\mathcal{A}(U) = \{\alpha \in L^2(U) \;:\; \alpha \text{ holomorphic}\}. \tag{2.3}$$

The anti-holomorphic Bergman space is denoted $\overline{\mathcal{A}(U)}$. We will also denote

$$\mathcal{A}_{\text{harm}}(U) = \{\alpha \in L^2(U) \;:\; \alpha \text{ harmonic}\}. \tag{2.4}$$

Observe that $\mathcal{A}(U)$ and $\overline{\mathcal{A}(U)}$ are orthogonal with respect to the inner product (2.2). In fact we have the direct sum decomposition

$$\mathcal{A}_{\text{harm}}(U) = \mathcal{A}(U) \oplus \overline{\mathcal{A}(U)}. \tag{2.5}$$

If we restrict the inner product to $\alpha, \beta \in \mathcal{A}(U)$ then since $* \overline{\beta} = i\overline{\beta}$, we have

$$(\alpha, \beta) = i \iint_U \alpha \wedge \overline{\beta}.$$

Denote the projections induced by this decomposition by

$$\begin{aligned} \mathbf{P}_U &: \mathcal{A}_{\mathrm{harm}}(U) \to \mathcal{A}(U) \\ \overline{\mathbf{P}}_U &: \mathcal{A}_{\mathrm{harm}}(U) \to \overline{\mathcal{A}(U)}. \end{aligned} \tag{2.6}$$

Let $f : U \to V$ be a biholomorphism. We denote the pull-back of $\alpha \in \mathcal{A}_{\mathrm{harm}}(V)$ under f by $f^*\alpha$. Explicitly, if α is given in local coordinates w by $a(w)\,dw + \overline{b(w)}\,d\bar{w}$ and $w = f(z)$, then the pull-back is given by

$$f^*\left(a(w)\,dw + \overline{b(w)}\,d\bar{w}\right) = a(f(z))f'(z)\,dz + \overline{b(f(z))f'(z)}\,d\bar{z}.$$

The Bergman spaces are all conformally invariant, in the sense that if $f : U \to V$ is a biholomorphism, then $f^*\mathcal{A}(V) = \mathcal{A}(U)$ and f^* preserves the inner product. Similar statements hold for the anti-holomorphic and harmonic spaces.

Definition 2.20. We define the space $\mathcal{A}^{\mathrm{e}}_{\mathrm{harm}}(U)$ as the subspace of exact elements of $\mathcal{A}_{\mathrm{harm}}(U)$, and similarly for $\mathcal{A}^{\mathrm{e}}(\Sigma)$ and $\overline{\mathcal{A}^{\mathrm{e}}(\Sigma)}$.

We also consider one-forms which have zero boundary periods, which we call semi-exact.

Definition 2.21. Let Σ be a bordered surface of type (g,n). We say that an L^2 one-form $\alpha \in \mathcal{A}_{\mathrm{harm}}(\Sigma)$ is *semi-exact* if for any simple closed curve γ homotopic to a boundary curve $\partial_k\Sigma$,

$$\int_\gamma \alpha = 0.$$

The class of semi-exact one-forms on Σ is denoted $\mathcal{A}^{\mathrm{se}}_{\mathrm{harm}}(\Sigma)$. The holomorphic and anti-holomorphic semi-exact one-forms are denoted by $\mathcal{A}^{\mathrm{se}}(\Sigma)$ and $\overline{\mathcal{A}^{\mathrm{se}}(\Sigma)}$ respectively.

The following spaces also play significant roles in this paper.

Definition 2.22. The *Dirichlet spaces of functions* are defined by

$$\begin{aligned} \mathcal{D}_{\mathrm{harm}}(U) &= \{f : U \to \mathbb{C}, f \in C^2(U), : \ df \in L^2(U) \ \text{ and } \ d * df = 0\}, \\ \mathcal{D}(U) &= \{f : U \to \mathbb{C} \ : \ df \in \mathcal{A}(U)\}, \text{ and} \\ \overline{\mathcal{D}(U)} &= \{f : U \to \mathbb{C} \ : \ df \in \overline{\mathcal{A}(U)}\}. \end{aligned}$$

We can define a degenerate inner product on $\mathcal{D}_{\mathrm{harm}}(U)$ by

$$(f,g)_{\mathcal{D}_{\mathrm{harm}}(U)} = (df, dg)_{\mathcal{A}_{\mathrm{harm}}(U)},$$

where the right hand side is the inner product (2.2) restricted to elements of $\mathcal{A}_{\mathrm{harm}}(U)$. The inner product can be used to define a seminorm on $\mathcal{D}_{\mathrm{harm}}(U)$, by letting

$$\|f\|^2_{\mathcal{D}_{\mathrm{harm}}(U)} := (df, df)_{\mathcal{A}_{\mathrm{harm}}(U)}.$$

We note that if one defines the *Wirtinger operators* via their local coordinate expressions

$$\partial f = \frac{\partial f}{\partial z}\, dz, \quad \overline{\partial} f = \frac{\partial f}{\partial \bar{z}}\, d\bar{z},$$

then the aforementioned inner product can be written as

$$(f, g)_{\mathcal{D}_{\text{harm}}(U)} = i \iint_U \left[\partial f \wedge \overline{\partial g} - \overline{\partial} f \wedge \partial \overline{g}\right]. \tag{2.7}$$

Although this implies that $\mathcal{D}(U)$ and $\overline{\mathcal{D}(U)}$ are orthogonal, there is no direct sum decomposition of $\mathcal{D}_{\text{harm}}(U)$ into $\mathcal{D}(U)$ and $\overline{\mathcal{D}(U)}$. This is because in general there exist exact harmonic one-forms whose holomorphic and anti-holomorphic parts are not exact.

Observe that the Dirichlet spaces are conformally invariant in the same sense as the Bergman spaces. That is, if $f : U \to V$ is a biholomorphism then

$$\mathbf{C}_f h = h \circ f$$

satisfies

$$\mathbf{C}_f : \mathcal{D}(V) \to \mathcal{D}(U)$$

and this is a seminorm preserving bijection. Similar statements hold for the anti-holomorphic and harmonic spaces.

We also note that if $h \in \mathcal{D}(U)$ and $\tilde{h}(z) = h \circ \phi^{-1}(z)$ is the expression for h in local coordinates $z = \phi(w)$ in an open set $\phi(U) \subseteq \mathbb{C}$, then we have the local expression

$$(h, h)_{\mathcal{D}(U)} = \iint_{\phi(U)} |\tilde{h}'(z)|^2 dA_z$$

where dA_z denotes Lebesgue measure in the z-plane. Similar expressions hold for the other Dirichlet spaces.

Next we gather some results from the theory of Sobolev spaces which we shall use in this paper.

Definition 2.23. For $s \in \mathbb{R}$, one defines the *Sobolev space* $H^s(\mathbb{R}^n)$, which consists of tempered distributions u such that

$$\|u\|^2_{H^s(\mathbb{R}^n)} := \|(1-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi < \infty,$$

where $\widehat{u}(\xi)$ is the Fourier transform of u defined by $\widehat{u}(\xi) = \int_{\mathbb{R}^n} u(x)\, e^{-ix\cdot\xi}\, dx$ and

$$(1-\Delta)^{s/2} u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s/2}\, \widehat{u}(\xi)\, e^{ix\cdot\xi}\, d\xi.$$

The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$, is the space of tempered distributions such that $\int_{\mathbb{R}^n} |\xi|^{2s}\, |\widehat{u}(\xi)|^2\, d\xi < \infty$.

The scales of Sobolev spaces that are of particular interest for us are $s = 1, \pm\frac{1}{2}$ (defined on various manifolds). For instance $H^1(\mathbb{R}^n)$ consists of the space of tempered distributions u for which

$$\begin{aligned}\|u\|_{H^1(\mathbb{R}^n)} &:= \left(\int_{\mathbb{R}^n} |\nabla u(x)|^2\, dx + \int_{\mathbb{R}^n} |u|^2 dx\right)^{\frac{1}{2}} \\ &=: \left(\|u\|^2_{\dot{H}^1(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)}\right)^{\frac{1}{2}} < \infty,\end{aligned} \tag{2.8}$$

and $H^{1/2}(\mathbb{R}^n)$ consists of the space of tempered distributions u for which

$$\begin{aligned}\|u\|_{H^{1/2}(\mathbb{R}^n)} &:= \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+1}} dxdy + \int_{\mathbb{R}^n} |u|^2 dx\right)^{\frac{1}{2}} \\ &=: \left(\|u\|^2_{\dot{H}^{1/2}(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)}\right)^{\frac{1}{2}} < \infty.\end{aligned} \tag{2.9}$$

The Sobolev space $H^s(\mathbb{S}^1)$, $s \geq 0$, will also play an important role in our investigations, whose definition we also recall. Given $f \in L^2(\mathbb{S}^1)$ one defines the Fourier coefficients and the Fourier series associated to f by

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-in\theta}\, d\theta, \quad f = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}, \tag{2.10}$$

where the convergence of the series is both in the L^2-norm and also pointwise almost everywhere. The Sobolev space $H^s(\mathbb{S}^1)$ is defined by

$$H^s(\mathbb{S}^1) = \left\{f \in L^2(\mathbb{S}^1) : \sum_{n=-\infty}^{\infty} \left(1+|n|^2\right)^s |\hat{f}(n)|^2 < \infty\right\}. \tag{2.11}$$

Like all other L^2-based Sobolev spaces, $H^s(\mathbb{S}^1)$ is a Hilbert space and given $f, g \in H^s(\mathbb{S}^1)$ their scalar product is given by

$$\langle f, g\rangle_{H^s(\mathbb{S}^1)} = \sum_{n=-\infty}^{\infty} \left(1+|n|^2\right)^s \hat{f}(n)\overline{\hat{g}(n)}, \tag{2.12}$$

and so

$$\|f\|_{H^s(\mathbb{S}^1)} = \left(\sum_{n=-\infty}^{\infty} \left(1+|n|^2\right)^s |\hat{f}(n)|^2\right)^{1/2}. \tag{2.13}$$

Of particular interest in this paper, are the functions in the Sobolev space $H^{1/2}(\mathbb{S}^1)$ for which one also has the analogue of (2.9), i.e.

$$\|f\|_{H^{1/2}(\mathbb{S}^1)} := \Big(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|f(z)-f(\zeta)|^2}{|z-\zeta|^2}\, |dz|\, |d\zeta| + \|f\|^2_{L^2(\mathbb{S}^1)}\Big)^{1/2}. \tag{2.14}$$

As was shown by J. Douglas [8], for a function $F \in \mathcal{D}_{\rm harm}(\mathbb{D})$ the restriction of F to $\mathbb{S}^1$ is in $H^{1/2}(\mathbb{S}^1)$, and if the boundary value of F is denoted by f then

$$\|F\|^2_{\mathcal{D}_{\rm harm}(\mathbb{D})} = \pi \int_0^{2\pi} \int_0^{2\pi} \frac{|f(z) - f(\zeta)|^2}{|z-\zeta|^2}\, |dz|\, |d\zeta|. \tag{2.15}$$

We shall also recall the following useful embedding result, whose proof can be found in [37].

Theorem 2.24. *If* $1 \leq p < \infty$ *and* $s \geq 0$ *with* $s + \frac{1}{p} \geq \frac{1}{2}$ *then one has the continuous inclusion* (*embedding*)

$$H^s(\mathbb{S}^1) \subset L^p(\mathbb{S}^1). \tag{2.16}$$

Now regarding Sobolev spaces on manifolds, we first recall the definition of Sobolev $H^s(M)$, $s \in \mathbb{R}$ for compact manifolds M, see e.g. [3].

Definition 2.25. Let M be an $n-$dimensional smooth compact manifold without boundary, with the smooth atlas (ϕ_j, U_j) and the corresponding smooth partition of unity ψ_j with $\psi_j \geq 0$, $\operatorname{supp} \psi_j \subset U_j$ and $\sum_j \psi_j = 1$. Given $s \geq 0$, the Sobolev spaces $H^s(M)$ are the space of complex-valued L^2 functions on M for which

$$\|f\|_{H^s(M)} := \sum_j \|(\psi_j f)\circ\phi_j^{-1}\|_{H^s(\mathbb{R}^n)} < \infty. \tag{2.17}$$

The homogeneous Sobolev space $\dot{H}^s(M)$ is defined using (2.17) by substituting $H^s(\mathbb{R}^n)$ with $\dot{H}^s(\mathbb{R}^n)$.

It is also well-known that different choices of the atlas and its corresponding partition of unity, produces norms that are equivalent with (2.17).

Next let X be a smooth compact n-dimensional manifold with smooth boundary $\mathrm{bd}(X)$ and fix a Riemannian structure on X. Use the Riemannian structure to construct a collar neighbourhood $N = \mathrm{bd}(X) \times I$ of the boundary $\mathrm{bd}(X)$ and denote the (inward) normal coordinate by $t \in I = [0,1]$. We may assume that X is a submanifold of a closed compact, smooth manifold M, which is the compact double of X.

Definition 2.26. Let X be a smooth compact n-dimensional manifold with boundary. We can regard X as a submanifold of a closed smooth n-dimensional manifold M (i.e. M is compact without boundary as above). Then the space $H^s(X)$ consists of the restrictions $\{\mathbf{R}u;\ u \in H^s(M)\}$ where $\mathbf{R} : L^2(M) \to L^2(X)$ denotes the *restriction operator* $u \mapsto u|_X$.

In this connection one also has the fundamental fact about Sobolev spaces on manifolds with boundary that asserts that the *trace map*, i.e. the map

$$\mathrm{Tr} : u \mapsto u|_{\mathrm{bd}(X)}$$

from $H^s(X) \to H^{s-\frac{1}{2}}(\mathrm{bd}(X))$ is continuous for $s > \frac{1}{2}$, see e.g. [3, Theorem 11.4, p 68].

Ahead, we will show that the border structure on a Riemann surface induces a smooth boundary in the Riemannian sense above, so that Sobolev trace can be applied. In this section, we will keep the notation $\mathrm{bd}(X)$ to denote the boundary in the sense above. Once it is established that the theory applies to the case of the border of a Riemann surface, we will return to the notation $\partial\Sigma$.

Occasionally, we will also use the invariance of the Sobolev space H^s under diffeomorphisms. We state this below as a lemma whose proof could be found in Lemma 1.3.3 in [12], or even more explicitly as Theorem 9.2.3 in [**?**], or by using interpolation between the well-known results for Sobolev spaces of integer scales.

Lemma 2.27. *Let $s \in \mathbb{R}$ and ψ be a diffeomorphism of a bounded open set $U_1 \subset \mathbb{R}^n$ onto another bounded open set $U_2 \subset \mathbb{R}^n$ such that $\psi \in \mathcal{C}^\infty(\mathrm{cl}(U_1))$ and $\psi^{-1} \in \mathcal{C}^\infty(\mathrm{cl}(U_2))$. Then one has*

$$\|f\circ\psi\|_{H^s(U_1)} \approx \|f\|_{H^s(U_2)}.$$

The following result is quite useful in connection to the boundedness of certain operators which will be introduced later. In fact this theorem enables us to turn our estimates into conformally invariant ones through suitable choices of the norms involved in the estimates.

Theorem 2.28. *Let X be a compact Riemannian manifold with smooth boundary, for which the homogeneous and inhomogeneous Sobolev spaces are well defined. Assume that $\mathscr{F}$ is a non-negative functional on $H^s(X)$, $s > 0$, with the following properties:*

1. *$\mathscr{F}$ is real-valued and for all $c \in \mathbb{C}$ and $f \in H^s(X)$, $\mathscr{F}(cf) = |c|\mathscr{F}(f)$;*
2. *For $f \in H^s(X)$, there exists a constant C (independent of f) such that*
$$0 \le \mathscr{F}(f) \le C\|f\|_{H^s(X)};$$
3. *For $f \equiv 1$ on $\mathrm{cl}(X)$ one has that $\mathscr{F}(f) \neq 0$.*

Then there are constants C_1 and C_2 such that for $f \in H^s(X)$ one has

$$C_1\Big(\|f\|^2_{\dot{H}^s(X)} + (\mathscr{F}(f))^2\Big)^{1/2} \le \|f\|_{H^s(X)} \le C_2\Big(\|f\|^2_{\dot{H}^s(X)} + (\mathscr{F}(f))^2\Big)^{1/2}. \quad (2.18)$$

Proof. Set $\Phi(f) := \Big(\|f\|^2_{\dot{H}^s(X)} + (\mathscr{F}(f))^2\Big)^{1/2}$. Then trivially one has that $\Phi(f_1 + f_2) \le \Phi(f_1) + \Phi(f_2)$, and for any $c \in \mathbb{C}$ one has $\Phi(cf) = |c|\Phi(f)$. Moreover Φ is injective, since if $\Phi(f) = 0$ then $\|f\|_{\dot{H}^s(X)} = 0$ and $\mathscr{F}(f) = 0$. The first equality yields that f = constant, and from the second inequality and the assumption on $\mathscr{F}$ it follows that $f = 0$. This shows that $\Phi(\cdot)$ defines a norm on $H^s(X)$. Furthermore since the continuity of $\mathscr{F}$ implies that $\Phi(f) \leq A\|f\|_{H^s(X)}$, a result based on Banach's open mapping theorem, see e.g. [23] Corollary 2.12b, yields that $\|f\| \leq B\Phi(f)$. Taking $C_1 = 1/A$ and $C_2 = B$ we obtain (2.18). □

A useful corollary of this result is the following

Corollary 2.29. *Let $F \in \mathcal{D}_{\mathrm{harm}}(\mathbb{D})$and let f denote the boundary value of F. Then one has*

$$\|f\|_{H^{1/2}(\mathbb{S}^1)} \approx |F(0)| + \|F\|_{\mathcal{D}_{\mathrm{harm}}(\mathbb{D})}. \tag{2.19}$$

Proof. Since $f \in H^{1/2}(\mathbb{S}^1)$, we know that $f \in L^2(\mathbb{S}^1)$ and so

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta},$$

with convergence almost everywhere, where $\hat{f}(n)$ is given by (2.10). Therefore, for the harmonic extension F of f, one has that $F(0) = \hat{f}(0)$ and using Parseval's identity we obtain

$$|F(0)| = |\hat{f}(0)| \leq \Big(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2\Big)^{1/2} = \frac{1}{\sqrt{2\pi}}\|f\|_{L^2(\mathbb{S}^1)}. \tag{2.20}$$

Hence using (2.20) and (2.14) among others, one can easily check that the functional

$$\mathscr{F}(f) := |F(0)|$$

satisfies all the conditions of Theorem 2.28. Hence Theorem 2.28 and equation (2.14) yield that

$$\|f\|_{H^{1/2}(\mathbb{S}^1)} \approx \Big(|F(0)|^2 + \int_0^{2\pi}\int_0^{2\pi} \frac{|f(z)-f(\zeta)|^2}{|z-\zeta|^2}\,|dz|\,|d\zeta|\Big)^{1/2}.$$

Finally, (2.15) and the elementary inequality $\frac{1}{\sqrt{2}}(|a|+|b|) \leq (|a|^2+|b|^2)^{1/2} \leq |a|+|b|$ shows that (2.19) is valid. □

We also record a rather general fact that is often useful in connection to various boundedness results involving Sobolev spaces, see e.g. Theorem 2.6 in [6] for a proof.

Theorem 2.30. *Let Ω be a domain whose boundary is locally the graph of a Lipschitz function (i.e. a Lipschitz domain). Then there exists a unique continuous linear mapping $\gamma : H^1(\Omega) \to L^2(\mathrm{bd}(\Omega))$ such that $\gamma(u) = u|_{\mathrm{bd}(\Omega)}$. In particular, one has the estimate*

$$\int_{\mathrm{bd}(\Omega)} |u|^2 \lesssim \|u\|^2_{H^1(\Omega)}. \tag{2.21}$$

Now let us turn to Sobolev spaces on bordered Riemann surfaces. Let $(\mathscr{R}, h)$ be a compact Riemann surface with metric h. In the case that $\mathscr{R}$ is the sphere or torus, we let h be the spherical or Euclidean metric respectively. Otherwise, h is the hyperbolic metric. Set $d\sigma(h) := \sqrt{|\det h_{ij}|}\,|dz|^2$ which is the area-element of $\mathscr{R}$, where h_{ij} are the components of the metric with respect to coordinates $z = x_1 + ix_2$. We define the inhomogeneous and homogeneous Sobolev norms and seminorms respectively of a function f defined on $\mathscr{R}$ as

$$\begin{aligned} \|f\|_{H^1(\mathscr{R})} &:= \Big(\iint_{\mathscr{R}} df \wedge * \overline{df} + \iint_{\mathscr{R}} |f|^2 d\sigma(h) \Big)^{\frac{1}{2}} \\ &=: \Big(\|f\|^2_{\dot{H}^1(\mathscr{R})} + \|f\|^2_{L^2(\mathscr{R})} \Big)^{\frac{1}{2}}. \end{aligned} \tag{2.22}$$

Observe that the Dirichlet seminorm and the homogeneous Sobolev seminorm $\|\cdot\|_{\dot{H}^1(\mathscr{R})}$ are given by the same expression up to a constant.

We also note that since any two smooth metrics on $\mathscr{R}$ have comparable determinants, choosing different metrics in the definitions above yield equivalent norms. Now if $\mathscr{R}$ is a compact Riemann surface and Σ is an open subset of $\mathscr{R}$ with analytic boundary $\partial\Sigma$, then the pull back of the metric h_{ij} under the inclusion map yields a metric on Σ. Using that metric, we can define the inhomogeneous and homogeneous Sobolev spaces $H^1(\Sigma)$ and $\dot{H}^1(\Sigma)$. However these definitions will a-priori depend on the choice of the metric induced by $\mathscr{R}$, due to the non-compactness of Σ, unless further conditions on Σ are specified.

We define Sobolev spaces on the border of a surface Σ using the intrinsic notion of border. Since the point is important, we embed it in a remark in order to refer to it later.

Remark 2.31 (Sobolev spaces on the border). Whenever we consider the Sobolev space $H^{1/2}(\partial\Sigma)$ in this paper, we assume that $\Sigma \subset \Sigma^d$ where Σ^d is the compact double, so that $\partial\Sigma$ is an analytic curve (and in particular smooth) and thus an embedded submanifold of R. Thus the charts on $\partial\Sigma$ can be taken to be restrictions of charts from Σ^d. Equivalently, the boundary $\partial\Sigma$ is endowed with the manifold structure obtained by treating it as the border of Σ.

Ahead we consider the case that $\Sigma \subseteq \mathscr{R}$ and the topological boundary of Σ in $\mathscr{R}$ is a quasicircle. Although the border of Σ can be identified as a set with the topological boundary of Σ, the inclusion map of the bordered surface Σ into $\mathscr{R}$ is not an embedding in the differential geometric sense. In this situation, we do not apply the Sobolev theory directly to the boundary $\partial\Sigma$ as a subset of $\mathscr{R}$, but consider instead the Sobolev space on the abstract border or double Σ^d as in Remark 2.31.

Regarding the homogeneous and inhomogeneous Sobolev spaces, it was proved in [26] that

Theorem 2.32. *Let $\mathscr{R}$ be a compact surface and let $\Sigma \subset \mathscr{R}$ be bounded by a closed analytic curve Γ. Fix a Riemannian metric $\Lambda_{\mathscr{R}}$ on $\mathscr{R}$ as follows. If $\mathscr{R}$ has genus $g > 1$ then let $\Lambda_{\mathscr{R}}$ be the hyperbolic metric; if $\mathscr{R}$ has genus 1 then let $\Lambda_{\mathscr{R}}$ be the Euclidean metric, and if $\mathscr{R}$ has genus 0 then let $\Lambda_{\mathscr{R}}$ be a spherical metric. Let $H^1(\Sigma)$ and $\dot{H}^1(\Sigma)$ denote the Sobolev spaces with respect to $\Lambda_{\mathscr{R}}$. Then $\dot{H}^1(\Sigma) = H^1(\Sigma)$ as sets.*

2.5. Harmonic measures.

We start with the definition of harmonic measure in the context of bordered Riemann surfaces.

Definition 2.33. Let ω_k, $k = 1, \dots, n$ be the unique harmonic function which is continuous on the closure of Σ and which satisfies

$$\omega_k = \begin{cases} 1 & \text{on } \partial_k\Sigma \\ 0 & \text{on } \partial_j\Sigma, \quad j \neq k. \end{cases}$$

The one-forms $d\omega_k$ are the *harmonic measures*. We denote the complex linear span of the harmonic measures by $\mathcal{A}_{\mathrm{hm}}(\Sigma)$. Moreover we define

$$* \mathcal{A}_{\mathrm{hm}}(\Sigma) = \{* \alpha : \alpha \in \mathcal{A}_{\mathrm{hm}}(\Sigma)\}.$$

By definition any element of $\mathcal{A}_{\mathrm{hm}}(\Sigma)$ is exact, and its anti-derivative ω is constant on each boundary curve. On the other hand, the elements of $* \mathcal{A}_{\mathrm{hm}}(\Sigma)$ are closed but not exact. Elements of $\mathcal{A}_{\mathrm{hm}}(\Sigma)$ and $* \mathcal{A}_{\mathrm{hm}}(\Sigma)$ extend real analytically to the border, in the sense that they are restrictions to Σ of harmonic one-forms on the double. In particular they are square-integrable, which explains our choice of notation. Thus to summarize:

Proposition 2.34. *Let Σ be a bordered surface of type (g, n). Then $\mathcal{A}_{\mathrm{hm}}(\Sigma) \subseteq \mathcal{A}^{\mathrm{e}}_{\mathrm{harm}}(\Sigma)$ and $* \mathcal{A}_{\mathrm{hm}}(\Sigma) \subseteq \mathcal{A}_{\mathrm{harm}}(\Sigma)$.*

Definition 2.35. The *boundary period matrix* Π_{jk} of a non-compact surface Σ of type (g, n) is defined by

$$\Pi_{jk} := \int_{\partial\Sigma} \omega_j * d\omega_k = \int_{\partial_j\Sigma} * d\omega_k.$$

Theorem 2.36. *If we let j, k run from 1 to n, omitting one fixed value m say, then the resulting matrix Π_{jk} is symmetric and positive definite.*

Proof. The matrix is symmetric, because

$$\Pi_{jk} - \Pi_{kj} = \int_{\partial\Sigma} \left(\omega_j * d\omega_k - \omega_k * d\omega_j\right) = \iint_{\Sigma} \left(\omega_j\, d * d\omega_k - \omega_k\, d * d\omega_j\right) = 0.$$

Now let $\lambda_1, \dots, \lambda_n$ denote fixed real numbers, where λ_m is omitted from the list. Define

$$\omega = \sum_{\substack{k=1 \\ k \neq m}}^{n} \lambda_k \omega_k$$

then using the fact that ω is harmonic we obtain (implicitly using Proposition 2.34)

$$\begin{aligned}\|d\omega\|^2 &= \iint_\Sigma d\omega \wedge * \, d\omega = \int_{\partial\Sigma} \omega \wedge * \, d\omega \\ &= \int_{\partial\Sigma} \left(\sum_{j\neq m} \lambda_j \omega_j \right) * \, d \left(\sum_{k \neq m} \lambda_k \omega_k \right) \\ &= \sum_{j\neq m} \sum_{k\neq m} \Pi_{jk} \lambda_j \lambda_k .\end{aligned}$$

Since $d\omega_1, \dots, d\omega_n$ (omitting $d\omega_m$) are linearly independent, this completes the proof. □

Thus Π_{jk}, $j, k = 1, \dots \hat{m}, \dots, n$ (here $\hat{(\cdot)}$ denotes omission) is an invertible matrix, and we can specify $n-1$ of the boundary periods of elements of $* \, \mathcal{A}_{\mathrm{hm}}(\Sigma)$.

2.6. Green's functions. Another basic notion which is of fundamental importance in our investigations is that of Green's functions.

Definition 2.37. Let Σ be a type (g, n) surface. For fixed $z \in \Sigma$, we define *Green's function of* Σ to be a function $G_\Sigma(w; z)$ such that

1. for a local coordinate ϕ vanishing at z the function $w \mapsto G_\Sigma(w; z) + \log |\phi(w)|$ is harmonic in an open neighbourhood of z;
2. $\lim_{w \to \zeta} G_\Sigma(w; z) = 0$ for any $\zeta \in \partial\Sigma$.

That such a function exists, follows from [1, II.3 11H, III.1 4D], considering Σ to be a subset of its double Σ^d.

Green's function is conformally invariant. That is, if Σ is of type (g, n), and $f : \Sigma \to \Sigma'$ is conformal, then

$$G_{\Sigma'}(f(w); f(z)) = G_\Sigma(w; z). \tag{2.23}$$

This follows from uniqueness of Green's function, and the fact that a biholomorphism extends to a homeomorphism of the boundary curves.

2.7. Quasisymmetric mappings. Throughout this paper we will use the concept of quasisymmetric mappings. Therefore, we recall the definitions of quasisymmetries in plane and on Riemann surfaces. We start by defining the quasisymmetric homeomorphisms of the circle.

Definition 2.38. An orientation-preserving homeomorphism h of $\mathbb{S}^1$ is called an *orientation-preserving quasisymmetric mapping*, iff there is a constant $k > 0$, such that for every θ, and every ψ not equal to a multiple of 2π, the inequality

$$\frac{1}{k} \leq \left| \frac{h(e^{i(\theta+\psi)}) - h(e^{i\theta})}{h(e^{i\theta}) - h(e^{i(\theta-\psi)})} \right| \leq k$$

holds. We say that h is an orientation-reversing quasisymmetry if $h \circ s$ is an orientation-preserving quasisymmetry where $s(e^{i\theta}) = e^{-i\theta}$.

A quasisymmetry is either an orientation-preserving or orientation-reversing quasisymmetry.

We generalize this to general Riemann surfaces of type (g, n).

Definition 2.39. Fix $k \in \{1, \dots, n\}$. Let $\tau : \mathbb{S}^1 \to \partial_k \Sigma$ be a homeomorphism. We say that τ is a *quasisymmetry* if there is a collar chart $\phi : U \to \mathbb{A}_{r,1}$ of $\partial_k \Sigma$ such that $\phi \circ \tau$ is a quasisymmetry in the sense of Definition 2.38. We say that τ is orientation-preserving (resp. orientation-reversing) when $\phi \circ \tau$ is orientation-preserving (resp. orientation-reversing).

Theorem 2.40. *Let $\tau : \mathbb{S}^1 \to \partial_k \Sigma$ be a homeomorphism for some fixed $k \in \{1, \dots, n\}$. If $\phi \circ \tau$ is a quasisymmetry of $\mathbb{S}^1$ for some collar chart ϕ of $\partial_k \Sigma$, then $\phi \circ \tau$ is a quasisymmetry of $\mathbb{S}^1$ for any collar chart ψ of $\partial_k \Sigma$.*

Proof. If ψ is another collar chart, then $\psi \circ \phi^{-1}$ is a conformal map from some collar neighbourhood of $\mathbb{S}^1$ to another collar neighbourhood of $\mathbb{S}^1$. It extends homeomorphically to the boundary by Theorem 2.12. Thus by Schwarz reflection $\psi \circ \phi^{-1}$ extends to a conformal map of a neighbourhood of $\mathbb{S}^1$. Thus $\psi \circ \tau = \psi \circ \phi^{-1} \circ \phi \circ \tau$ is also a quasisymmetry. □

In a similar way, we can define the notion of analytic parametrization.

Definition 2.41. We say that τ is an *analytic parametrization* if $\phi \circ \tau$ is analytic for any collar chart ϕ.

3. Conformally non-tangential limits and overfare of harmonic functions

3.1. About this section. Since there are many technical details in proving that overfare is well-defined and bounded, we give here an overview of the constructions and the argument.

First, we need a notion of boundary values; these are what we call conformally non-tangential boundary values. They are defined in Section 3.2; briefly, we use a collar chart to map the function near the boundary to the disk, and apply Beurling's theorem on non-tangential boundary values of Dirichlet bounded functions. We then show that this is independent of the choice of collar chart.

To prove that the overfare process makes sense, it must be shown that the set of possible boundary values of Dirichlet bounded harmonic functions is the same from either side. This requires, among other things, showing that a set which is potential-theoretically negligible from the point of view of Σ_1 is also potential-theoretically negligible from the point of view of Σ_2. To be precise, a negligible set from the point of view of Σ_1 (resp. Σ_2) is a Borel set whose image under a collar chart in Σ_1 (resp. Σ_2) is a set of logarithmic capacity zero. We call such sets "null sets" with respect to Σ_1 (resp. Σ_2). The proof that null sets with respect to Σ_1 are also null with respect to Σ_2 is accomplished by cutting and pasting neighbourhoods of the boundary, applying a chart, and using the corresponding result in the plane. This is done in Section 3.6.

We will also prove that the overfare operator exists and is bounded, using sewing techniques. The proof proceeds in steps. First, we show that a certain "bounce operator" is bounded. This bounce operator acts entirely within one surface, say Σ_1. It takes Dirichlet bounded functions defined on a collar neighbourhood of the collection of quasicircles, and produces the unique Dirichlet bounded function on the Riemann surface Σ_1 with the same boundary values. We show in Section 3.5 that this operator is bounded, by using the existence and continuous dependence of solutions to the Dirichlet problem together with the fact that the Sobolev trace is bounded. Then, we define a "local" overfare as follows. Given a function defined in a collar neighbourhood of a boundary curve in Σ_1, we cut out a tubular neighbourhood of a quasicircle, and map it into the plane with a doubly connected chart. Using the fact that bounce and overfare are bounded in the plane, we obtain a bounded map taking Dirichlet bounded functions on a collar neighbourhood in Σ_1 to Dirichlet bounded functions in a collar neighbourhood in Σ_2.

The overfare operator is then shown to be bounded in Section 3.6 by first overfaring locally and then applying the bounce operator on Σ_2. Since every step is bounded, this will complete the proof.

In previous works of the authors, only one curve was involved. This meant that constant functions overfare to constant functions. For this reason, it was sufficient to work with the Dirichlet seminorm. However, if there are many curves, it is possible that many constants are involved, and indeed it is even possible that the overfare of a locally constant function is a non-constant function. It is then possible to drive up the Dirichlet seminorm on one side while it is unchanged on the other.

If the originating surface is connected, this problem does not arise. In this case, we show that overfare is bounded with respect to the Dirichlet seminorm for general quasicircles. On the other hand, to control the constants, we need to work with a true norm. We introduce a conformally invariant norm in Section 3.3, which can be given in several equivalent forms. We show that for quasicircles with greater regularity the overfare is bounded with respect to this true norm. This conformally invariant norm also plays an important role in the theory of boundary values of L^2 harmonic one-forms established in the second paper in this series. In Section 3.3 we also establish two "anchor lemmas" which are necessary to rigorously define boundary integrals of pairings of Dirichlet bounded functions with L^2 forms. This is necessary for definition of the norm which controls constants, and will also play a role in the sequels to this paper [30, 31, 32].

3.2. CNT limits and boundary values of functions and forms. In this section, we define a notion of non-tangential limit which is conformally invariant. Existence of this limit is independent of coordinate. In a sense, this is the natural notion of non-tangential limit on the border of a Riemann surface. The main idea is that any border chart determines a notion of non-tangential approach to a point on the boundary, and the compatibility of border charts implies that this notion is independent of chart.

We now give the precise definition. First, we recall the definition of non-tangential limit on the upper half plane and the disk $\mathbb{D}$. For $\theta \in (0, \pi/2)$ and $p \in \partial\mathbb{H}$ define the wedge

$$V_{p,\theta} = \{z \in \mathbb{H} : \pi/2 - \theta < \arg(z - p) < \pi/2 + \theta\}.$$

Let $h : U \to \mathbb{C}$ be a function defined on an open set U in $\mathbb{H}$ which contains a half disk $D_r = \{z : |z - p| < r, z \in \mathbb{H}\}$.

Definition 3.1. We say that h has a *non-tangential limit at p* if

$$\lim_{z \to p} h|_{V_{p,\theta} \cap U}$$

exists for every $\theta \in (0, \pi/2)$.

Similarly, we can define non-tangential limit for functions h on open subsets U of $\mathbb{D}$ containing a set $D_r = \{z : |z - p| < r, z \in \mathbb{D}\}$. A non-tangential wedge in $\mathbb{D}$ with vertex at $p \in \mathbb{S}^1$ is a set of the form

$$W(p, M) = \{z \in \mathbb{D} : |p - z| < M(1 - |z|)\}$$

for some $M \in (1, \infty)$. We say that a function $h : \mathbb{D} \to \mathbb{C}$ has a non-tangential limit at p if the limit of $h|_{W(p,M) \cap U}$ as $z \to p$ exists for all $M \in (1, \infty)$. One may equivalently use Stolz angles, that is sets of the form

$$S(p, \alpha) = \{z : \arg(1 - \bar{p}z) < \alpha, |z| < \rho \cos \alpha\}$$

where $\alpha \in (0, \pi/2)$ [18, p6].

It is easily seen that if $T : \mathbb{D} \to \mathbb{D}$ is a disk automorphism, then h has a non-tangential limit at p if and only if $h \circ T$ has a non-tangential limit at $T(p)$. A similar statement holds for non-tangential limits in the upper half plane. Finally, observe that if T is a Möbius transformation from $\mathbb{D}$ to $\mathbb{H}$ then a function h on a subset of the upper half plane has a non-tangential limit at p if and only if $h \circ T$ has a non-tangential limit at $T^{-1}(p)$.

We now define conformally non-tangential limits. Let U be an open subset of Σ and let $h : U \to \mathbb{C}$. Let $p \in \partial\Sigma$. We say that h is "defined near p" if there is a boundary chart $\phi : V \to \text{cl}(\mathbb{H})$ such that $\phi(U)$ contains a half-disk $D_r = \{z : |z - p| < r, \ z \in \mathbb{H}\}$.

Definition 3.2. Let Σ be a Riemann surface with border $\partial\Sigma$. Fix $p \in \partial\Sigma$ and let $h : U \to \mathbb{C}$ be defined near p. We say that h has a conformally non-tangential limit at p if there is a boundary chart $\phi : V \to \text{cl}(\mathbb{H})$ such that $p \in V$ and $h \circ \phi^{-1}$ has a non-tangential limit at $\phi(p)$.

We will use the acronym CNT in place of "conformally non-tangential".

The following theorem shows that the existence of the CNT limit does not depend on the chart, in the sense that the condition of the definition holds either for all boundary charts or none.

Proposition 3.3. *For fixed $p \in \partial\Sigma$, let $h : U \to \mathbb{C}$ be defined near p and let h have a* CNT *limit equal to ζ at p. Then the* CNT *limit is independent of the boundary chart used in* Definition 3.2. *That is, for any boundary chart $\psi : W \to \mathbb{H}$, $h \circ \psi^{-1}$ has a non-tangential limit equal to ζ at $\psi(p)$. The same claims holds for boundary charts $\psi : W \to \mathbb{D}^+$.*

Proof. Assume that $h \circ \phi^{-1}$ has a non-tangential limit equal to ζ at $\phi(p)$ for some boundary chart $\phi : V \to \mathbb{H}$. Let $\psi : W \to \mathbb{H}$ be any other boundary chart near p. By the Schwarz reflection principle, $\phi \circ \psi^{-1}$ extends to a biholomorphism from an open neighbourhood of $\psi(p)$ to an open neighbourhood of $\phi(p)$. In particular, for any non-tangential wedge $V_{\psi(p),\theta}$ there is a disk D at $\phi(p)$ such that $\phi \circ \psi^{-1}(D \cap V_{\psi(p),\theta})$ is contained in a non-tangential wedge at $\phi(p)$. Thus the limit as z approaches $\psi(p)$ of $h \circ \psi^{-1} = h \circ \phi \circ \phi \circ \psi^{-1}$ within $D \cap V_{\psi(p),\theta}$ equals ζ. □

It follows immediately from the definition of CNT limits that they are conformally invariant. Although this is a simple consequence it deserves to be highlighted.

Theorem 3.4 (Conformal invariance of CNT limits)**.** *Let Σ be a bordered Riemann surface and $h : U \to \mathbb{C}$ be a function defined near $p \in \partial\Sigma$. If $F : \Sigma_1 \to \Sigma$ is a conformal map, then h has a* CNT *limit of ζ at p if and only if $h \circ F$ has a* CNT *limit of ζ at $F^{-1}(p)$.*

Next, we define a potentially-theoretically negligible set on the border which we call a null set. We first need a lemma.

Lemma 3.5. *Let Σ be a type (g, n) bordered surface and let $\phi_k : U_k \to \mathbb{A}_{r_k,1}$ be collar charts of a boundary curve $\partial_j\Sigma$ for $k = 1, 2$ and some fixed $j \in \{1, \dots, n\}$. Let $I \subset \partial_j\Sigma$ be a Borel set. Then $\phi_1(I)$ has logarithmic capacity zero if and only if $\phi_2(I)$ has logarithmic capacity zero.*

Proof. If $K \subset \mathbb{S}^1 = \{z : |z| = 1\}$ is a Borel set of logarithmic capacity zero, and ϕ is a quasisymmetry, then $\phi(K)$ has logarithmic capacity zero [25, Theorem 2.9]. Since the inverse of a quasisymmetric map is also a quasisymmetry (and in particular a homeomorphism), we see that a Borel set K has logarithmic capacity zero if and only if $\phi(K)$ is a Borel set of logarithmic capacity zero.

Now let $\phi_1 : U_1 \to \mathbb{A}_{r_1,1}$ and $\phi_2 : U_2 \to \mathbb{A}_{r_2,1}$ be collar charts such that U_1 and U_2 are in Σ. By Lemma 2.12, $\phi_1 \circ \phi_2^{-1}$ has a homeomorphic extension to $\mathbb{S}^1$. By the Schwarz reflection principle, it has an extension to a conformal map of an open neighbourhood of $\mathbb{S}^1$, so it is an analytic diffeomorphism of $\mathbb{S}^1$ and in particular a quasisymmetry. Thus $\phi_2(I)$ has logarithmic capacity zero if and only if $\phi_1(I)$ has capacity zero. This completes the proof. □

The previous lemma motivates and justifies the following definition.

Definition 3.6. Let Σ be a bordered Riemann surface of type (g, n). We say that a Borel set $I \subset \partial_k\Sigma$ is a *null set* if $\phi(I)$ is a set of logarithmic capacity zero in $\mathbb{S}^1$

for some collar chart ϕ of $\partial_k\Sigma$. We say that a Borel set I in $\partial\Sigma$ is null if it is a union of null sets $I_k \subset \partial_k\Sigma$, $k = 1, \dots, n$.

We also have the following two results:

Proposition 3.7. *If I_1 and I_2 are null in $\partial_k\Sigma$ then $I_1 \cup I_2$ is null.*

Proof. It is enough to show that the union of Borel sets I_1 and I_2 of logarithmic capacity zero in $\mathbb{S}^1$ are of logarithmic capacity zero. By Choquet's theorem, the outer capacity of I_1 and I_2 equal their capacity. Since outer capacity is subadditive, the outer capacity of $I_1 \cup I_2$ is zero. The claim follows from another application of Choquet's theorem. □

Harmonic functions which are Dirichlet bounded near a border have CNT boundary values except possibly on a null set.

Theorem 3.8. *Let Σ be a bordered Riemann surface of type (g, n). Let U_k be a collar neighbourhood of $\partial_k\Sigma$ for some $k \in \{1, \dots, n\}$. If $h \in \mathcal{D}_{\text{harm}}(U_k)$ then h has* CNT *boundary values on $\partial_k\Sigma \backslash I$ for some null set $I \subset \partial_k\Sigma$.*

Proof. By conformal invariance of the Dirichlet space and CNT boundary values (Theorem 3.4), it is enough to prove this for an annulus in the plane, which is a special case of [26, Theorem 3.12]. □

Remark 3.9. The non-tangential boundary values agree with the Sobolev trace up to a set of measure zero, if the boundary is sufficiently regular. This holds for example if we treat the border as an analytic curve in the double.
In fact if one has an (ε, δ) domain Ω (in the plane these are quasidisks) with Ahlfors-regular boundary in the sense of Definition 1.1 of [5], then using Theorem 8.7 (iii) in [5] and taking $s = 1$, $p = 2$ and $n = 2$, we have that their condition $s - \frac{n-d}{p} = 1 - \frac{2-1}{2} = \frac{1}{2} \in (0, \infty)$ is satisfied. Thus, Theorem 8.7 (iii) in [5] yields that the Sobolev trace belonging to $H^{1/2}(\partial\Omega)$ agrees almost everywhere (since the 1-dimensional Hausdorff measure on $\partial\Omega$ is the 1-dimensional Lebesgue measure) with the non-tangential limit of the function $h \in H^1(\Omega)$. Note that chord-arc domains, are examples of (ε, δ) domains with Ahlfors-regular boundary.

Theorem 3.10. *Let Σ be a bordered surface of type (g, n). If $h \in \mathcal{D}_{\text{harm}}(\Sigma)$, then there is a null set $I \subset \partial\Sigma$ such that h has* CNT *boundary values on $\Sigma \backslash I$. If H is any element of $\mathcal{D}_{\text{harm}}(\Sigma)$ with* CNT *boundary values which agree with those of h except possibly on a null set J, then $h = H$.*

Proof. The first claim follows directly from Theorem 3.8. For the uniqueness part, it is well-known that if X is a smooth compact Riemannian manifold with boundary, then the Dirichlet problem

$$\begin{cases} \Delta u = 0 \\ u|_{\partial X} = f \in H^{1/2}(\partial X) \end{cases} \tag{3.1}$$

has a unique solution that satisfies

$$\|u\|_{H^1(X)} \leq C\|f\|_{H^{1/2}(\partial X)},$$

see e.g. [36, Proposition 1.7, p 382]. Using this together with Remark 3.9 it follows that if $H = h$ up to a null set on $\partial\Sigma$ then $h = H$. □

A suitable adaptation of the proof of [26, Theorem 3.17] also yields

Theorem 3.11. *Let Σ be a bordered surface of type (g, n) and let $U_k \subseteq \Sigma$ be collar neighbourhoods of $\partial_k\Sigma$ for $k = 1, \ldots, n$. Let $h_k \in \mathcal{D}_{\text{harm}}(U_k)$ for $k = 1, \ldots, n$. There is a function $H \in \mathcal{D}_{\text{harm}}(\Sigma)$ whose* CNT *boundary values agree with those of h_k on $\partial_k\Sigma$ up to a null set for each $k = 1, \ldots, n$.*

We thus make the following definition.

Definition 3.12. Let Σ be a Riemann surface and let Γ be a finite collection of borders of Σ each of which is homeomorphic to $\mathbb{S}^1$. Given functions $h_k : \Gamma\backslash I_k \to \mathbb{C}$ where I_1 and I_2 are null sets, we say that $h_1 \sim h_2$ if h_1 and h_2 are both defined on $\Gamma\backslash I$ for some null set I and $h_1 = h_2$ on $\Gamma\backslash I$. The *Osborn space* $\mathcal{H}(\Gamma)$ is the set of equivalence classes of such functions.

Remark 3.13. It follows from the results of this section that every element of $H^{1/2}(\Gamma)$, which is defined almost everywhere, has a unique extension to an element of $\mathcal{H}(\Gamma)$ which is defined except possibly on a null set.

3.3. Anchor lemmas, boundary integrals, and a conformally invariant Dirichlet norm. Having defined the notion of CNT boundary values in the previous section, we establish two lemmas which allow us to consistently define integrals of the form

$$\int_\Gamma \alpha\, h$$

where Γ is a boundary curve of a Riemann surface, α is an L^2 harmonic one-form in a collar neighbourhood of Γ, and h is a harmonic function with finite Dirichlet norm in a collar neighbourhood of Γ. Moreover the integral, as far as h is concerned, depends only on the CNT boundary values of h on Γ.

We do this by evaluating the integral along curves which approach Γ in the limit. Let Σ be a Riemann surface of type (g, n), Γ_k be one of its boundary curves, and $\phi : A \to \mathbb{A}$ be a collar chart for Γ_k. By Remark 2.13, setting $C_r = \{z : |z| = r\}$ for $r \in (0, 1)$

$$\Gamma_k^r = \phi(C_r) \tag{3.2}$$

is an isotopy of analytic Jordan curves on $[R, 1]$ for some $R \in (0, 1)$, such that $\Gamma_k^1 = \Gamma_k$.

The following two lemmas show that the limiting integrals are well-defined in the sense that they are independent of the choice of limiting curves (the first anchor lemma), and depend only on the boundary values (the second anchor lemma).

Lemma 3.14 (First anchor lemma)**.** *Let $\phi : A \to \mathbb{A}$ be a collar chart of Γ_k in Σ_1, and let $\Gamma_k^r = \phi(C_r)$. Let $\alpha \in \mathcal{A}(A)$. For any $h \in \mathcal{D}_{\mathrm{harm}}(A)$*

$$\lim_{r \nearrow 1} \int_{\Gamma_k^r} \alpha(w) h(w)$$

exists. Furthermore, this quantity is independent of the collar chart.

Proof. Existence follows from Stokes' theorem, since

$$\lim_{r \nearrow 1} \int_{\Gamma_k^r} \alpha(w) h(w) = \int_{\Gamma_k} \alpha(w) h(w) + \iint_{A_r} \alpha \wedge \overline{\partial} h(w). \tag{3.3}$$

where A_r is the region bounded by Γ_k^r and Γ_k. This existence argument of course applies to any choice of collar chart.

We need to show that it gives the same result regardless of the choice. By change of variables, it is enough to prove this in the situation that one of the collar charts ϕ is of $\Gamma_k = \mathbb{S}^1$ as a boundary of $A = \mathbb{A}_{r,1}$, and $\phi = \mathrm{Id}$. The curves Γ_k^r are then just $|z| = r$. Let $\phi' : A' \to \mathbb{A}'$ be some other collar chart of $\mathbb{S}^1$. Let γ_k^r denote the isotopy induced by ϕ'.

Fix any $\varepsilon > 0$ and choose R such that

$$\left| \lim_{r \nearrow 1} \int_{\gamma_k^r} \alpha(w) h(w) - \int_{\gamma_k^R} \alpha(w) h(w) \right| < \varepsilon/2$$

and

$$\|\alpha\|_{\mathcal{A}(A'_R)} \|\overline{\partial} h\|_{\overline{\mathcal{A}(A'_R)}} < \varepsilon/2 \tag{3.4}$$

where A'_R is the region bounded by $\mathbb{S}^1$ and γ_k^R. Since γ_k^R is compact, $|z|$ has a maximum $M < 1$ on γ_k^R. For any $r > M$, Γ_k^r is contained in A'_R and does not intersect γ_k^R. If we let B denote the region bounded by these two curves, then $B \subseteq A'_R$. Therefore using Cauchy-Schwarz's inequality we deduce that

$$\begin{aligned}
\left| \lim_{r \nearrow 1} \int_{\gamma_k^r} \alpha(w) h(w) - \int_{\Gamma_k^r} \alpha(w) h(w) \right| &\le \left| \lim_{r \nearrow 1} \int_{\gamma_k^r} \alpha(w) h(w) - \int_{\gamma_k^R} \alpha(w) h(w) \right| \\
&\quad + \left| \int_{\gamma_k^R} \alpha(w) h(w) - \int_{\Gamma_k^r} \alpha(w) h(w) \right| \\
&< \frac{\varepsilon}{2} + \left| \iint_B \alpha(w) \wedge \overline{\partial} h(w) \right| \\
&\le \frac{\varepsilon}{2} + \|\alpha\|_{\mathcal{A}(A'_R)} \|\overline{\partial} h\|_{\overline{\mathcal{A}(A'_R)}}
\end{aligned}$$

which by (3.4) proves the claim. □

Henceforth we will denote this limiting integral by

$$\int_{\Gamma_k} \alpha(w) h(w) \quad \text{or} \quad \int_{\partial_k \Sigma} \alpha(w) h(w)$$

if $\Gamma_k = \partial_k \Sigma$, where the notation is justified by Lemma 3.14.

We now show that the integral depends only on the CNT boundary values of the harmonic function h in the integral above.

Lemma 3.15 (Second anchor lemma)**.** *Let A be a collar neighbourhood of Γ_k in Σ_1 for some $k \in \{1, \dots, n\}$. If h_1 and h_2 are any two elements of $\mathcal{D}_{\mathrm{harm}}(A)$ with the same* CNT *boundary values on Γ_k up to a null set, then for any $\alpha \in \mathcal{A}(A)$*

$$\int_{\partial_k \Sigma} \alpha(w) h_1(w) = \int_{\partial_k \Sigma} \alpha(w) h_2(w).$$

Proof. By Lemma 3.14 we may use any collar chart to determining a limiting sequence of curves. By Proposition 2.16 we can find a collar chart whose domain is in A. Since the integral along a curve is invariant under composition with a conformal map, it is enough to prove this for $\Gamma_k = \mathbb{S}^1$ and $A = \mathbb{A}_{r,1}$ for some r, with limiting curves Γ_k^r given by $|z| = r$. We can apply [27, Theorem 4.7] or [28, Lemma 3.21] to $(h_1 - h_2)$ in this case. □

In summary, the limiting integral of h against any $\alpha \in \mathcal{A}(A)$ exists and depends only on the CNT boundary values of h.

Remark 3.16. We will often consider the situation where the Riemann surface Σ is a subset of a compact surface $\mathscr{R}$, where the boundary is irregular (such as a quasicircle). However the anchor lemmas involve only the assumption that the boundary is a border (and hence, a collar chart exists). In particular, no reference is made to any outside surface, and thus they apply in the situation above.

We now obtain two explicit collar charts which arise naturally from the harmonic measure and Green's function. These two canonical collar charts are very useful in association with the evaluation of certain boundary integrals. The first lemma tackles the case of the collar chart from harmonic measure.

Lemma 3.17 (Collar chart from harmonic measure)**.** *Let Σ be a type (g, n) Riemann surface for $n > 1$. Let ω_k be the harmonic function which is one on $\partial_k \Sigma$ and 0 on the other boundary curves. Let ψ be the multi-valued holomorphic function with real part $\omega_k - 1$ and set*

$$i a_k = i \int_{\partial_k \Sigma} * \, d\omega_k.$$

Then

$$\phi(z) = \exp\left(2\pi\psi / a_k\right)$$

is a collar chart on some collar neighbourhood U of $\partial_k \Sigma$. Furthermore

$$* \, d\omega_k = \frac{a_k}{2\pi} \phi^* d\theta,$$

and thus for any $h \in \mathcal{D}_{\mathrm{harm}}(U)$ we have

$$\int_{\partial_k \Sigma} h * d\omega_k = \frac{a_k}{2\pi} \int_{\mathbb{S}^1} h \circ \phi(e^{i\theta}) d\theta.$$

Proof. It is clear that ϕ takes level curves $\omega_k = 1 - \epsilon$ to curves $|z| = e^{-\epsilon}$ for ϵ sufficiently small. Observe that $d\mathrm{Re}\psi = d\omega_k$, so the harmonic conjugate of $\omega_k - 1$ is a primitive of $* \, d\omega_k$. By the definition of a_k, this shows that ϕ is single-valued. An application of the argument principle shows that the map is a bijection for some collar neighbourhood defined by $0 < \epsilon < s$ for some fixed s. This proves the first claim.

The second claim follows from

$$d\theta = d\,\mathrm{Im}\log\psi = \frac{2\pi}{a_k} * d\omega_k.$$

The final claim follows from a change of variables and the second claim. □

We also have

Lemma 3.18 (Collar chart from Green's function)**.** *Let Σ be a type (g, n) Riemann surface and let G_Σ be Green's function of Σ. For fixed p, let $\psi(w)$ be the multi-valued holomorphic function with real part $G_\Sigma(w; p)$. Setting*

$$ia_k = i\int_{\partial_k\Sigma} * \, dG_\Sigma(\cdot\,; p)$$

it holds that

$$\phi(w) = \exp(2\pi\psi(w)/a_k)$$

is a collar chart on some open neighbourhood U of $\partial_k\Sigma$.

Proof. The proof is similar to that of the above, and can be found in [26]. □

The important property of these two collar charts is that the limiting curves are level curves of the harmonic measure and Green's function respectively.

A useful application is the following extension of the well-known reproducing property of Green's function.

Proposition 3.19. *Let Σ be a type (g, n) Riemann surface and let G_Σ be its Green's function. Let Γ^p_ϵ denote the level curves of Green's function for any fixed $p \in \Sigma$. For any $h \in \mathcal{D}_{\mathrm{harm}}(\Sigma)$*

$$h(z) = -\frac{1}{2\pi}\int_{\partial\Sigma} * \, d_w G_\Sigma(w; z) h(w) = -\lim_{\epsilon\searrow 0}\frac{1}{2\pi}\int_{\Gamma^p_\epsilon} * \, d_w G_\Sigma(w; z) h(w).$$

We also have

$$h(z) = -\frac{1}{\pi i}\int_{\partial\Sigma} \partial_w G_\Sigma(w; z) h(w).$$

Proof. We prove the first displayed equation. By Lemma 3.14, the integral on the left is well-defined, that is, the right hand side is the same no matter what the choice of p is. Thus we may assume that $p = z$. In that case, Stokes' theorem and the harmonicity of h yield that

$$\int_{\Gamma^p_\epsilon} G_\Sigma * dh = \epsilon\int_{\Gamma^p_\epsilon} * \, dh = 0.$$

From here, the proof proceeds in the usual way using Green's identity

$$\int_{\Gamma_\epsilon^p} (G_\Sigma(w;p) * dh(w) - h(w) * dG_\Sigma(w;p)) = \int_{\gamma_r} (G_\Sigma(w;p) * dh(w) - h(w) * dG_\Sigma(w;p))$$

where γ_r is a curve $|w-p| = r$ in some local coordinate, and letting $r \searrow 0$.

To prove the second displayed equation, choose the limiting curves to be level curves of $G_\Sigma(\cdot\,;z)$; again, this can be done by Lemma 3.14. Along such curves $dG_\Sigma = 0$, so that

$$\partial_w G_\Sigma(w;z) = \frac{i}{2} * d_w G_\Sigma$$

by equation (2.1), which proves the claim. □

Note that, this is usually written in terms of an integral around the boundary, under the assumption that h is more regular.

3.4. Conformally invariant Sobolev norms. We now define conformally invariant norms on Dirichlet-bounded harmonic functions. The idea is motivated by the case of the disk as follows. To turn the Dirichlet semi-norm of harmonic functions h on the disk into a norm, one usually adds a value at a point to the norm, say $|h(0)|$, which equals

$$\frac{1}{2\pi}\left|\int_{\mathbb{S}^1} h(e^{i\theta})\,d\theta\right|.$$

But on a Riemann surface we do not have a distinguished point or a canonically determined measure $d\theta$ on the borders. We therefore replace $d\theta$ with more general quantities associated to harmonic measures or Green's function.

Given $h \in \mathcal{D}_{\text{harm}}(\Sigma)$ we set

$$\mathscr{H}_k := \int_{\partial_k \Sigma} h_k * d\omega_k.$$

In the case that $n = 1$, fix a point $p \in \Sigma \setminus U_1$ and define instead

$$\mathscr{H}_1 := \int_{\partial_1 \Sigma} h_1 * dG_\Sigma(w,p), \tag{3.5}$$

where $G_\Sigma(w,p)$ is Green's function of Σ.

Definition 3.20. Set $U = U_1 \cup \cdots \cup U_n$ as above. Let $H^1_{\text{conf}}(U)$ be the harmonic Dirichlet space $\mathcal{D}_{\text{harm}}(U)$ endowed with the norm

$$\|h\|_{H^1_{\text{conf}}(U)} := \left(\|h\|^2_{\mathcal{D}_{\text{harm}}(U)} + \sum_{k=1}^n |\mathscr{H}_k|^2\right)^{\frac{1}{2}}. \tag{3.6}$$

Assuming the Riemann surface Σ is connected, we can choose any fixed boundary curve $\partial_n\Sigma$ say, and define the norm

$$\|h\|_{H^1_{\mathrm{conf}}(\Sigma)} := \left(\|h\|^2_{\mathcal{D}_{\mathrm{harm}}(\Sigma)} + |\mathscr{H}_n|^2\right)^{1/2}, \tag{3.7}$$

(where any of the $\mathscr{H}_k$ could in fact be used in place of $\mathscr{H}_n$).

We can also use Green's function to define the norm in the case that $n > 1$, as the following lemma shows. The different characterizations will be of use to us later.

Lemma 3.21. *Let Σ be a connected Riemann surface of type (g, n). For any fixed point $p \in \Sigma$, the norms given by*

$$\begin{aligned}\|h\|^2_{\mathcal{D}_{\mathrm{harm}}(\Sigma),p} &= \|h\|^2_{\mathcal{D}_{\mathrm{harm}}(\Sigma)} + |h(p)|^2 \\ &= \|h\|^2_{\mathcal{D}_{\mathrm{harm}}(\Sigma)} + \left|\lim_{\varepsilon\searrow 0}\frac{1}{\pi i}\int_{\Gamma_\varepsilon}\partial_w G_\Sigma(w;p)h(w)\right|^2,\end{aligned}$$

where Γ_ε are the level curves of Green's function based at p, and the $H^1_{\mathrm{conf}}(\Sigma)$ norm are equivalent.

Proof. If $n = 1$ there is nothing to prove. First we note that if $U \subset \Sigma$ is a small neighbourhood of $p \in \Sigma$ then by the mean-value theorem for harmonic functions and Jensens inequality we have that $|h(p)|^2 \lesssim \|h\|^2_{L^2(\Sigma)}$, which confirms condition (2) of Theorem 2.28. Therefore, since conditions (1) and (3) of that theorem are also trivially satisfied, the Lemma follows. □

Theorem 3.22. *Let Σ be a connected Riemann surface of type (g, n). Then, the $H^1_{\mathrm{conf}}(\Sigma)$ norm is equivalent to the $H^1(\Sigma)$ norm. In particular, any choice of boundary curve in the definition of $H^1_{\mathrm{conf}}(\Sigma)$ leads to an equivalent norm.*

Proof. We note that for any integer $0 \le k \le n$, $|\mathscr{H}_k| \ge 0$, $\int_{\partial_k\Sigma} * d\omega_k = -\int_{\gamma_k} * d\omega_k \neq 0$, and $|\mathscr{H}_k| \le C\|h\|_{H^1(\Sigma)}$, by the Cauchy-Schwarz inequality and (2.21). Similarly in the case that $n = 1$ and we use $* dG_\Sigma$ to define $\mathscr{H}_1$. Therefore Theorem 2.28 yields the desired result. □

3.5. The bounce operator. Let Σ be a bordered surface of type (g, n) and let $U_k \subseteq \Sigma$ be collar neighbourhoods of $\partial_k\Sigma$ for $k = 1, \ldots, n$. Let $h_k \in \mathcal{D}_{\mathrm{harm}}(U_k)$ for $k = 1, \ldots, n$. Recall that by Theorems 3.10 and 3.11, there is a unique $H \in \mathcal{D}_{\mathrm{harm}}(\Sigma)$ whose CNT boundary values agree with those of h_k on $\partial_k\Sigma$ up to a null set for each $k = 1, \ldots, n$. This fact allows us to define the following operator, which plays a major role in what follows.

Definition 3.23. Set $U = U_1 \cup \cdots \cup U_n$ and let $h : U \to \mathbb{C}$ be the function whose restriction to U_k is h_k for each $k = 1, \ldots, n$. We define

$$\begin{aligned}\mathbf{G}_{U,\Sigma} : \mathcal{D}_{\mathrm{harm}}(U) &\to \mathcal{D}_{\mathrm{harm}}(\Sigma)\\ h &\mapsto H\end{aligned}$$

where H is the unique element of $\mathcal{D}_{\text{harm}}(\Sigma)$ with CNT boundary values agreeing with h up to a null set, which exists by Theorem 3.10. We call this operator the *bounce operator*.

By conformal invariance of CNT limits, the bounce operator is conformally invariant, that is, if $f : \Sigma \to \Sigma'$ is a biholomorphism and $f(U) = U'$, then

$$\mathbf{G}_{U,\Sigma}\mathbf{C}_f = \mathbf{C}_f\mathbf{G}_{U',\Sigma'}. \tag{3.8}$$

Theorem 3.24 (Boundedness of bounce operator)**.** *Let Σ and U_k be as above for $k = 1, \ldots, n$. Then $\mathbf{G}_{U,\Sigma}$ is bounded from $H^1_{\text{conf}}(U)$ to $H^1_{\text{conf}}(\Sigma)$.*

Remark 3.25. Note that a proof of the special case of Theorem 3.24 can be found in [28], Theorem 4.6.

Proof. The goal is to show that if $U = U_1 \cup \cdots \cup U_n$ and if $h : U \to \mathbb{C}$ is a function in $\mathcal{D}_{\text{harm}}(U)$ whose restriction to U_k is h_k for each $k = 1, \ldots, n$, then

$$\|\mathbf{G}_{U,\Sigma}\, h\|_{H^1_{\text{conf}}(\Sigma)} \lesssim \|h\|_{H^1_{\text{conf}}(U)},$$

for $h \in \mathcal{D}_{\text{harm}}(U)$. First, observe that we can assume that the inner boundary of U_k is analytic. To see this, let $U'_k \subseteq U_k$ be a collar neighbourhood whose inner boundary is analytic. Since $\| \left. h_k \right|_{U'_k} \|_{H^1_{\text{conf}}(U'_k)} \leq \|h_k\|_{H^1_{\text{conf}}(U_k)}$, it is enough to show that $\mathbf{G}_{U,\Sigma}$ is bounded with respect to the $H^1_{\text{conf}}(U')$ norm, where $U' = U'_1 \cup \cdots \cup U'_n \subset U$. In what follows, we relabel the new sets by removing the primes.

Next, observe that because CNT boundary values and the Dirichlet norms are conformally invariant, it is enough to prove this for analytic strip-cutting curves $\partial_k\Sigma$, and this can be arranged for example by embedding Σ in its double. Thus, we can assume that ∂U_k is analytic.
Furthermore by the result on the unique Sobolev extension, see e.g. [36, Proposition 4.5, p 356] and the fact that $\partial_k\Sigma \subsetneq \partial U_k$, yields that

$$\| \left. h \right|_{\partial_k\Sigma} \|_{H^{1/2}(\partial_k\Sigma)} \leq \| \left. h \right|_{\partial_k\Sigma} \|_{H^{1/2}(\partial U_k)} = \|h_k\|_{H^1(U_k)}. \tag{3.9}$$

Also, since $\partial\Sigma = \cup_{k=1}^n \partial_k\Sigma$, given the Dirichlet data $\left. h \right|_{\partial_k\Sigma}$, $k = 1, \ldots, n$, on each of the boundary components, Theorem 3.10 yields that the unique harmonic extension H of the boundary values $\left. h \right|_{\partial_k\Sigma}$ satisfies

$$\|H\|_{H^1(\Sigma)} \lesssim \sum_{k=1}^{n} \| \left. h \right|_{\partial_k\Sigma} \|_{H^{1/2}(\partial_k\Sigma)}. \tag{3.10}$$

Now since $H = \mathbf{G}_{U,\Sigma}\, h$, using (3.9) and (3.10) one has

$$\|\mathbf{G}_{U,\Sigma}\, h\|_{H^1(\Sigma)} \lesssim \sum_{k=1}^{n} \|h_k\|_{H^1(U_k)} \lesssim \|h\|_{H^1(U)}. \tag{3.11}$$

Now let $\mathscr{F}(h) := \left(\sum_{k=1}^n |\mathscr{H}_k|^2 \right)^{1/2}$ then $\mathscr{F}$ is clearly non-negative, and (3.12) yields that

$$\mathscr{F}(1) = \Big(\sum_{k=1}^{n} \left| \int_{\partial_k \Sigma} * \, d\omega_k \right|^2 \Big)^{1/2} \neq 0.$$

Assume for the moment that the inner boundary of U_k is an analytic curve γ_k (this can be arranged by taking a level curve of the collar chart of U_k and considering a new domain $U_k' \subseteq U_k$). By Stokes' theorem we have

$$\int_{\partial_k \Sigma} h_k * d\omega_k := \iint_{U_k} dh_k \wedge * \, d\omega_k - \int_{\gamma_k} h_k * d\omega_k \tag{3.12}$$

where we give γ_k the same orientation as $\partial_k \Sigma$. Furthermore the definition (3.12), the Cauchy-Schwarz inequality, (3.9) and Theorem 2.30 yield that

$$\mathscr{F}(h) \leq \sum_{k=1}^{n} | \iint_{U_k} dh_k \wedge * \, d\omega_k | + \sum_{k=1}^{n} | \int_{\gamma_k} h_k * d\omega_k | \lesssim \|h\|_{H^1(U)} \tag{3.13}$$

Now if we had to choose new domains $U_k' \subseteq U_k$ to arrange that γ_k were analytic, since $\|h_k\|_{H^1(U_k')} \leq \|h_k\|_{H^1(U_k)}$ we see that (3.13) holds in general.

This shows that $\mathscr{F}$ is a bounded linear functional on $H^1(U)$ and thereby the conditions of Theorem 2.28 are all satisfied. Hence using (3.11) and Theorem 2.28 we obtain

$$\begin{aligned} \|\mathbf{G}_{U,\Sigma}\, h\|_{H^1(\Sigma)} &\lesssim \|h\|_{H^1(U)} \lesssim \left(\|h\|^2_{\dot{H}^1(U)} + (\mathscr{F}(h))^2 \right)^{1/2} \\ &\lesssim \left(\|h\|^2_{\mathcal{D}_{\text{harm}}(U)} + \sum_{k=1}^{n} |\mathscr{H}_k|^2 \right)^{1/2} \lesssim \|h\|_{H^1_{\text{conf}}(U)}. \end{aligned} \tag{3.14}$$

Now Theorem 3.22 on equivalence of the norms ends the proof of the boundedness of the bounce operator. □

Now as an illuminating example, choose $\Sigma = \mathbb{D}$ and $U = \mathbb{A}$ where $\mathbb{A} = \mathbb{A}_{r,1}$. Choosing $p = 0$ in (3.5), we observe that $* \, dG = d\theta$ where θ is angle in polar coordinates $z = re^{i\theta}$ on $\mathbb{D}$. Thus

$$\mathscr{H}_1 = \int_{\mathbb{S}^1} h(e^{i\theta})\, d\theta,$$

that is, it is just the constant term in the Fourier expansion of the trace of h to the boundary. Using this fact it is elementary to show that

Proposition 3.26. *The subset $\mathbf{G}_{\mathbb{A},\mathbb{D}}\mathcal{D}(\mathbb{A})$ is dense in $H^1_{\text{conf}}(\mathbb{D})$.*

Proof. Given $f \in H^1_{\text{conf}}(\mathbb{D})$ and $\varepsilon > 0$, take a polynomial $p(z) \in \mathcal{D}(\mathbb{A})$ such that $\|f - p\|_{H^1_{\text{conf}}(\mathbb{D})} < \varepsilon$. Now since $\mathbf{G}_{\mathbb{A},\mathbb{D}} f = f$, Theorem 3.24 yields that

$$\|f - \mathbf{G}_{\mathbb{A},\mathbb{D}}\, p\|_{H^1_{\text{conf}}(\mathbb{D})} = \|\mathbf{G}_{\mathbb{A},\mathbb{D}}(f - p)\|_{H^1_{\text{conf}}(\mathbb{D})} \lesssim \|f - p\|_{H^1_{\text{conf}}(\mathbb{D})} < \varepsilon, \tag{3.15}$$

which proves the desired density. □

In order to prove a density result in the case of many boundary curves, we need the following lemma.

Lemma 3.27. *Let Σ be a Riemann surface of type (g, n). For any collar neighbourhood $\phi : U \to \mathbb{A}$ of a border $\partial_k \Sigma$, the map*

$$\mathbf{C}_\phi : H^1_{\mathrm{conf}}(U) \to H^1_{\mathrm{conf}}(\mathbb{A})$$

is a bounded isomorphism.

Proof. Note that we can treat $\partial_k \Sigma$ as an analytic curve in the double and in fact there is a biholomorphism of a doubly-connected neighbourhood of $\partial_k \Sigma$ in the double to a doubly-connected neighbourhood of $\mathbb{S}^1$. So after localizations and partition of unity on the boundary structure of Σ, and using Theorem 2.28, matters reduce to Lemma 2.27. □

We will also need the following when we study overfare in the next section.

Proposition 3.28. *Let Σ be a type (g, n) Riemann surface. Let $\phi_k : U_k \to \mathbb{A}_k$ be a collection of collar charts of the boundaries $\partial_k \Sigma$ for $k = 1, \ldots, n$ and let $U = U_1 \cup \cdots \cup U_n$. Then the restriction map*

$$\mathbf{R}_{\Sigma,U} : H^1_{\mathrm{conf}}(\Sigma) \to H^1_{\mathrm{conf}}(U)$$

is bounded.

Proof. This follows from the definitions of the norms (3.6), (3.7) and Theorem 3.22. □

We may now prove the following:

Theorem 3.29. *Let Σ be a type (g, n) Riemann surface. Let $U = U_1 \cup \cdots \cup U_n$ be a union of collar neighbourhoods U_k of $\partial_k \Sigma$. Then $\mathbf{G}_{U,\Sigma}\mathcal{D}(U)$ is dense in $H^1_{\mathrm{conf}}(\Sigma)$.*

Proof. The proof relies on a factorization trick. Let $F_k : U_k \to \mathbb{A}_k$ be collar charts, and denote $\mathbb{A}^n = \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ and $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$, and define $F : U \to \mathbb{A}$ by $F(z) = (F_1(z), \ldots, F_n(z))$. Define the restriction maps

$$\begin{aligned} \mathbf{R}_{\Sigma,U} : \mathcal{D}_{\mathrm{harm}}(\Sigma) &\to \mathcal{D}_{\mathrm{harm}}(U) \\ h &\mapsto h|_U \end{aligned}$$

and similarly

$$\begin{aligned} \mathbf{R}_{\mathbb{D}^n,\mathbb{A}^n} : \bigoplus^n \mathcal{D}_{\mathrm{harm}}(\mathbb{D}) &\to \bigoplus_{k=1}^n \mathcal{D}_{\mathrm{harm}}(\mathbb{A}_k) \\ (h_1, \ldots, h_n) &\mapsto \left(h_1|_{\mathbb{A}_1}, \ldots, h_n|_{\mathbb{A}_n} \right). \end{aligned}$$

Now

$$\mathbf{C}_F \mathbf{R}_{\mathbb{D}^n,\mathbb{A}^n} : H^1_{\mathrm{conf}}(\mathbb{D}^n) \to H^1_{\mathrm{conf}}(U)$$

is bounded by Lemma 3.27, where we put the direct sum norm on $H^1_{\mathrm{conf}}(\mathbb{D}^n)$. Similarly

$$\mathbf{C}_{F^{-1}} \mathbf{R}_{\Sigma,U} : H^1_{\mathrm{conf}}(\Sigma) \to H^1_{\mathrm{conf}}(\mathbb{A}^n)$$

is bounded. Thus

$$\rho = \mathbf{G}_{U,\Sigma}\,\mathbf{C}_F\,\mathbf{R}_{\mathbb{D}^n,\mathbb{A}^n} : H^1_{\mathrm{conf}}(\mathbb{D}^n) \to H^1_{\mathrm{conf}}(\mathbb{D}^n)$$

is bounded by Theorem 3.24, as is

$$\rho^{-1} = \mathbf{G}_{\mathbb{A}^n,\mathbb{D}^n}\mathbf{C}_{F^{-1}}\mathbf{R}_{\Sigma,U}.$$

The fact that this is the inverse of ρ follows from conformal invariance of CNT boundary values.

Again by conformal invariance of CNT boundary values and the definition of the bounce operator, we have the factorization

$$\mathbf{G}_{U,\Sigma} = \rho\,\mathbf{G}_{\mathbb{A}^n,\mathbb{D}^n}\,\mathbf{C}_{F^{-1}}.$$

Since $\mathbf{C}_{F^{-1}}$ is a bounded invertible map by Lemma 3.27, and we have shown that ρ is a bounded invertible map, then density follows from Proposition 3.26. □

3.6. Overfare of harmonic functions. In this section, we show that given a collection of quasicircles Γ separating a Riemann surface $\mathscr{R}$ into two components Σ_1 and Σ_2, given $h_1 \in \mathcal{D}_{\mathrm{harm}}(\Sigma_1)$ there is an $h_2 \in \mathcal{D}_{\mathrm{harm}}(\Sigma_2)$ with the same boundary values up to a negligible set. We call h_2 the overfare of h_1.

We saw that for the Dirichlet space, the negligible sets are null sets. However, a null set with respect to Σ_1 is not obviously null with respect to Σ_2. we show that this is true for quasicircles. Furthermore, for quasicircles, the overfare exists and is a bounded map with respect to the Dirichlet seminorms, when the originating surface Σ_1 is connected. We will also show that that the overfare map is bounded with respect to H^1_{conf} in the general case, if we assume that the quasicircle is more regular. As we shall see ahead, the so-called Weil-Petersson class quasicircles are sufficient for this purpose.

Definition 3.30. We say that a simple closed curve in $\bar{\mathbb{C}}$ is a *quasicircle* if it is the image of $\mathbb{S}^1$ under a quasiconformal map of the plane.

A simple closed curve Γ in a Riemann surface R is a quasicircle if there is an open set U containing Γ and a biholomorphism $\phi : U \to \mathbb{A}$ where $\mathbb{A}$ is an annulus in $\mathbb{C}$, such that $\phi(\Gamma)$ is a quasicircle.

By definition, a quasicircle is a strip-cutting Jordan curve.

There is a class of quasicircles, called *Weil-Petersson quasicircles*, that arise naturally and frequently in geometric function theory, Teichmüller theory, the theory of Schramm-Loewner evolution, and conformal field theory.

Definition 3.31. We say that a quasicircle in $\bar{\mathbb{C}}$ is a Weil-Petersson class quasicircle (or WP quasicircle) if there is a conformal map $f : \mathbb{D} \to \Omega$ where Ω is one of the connected components of the complement, such that the Schwarzian derivative $S(f) = f'''/f' - 3/2(f''/f')^2$ satisfies

$$\iint_{\mathbb{D}} (1-|z|^2)^2 |S(f)|^2 \frac{d\bar{z} \wedge dz}{2i} < \infty.$$

We say that a quasicircle Γ in a Riemann surface $\mathscr{R}$ is a WP class quasicircle if there is an open set U containing Γ and a biholomorphism $\phi : U \to \mathbb{A}$ where $\mathbb{A}$ is an annulus, such that $\phi(\Gamma)$ is a WP quasicircle.

One characterization of WP quasicircles is that Γ is a WP quasicircle if and only if the f in the definition above has a quasiconformal extension whose Beltrami differential is $L^2_{\mathrm{hyp}}(\mathbb{D}^-)$ where $\mathbb{D}^- = \{z : |z| > 1\} \cup \{\infty\}$, and $L^2_{\mathrm{hyp}}(\mathbb{D}^-)$ is the set of L^2 functions with respect to the hyperbolic metric of the disk. As with the case of general quasicircles, there are in fact a large number of characterizations of WP quasicircles. C. Bishop [4] has listed over twenty, many of which are new. His paper also contains other far-reaching generalizations of the concept of WP quasicircles to higher dimensions.

Having the definition of quasicircles at hand, we consider the following situation.

Definition 3.32. Let $\mathscr{R}$ be a compact Riemann surface, and let $\Gamma_1, \dots, \Gamma_m$ be a collection of quasicircles in $\mathscr{R}$. Denote $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$. We say that Γ *separates* $\mathscr{R}$ into Σ_1 and Σ_2 if

(1) there are doubly-connected neighbourhoods U_k of Γ_k for $k = 1, \dots, n$ such that $U_k \cap U_j$ is empty for all $j \neq k$,
(2) one of the two connected components of $U_k \backslash \Gamma_k$ is in Σ_1, while the other is in Σ_2;
(3) $\mathscr{R} \backslash \Gamma = \Sigma_1 \cup \Sigma_2$;
(4) $\mathscr{R} \backslash \Gamma$ consists of finitely many connected components;
(5) Σ_1 and Σ_2 are disjoint.

Briefly, Σ_1 and Σ_2 are the two “sides” of Γ, and each side is a finite union of Riemann surfaces.

Proposition 3.33. *Let $\mathscr{R}$ be a compact Riemann surface and $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ be a collection of quasicircles separating $\mathscr{R}$ into Σ_1 and Σ_2. Then Σ_1 and Σ_2 are each a finite union of bordered surfaces. For $k = 1, 2$, the inclusion map of Σ_k into $\mathscr{R}$ extends continuously to the border $\partial_k \Sigma$, and this extension is a homeomorphism onto Γ.*

Proof. This follows immediately from Theorem 2.8. □

Thus, we will identify $\partial\Sigma_1$ and $\partial\Sigma_2$ pointwise with Γ. It is important to note that the border structure is entirely independent of the inclusion map, and furthermore the border structures induced by Σ_1 and Σ_2 do not agree in general (unless the curves are analytic). In particular, a border chart in Σ_1 does not in general extend to a chart in $\mathscr{R}$ which is also a border chart of Σ_2, unless the curves Γ_k are analytic.

It is not obvious that a null set in $\partial\Sigma_1$ is null in $\partial\Sigma_2$, even though they are the same set. This holds for quasicircles.

Theorem 3.34. *Let $\mathscr{R}$ be a Riemann surface (not necessarily compact) and $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ be a collection of quasicircles separating $\mathscr{R}$ into Σ_1 and Σ_2. Then $I \subseteq \Gamma$ is null in $\partial\Sigma_1$ if and only if I is null in $\partial\Sigma_2$.*

Proof. It is enough that this is true for a single boundary curve Γ_k. Let $\eta : U \to V$ be a doubly-connected chart in a neighbourhood of Γ. By shrinking U if necessary, we can assume that U is bounded by analytic curves γ_1 and γ_2 in Σ_1 and Σ_2 respectively, and that η has a conformal extension to an open set containing the closure of U so that $\eta(\gamma_1)$ and $\eta(\gamma_2)$ are analytic curves in $\mathbb{C}$. Let $\phi : A \to \mathbb{A}$ be a collar chart in a neighbourhood of Γ in Σ_1 and $\psi : B \to \mathbb{B}$ be a collar chart in a neighbourhood of Γ in Σ_2. Let Ω^+ denote the bounded component of the complement of $\eta(\Gamma)$ in $\bar{\mathbb{C}}$ and Ω^- denote the unbounded component. We assume that η takes $U \cap \Sigma_1$ into Ω^+, again by composing with $z \mapsto 1/z$ if necessary. Finally, by possibly shrinking the domain of η again, we can assume that the analytic curve γ_1 is contained in the domain of ϕ.

Thus, $\phi\circ\eta^{-1}$ is a conformal map of a collar neighbourhood W of $\eta(\Gamma)$ in Ω^+ onto a collar neighbourhood of $\mathbb{S}^1$ in $\mathbb{D}$, whose inner boundary $\phi(\gamma_1)$ is an analytic curve. By the previous paragraph it has a conformal extension to an open neighbourhood of $\eta(\gamma_1)$, and thus the restriction of $\phi\circ\eta^{-1}$ is an analytic diffeomorphism from $\eta(\gamma_1)$ to $\phi(\gamma_1)$. Thus if we let W' be the simply connected set in Ω^+ bounded by $\eta(\gamma_1)$, then there is a quasiconformal map F of W' with a homeomorphic extension to $\eta(\gamma_1)$ equalling $\psi\circ\eta^{-1}$. The map

$$\Phi(z) = \begin{cases} F(z) & z \in W' \\ \phi\circ\eta^{-1}(z) & z \in W \cup \eta(\gamma_1) \end{cases} \tag{3.16}$$

is therefore a quasiconformal map from Ω^+ to $\mathbb{D}$. A similar argument shows that $\psi\circ\eta^{-1}$ has a quasiconformal extension to a map from Ω^- to $\mathbb{D}$.

Since $\eta(\Gamma)$ is a quasicircle, there is a quasiconformal reflection r of the plane which fixes each point in $\eta(\Gamma)$. Thus $\psi\circ\eta^{-1}\circ r\circ(\phi\circ\eta^{-1})^{-1}$ has an extension to an (orientation reversing) quasiconformal self-map of the disk. Thus it extends continuously to a quasisymmetry of $\mathbb{S}^1$, which takes Borel sets of capacity zero to Borel sets of capacity zero. Furthermore, on $\mathbb{S}^1$, this map equals $\psi\circ\phi^{-1}$. Since the same argument applies to $\phi\circ\psi^{-1}$, we have shown that $\phi(I)$ has capacity zero in $\mathbb{S}^1$ if and only if $\psi(I)$ has capacity zero in $\mathbb{S}^1$. This completes the proof. □

Definition 3.35. In the case that Σ is a finite union of connected Riemann surfaces $\Sigma_1, \ldots, \Sigma_s$, we define the Dirichlet seminorm on these components by

$$\|h\|_{\mathcal{D}_{\text{harm}}(\Sigma)} := \sum_{k=1}^{s} \left\| h|_{\Sigma_k} \right\|_{\mathcal{D}_{\text{harm}}(\Sigma_k)}$$

and similarly for the holomorphic and anti-holomorphic Dirichlet spaces, Bergman spaces, etc.

The overfare map was shown to exist and be bounded in the plane.

Theorem 3.36. *Let Γ be a quasicircle in $\bar{\mathbb{C}}$, and let Ω_1 and Ω_2 be the connected components of the complement of Γ. For all $h_1 \in \mathcal{D}_{\text{harm}}(\Omega_1)$ there is an $h_2 \in$*

$\mathcal{D}_{\text{harm}}(\Omega_2)$ *whose* CNT *boundary values agree with those of* h_1 *up to a null set, and one has the estimate*

$$\|h_2\|_{\mathcal{D}_{\text{harm}}(\Omega_2)} \lesssim \|h_1\|_{\mathcal{D}_{\text{harm}}(\Omega_1)}.$$

Proof. See [26] Theorem 3.25. □

In particular, we have well-defined operators

$$\mathbf{O}_{\Omega_1,\Omega_2} : \mathcal{D}_{\text{harm}}(\Sigma_1) \to \mathcal{D}_{\text{harm}}(\Sigma_2)$$

and

$$\mathbf{O}_{\Omega_2,\Omega_1} : \mathcal{D}_{\text{harm}}(\Sigma_2) \to \mathcal{D}_{\text{harm}}(\Sigma_1).$$

If the quasicircle is more regular, we can also control the H^1_{conf} norm.

Definition 3.37 (BZM quasicircle)**.** Let Γ be a quasicircle in $\bar{\mathbb{C}}$, and let Ω_1 and Ω_2 denote the connected components of the complement. We say that Γ is a *bounded zero mode quasicircle* (BZM for short), if $\mathbf{O}_{\Omega_1,\Omega_2}$ and $\mathbf{O}_{\Omega_2,\Omega_1}$ are bounded with respect to $H^1_{\text{conf}}(\Omega_k)$.

A quasicircle Γ in a compact Riemann surface $\mathscr{R}$ is called an BZM quasicircle if there is an open set U containing Γ and a conformal map $\phi : U \to \mathbb{A}$ onto an annulus $\mathbb{A} \subseteq \mathbb{C}$ such that $\phi(\Gamma)$ is a BZM quasicircle.

In this connection we have the following theorem which is built upon deep results regarding flows of Sobolev-vector fields on the unit circle, and also a basic result regarding the action of the group of quasisymmetries of the unit circle, by bounded automorphisms on the homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{S}^1)$ (the action is essentially a composition).

Theorem 3.38. WP *quasicircles are* BZM *quasicircles.*

Proof. It is enough to show that for a WP-class quasisymmetric homeomorphism of the circle ϕ, the composition operator $\mathbf{C}_\phi$ is bounded on the Sobolev space $H^{1/2}(\mathbb{S}^1)$.

Here we note that by the results of A. Figalli [10] and F. Gay-Balmaz and T. Ratiu [11], for a WP-class quasisymmetry ϕ on $\mathbb{S}^1$ both ϕ and its inverse ϕ^{-1} are in $H^{3/2-\varepsilon}(\mathbb{S}^1)$ for all $\varepsilon > 0$. In particular, $\phi^{-1} \in H^1(\mathbb{S}^1)$ in the case that $\varepsilon = 1/2$. Therefore using change of variables and Cauchy-Schwarz's inequality one has

$$\begin{aligned} \|\mathbf{C}_\phi f\|^2_{L^2(\mathbb{S}^1)} &= \int_{\mathbb{S}^1} |f\circ\phi|^2 = \int_{\mathbb{S}^1} |f|^2|(\phi^{-1})'| \leq \\ \Big(\int_{\mathbb{S}^1} |f|^4\Big)^{1/2}&\Big(\int_{\mathbb{S}^1} |(\phi^{-1})'|^2\Big)^{1/2} \leq \|f\|^2_{L^4(\mathbb{S}^1)}\|\phi^{-1}\|_{H^1(\mathbb{S}^1)}. \end{aligned} \tag{3.17}$$

Now if $f \in H^{1/2}(\mathbb{S}^1)$, then the Sobolev embedding (2.16) with $p = 4$ and $s = \frac{1}{2}$ yields that

$$\|f\|_{L^4(\mathbb{S}^1)} \lesssim \|f\|_{H^{1/2}(\mathbb{S}^1)}. \tag{3.18}$$

Thus taking the square root of both sides of (3.17), and using (3.18), we obtain for $f \in H^{1/2}(\mathbb{S}^1)$ that

$$\|\mathbf{C}_\phi f\|_{L^2(\mathbb{S}^1)} \lesssim \|f\|_{H^{1/2}(\mathbb{S}^1)} \|\phi^{-1}\|^{1/2}_{H^1(\mathbb{S}^1)} \lesssim \|f\|_{H^{1/2}(\mathbb{S}^1)}. \tag{3.19}$$

Moreover, by a result of Vodop'yanov-Nag-Sullivan [38] and [16], we also know that

$$\|\mathbf{C}_\phi f\|_{\dot{H}^{1/2}(\mathbb{S}^1)} \lesssim \|f\|_{\dot{H}^{1/2}(\mathbb{S}^1)}. \tag{3.20}$$

Consequently (3.19) and (3.20) yield that

$$\begin{aligned}\|\mathbf{C}_\phi f\|_{H^{1/2}(\mathbb{S}^1)} &\approx \|\mathbf{C}_\phi f\|_{\dot{H}^{1/2}(\mathbb{S}^1)} + \|\mathbf{C}_\phi f\|_{L^2(\mathbb{S}^1)} \\ &\lesssim \|f\|_{\dot{H}^{1/2}(\mathbb{S}^1)} + \|f\|_{H^{1/2}(\mathbb{S}^1)} \lesssim \|f\|_{H^{1/2}(\mathbb{S}^1)}.\end{aligned} \tag{3.21}$$

Note also that the hidden constants in the final estimates of (3.19) and (3.20) only depend on ϕ, which yield the same thing for the last estimate in (3.21). □

Remark 3.39. Regarding the implicit exponentials in the calculations above, let us assume that the angle in the image of the quasisymmetric homeomorphism $\chi : \mathbb{S}^1 \to \mathbb{S}^1$ is $\psi(\theta)$, so that

$$e^{i\psi} = \chi(e^{i\theta}).$$

One could write the estimate in terms of either ϕ or χ. If $\chi'(z)$ denotes the derivative of χ with respect to z, and $\dot{\psi}$ denotes the derivative with respect to θ, we would have

$$\dot{\psi} = \chi'(e^{i\theta}) \frac{e^{i\theta}}{\chi(e^{i\theta})}.$$

From this it immediately follows that

$$|\dot{\psi}| = |\chi'(e^{i\theta})|.$$

In particular the outcome of the estimate is unaffected by the choice.

The next three theorems concern existence and boundedness of the overfare operator for general curve complexes. Their proofs are somewhat involved and will be approached together in stages.

Theorem 3.40 (Existence of overfare)**.** *Let $\mathscr{R}$ be a compact Riemann surface and let $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ be a collection of quasicircles separating $\mathscr{R}$ into Σ_1 and Σ_2. Let $h_1 \in \mathcal{D}_{\mathrm{harm}}(\Sigma_1)$. There is a $h_2 \in \mathcal{D}_{\mathrm{harm}}(\Sigma_2)$ whose* CNT *boundary values agree with those of h_1 up to a null set, and this h_2 is unique.*

This theorem, which we will prove shortly, allows the definition of the overfare operator, which plays an important role in the scattering theory that is developed in [31, 32].

Definition 3.41. With the assumption of Theorem 3.40, we define the *overfare operator* $\mathbf{O}_{\Sigma_1,\Sigma_2}$ by

$$\begin{aligned}\mathbf{O}_{\Sigma_1,\Sigma_2} : \mathcal{D}_{\mathrm{harm}}(\Sigma_1) &\to \mathcal{D}_{\mathrm{harm}}(\Sigma_2) \\ h_1 &\mapsto h_2\end{aligned}$$

One obviously has that

$$\mathbf{O}_{\Sigma_2,\Sigma_1}\mathbf{O}_{\Sigma_1,\Sigma_2} = \mathrm{Id}$$

and of course one can switch the roles of Σ_1 and Σ_2.

The overfare operator is conformally invariant. That is, if $f : \mathscr{R} \to \mathscr{R}'$ is a biholomorphism and we set $f(\Sigma_k) = \Sigma_k'$ for $k = 1, 2$ then it follows immediately from conformal invariance of CNT limits that

$$\mathbf{O}_{\Sigma_1,\Sigma_2}\mathbf{C}_f = \mathbf{C}_f\mathbf{O}_{\Sigma_1',\Sigma_2'}. \tag{3.22}$$

Notation. If Σ_1 and Σ_2 are clear from context, we will denote the overfare operator by $\mathbf{O}_{1,2}$.

We will also obtain two results on boundedness of this operator with respect to H^1_{conf} and the Dirichlet seminorm.

Theorem 3.42 (Bounded overfare theorem for BZM quasicircles)**.** *Let $\mathscr{R}$ be a compact Riemann surface and let $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ be a collection of* BZM *quasicircles separating $\mathscr{R}$ into Σ_1 and Σ_2. There is a constant C such that*

$$\|\mathbf{O}_{1,2}h\|_{H^1_{\mathrm{conf}}(\Sigma_2)} \leq C\|h\|_{H^1_{\mathrm{conf}}(\Sigma_1)}$$

for all $h \in \mathcal{D}_{\mathrm{harm}}(\Sigma_1)$.

One can also obtain Dirichlet boundedness for general quasicircles, but one must assume that the originating surface is connected.

Theorem 3.43 (Bounded overfare theorem for general quasicircles)**.** *Let $\mathscr{R}$ be a compact Riemann surface and let $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ be a collection of quasicircles separating $\mathscr{R}$ into Σ_1 and Σ_2. Assume that Σ_1 is connected. There is a constant C such that*

$$\|\mathbf{O}_{1,2}h\|_{\mathcal{D}_{\mathrm{harm}}(\Sigma_2)} \leq C\|h\|_{\mathcal{D}_{\mathrm{harm}}(\Sigma_1)}$$

for all $h \in \mathcal{D}_{\mathrm{harm}}(\Sigma_1)$.

The remainder of the section is dedicated to proving these three theorems.

Lemma 3.44. *Let $\mathscr{R}$ be a Riemann surface and let Γ be a quasicircle in $\mathscr{R}$. Let $\phi : U \to \mathbb{A}$ be a doubly-connected chart, and let U_1, U_2 be the connected components of $U \backslash \Gamma$. There is an operator*

$$\mathbf{O}(\phi)_{1,2} : \mathcal{D}_{\mathrm{harm}}(U_1) \to \mathcal{D}_{\mathrm{harm}}(U_2)$$

such that the CNT *boundary values of $\mathbf{O}(\phi)_{1,2}h$ agree with those of h up to a null set, and a C such that*

$$\|\mathbf{O}(\phi)_{1,2}h\|_{\mathcal{D}_{\mathrm{harm}}(U_2)} \leq C\|h\|_{\mathcal{D}_{\mathrm{harm}}(U_1)}.$$

If Γ is a BZM *quasicircle, then there is a C' such that for all $h \in \mathcal{D}_{\mathrm{harm}}(U_1)$*

$$\|\mathbf{O}(\phi)_{1,2}h\|_{H^1_{\mathrm{conf}}(U_2)} \leq C'\|h\|_{H^1_{\mathrm{conf}}(U_1)}.$$

Proof. Let Ω_1 and Ω_2 be the connected components of $\bar{\mathbb{C}}\backslash\phi(\Gamma)$ containing $\phi(U_1)$ and $\phi(U_2)$ respectively. We then have a bounded overfare $\mathbf{O}_{\Omega_1,\Omega_2} : \mathcal{D}_{\text{harm}}(\Omega_1) \to \mathcal{D}_{\text{harm}}(\Omega_2)$ by Theorem 3.36. Furthermore, the bounce operator $\mathbf{G}_{\phi(U_k),\Omega_k}$ is bounded with respect to $\mathcal{D}_{\text{harm}}$ by [28, Theorem 4.6]. Defining

$$h_2 = \mathbf{C}_\phi \mathbf{R}_{\Omega_2,U_2} \mathbf{O}_{\Omega_1,\Omega_2} \mathbf{G}_{\phi(U_1),\Omega_1} \mathbf{C}_{\phi^{-1}} h_1, \tag{3.23}$$

by conformal invariance of the Dirichlet seminorm we have proven the first claim. The second claim follows by definition of BZM quasicircles, using Theorem 3.24, and Proposition 3.28. □

We call (3.23) the local overfare of induced by ϕ. It is non-canonical in the sense that it depends on ϕ. Since the values on the other boundaries of U are not specified, the local overfare is not unique.

On the other hand, the overfare to Σ_2 is unique. By combining local overfare with the bounce operator, we can show that the overfare exists.

Proof. (of Theorem 3.40). Let $\phi_k : U^k \to \mathbb{A}^k$ be the doubly-connected charts corresponding to the curves $\Gamma_1, \dots, \Gamma_m$. Denote $U^k_j = U^k \cap \Sigma_j$. Given $h \in \mathcal{D}_{\text{harm}}(\Sigma_1)$, Lemma 3.44 produces a collection of functions $H^k_2 \in \mathcal{D}_{\text{harm}}(U^k_2)$ whose boundary values agree with h.

For each connected component Σ^j_2 of Σ_2, let $\hat{U}_j$ denote the union of those U^k_2 which lie in this component. We now apply the bounce operator $\mathbf{G}_{\hat{U}_j,\Sigma^j_2} : \mathcal{D}_{\text{harm}}(\hat{U}_j) \to \mathcal{D}_{\text{harm}}(\Sigma^j_2)$ on each component separately to obtain a harmonic function in $\mathcal{D}_{\text{harm}}(\Sigma_2)$ whose CNT boundary values agree with h. □

We now prove the boundedness for BZM quasicircles.

Proof. (of Theorem 3.42). The idea is the same as in the previous proof, except that we must keep track of the bounds. Let $\phi_k : U^k \to \mathbb{A}^k$ be doubly-connected charts corresponding to the curves $\Gamma_1, \dots, \Gamma_m$, and let U^k_l be the components of $U_k\backslash\Gamma$ in Σ_l for $l = 1, 2$. Let $C = \sup\{C_1, \dots, C_m\}$ where $C_1, \dots, C_m$ are the constants in the second estimate of Lemma 3.44 for the local overfares from $\mathcal{D}_{\text{harm}}(U^k_1)$ to $\mathcal{D}_{\text{harm}}(U^k_2)$ determined by ϕ_k for $k = 1, \dots, m$. For any $h_1 \in \mathcal{D}_{\text{harm}}(\Sigma_1)$ we have therefore a collection of functions $H^k_2 \in \mathcal{D}_{\text{harm}}(U^k_2)$ such that

$$\left\|H^k_2\right\|_{H^1_{\text{conf}}(U^k_2)} \leq C \left\| h_1|_{U^k_1} \right\|_{H^1_{\text{conf}}(U^k_1)} \leq C\|h_1\|_{H^1_{\text{conf}}(\Sigma_1)} \tag{3.24}$$

where we have also used Proposition 3.28.

Now let $\Sigma^1_2, \dots, \Sigma^s_2$ be the connected components of Σ_2. For each fixed $j \in 1, \dots, s$, let $\hat{U}_j$ be the union of those U^k_2 which are in Σ^j_2, and let h^j_2 be the function whose restriction to $\hat{U}_j$ agrees with the corresponding functions H^k_2. By Theorem 3.24 there is a constant C'_j such that

$$\|\mathbf{G}_{\hat{U}_j,\Sigma_2} h^j_2\|_{H^1_{\text{conf}}(\Sigma_2)} \leq C'_j \|h^j_2\|_{H^1_{\text{conf}}(\hat{U}_j)}. \tag{3.25}$$

Combining (3.24) and (3.25) we obtain

$$\|\mathbf{G}_{\hat{U}_j,\Sigma_2^j} h_2^j\|_{H^1_{\mathrm{conf}}(\Sigma_2^j)} \leq mCC'_j \|h_1\|_{H^1_{\mathrm{conf}}(\Sigma_1)}$$

(where the m appears because there are at most m curves bounding the component Σ_2^j).

Set $C' = \sup\{mCC'_1, \dots, mCC'_s\}$. If we now let h_2 be the function on Σ_2 whose restriction to Σ_2^j is $\mathbf{G}_{\hat{U}_j,\Sigma_2^j} h_2^j$ for $j = 1, \dots, s$, we have that the CNT boundary values of h_2 agree with those of h_1 and

$$\|h_2\|_{H^1_{\mathrm{conf}}(\Sigma_2)} = \sum_{j=1}^{s} \|\mathbf{G}_{\hat{U}_j,\Sigma_2^j} h_2^j\|_{H^1_{\mathrm{conf}}(\Sigma_2^j)} \leq sC' \|h_1\|_{H^1_{\mathrm{conf}}(\Sigma_1)}.$$

□

To prove boundedness with respect to the Dirichlet seminorm, we require three lemmas.

Lemma 3.45. *Let $\mathscr{R}$ be a compact Riemann surface and Γ be a collection of quasicircles separating $\mathscr{R}$ into components Σ_1 and Σ_2. Assume that Σ_1 is connected. If Γ has the property that*

$$\|\mathbf{O}_{1,2} h\|_{H^1_{\mathrm{conf}}(\Sigma_2)} \leq K \|h\|_{H^1_{\mathrm{conf}}(\Sigma_1)}$$

then Γ also has the property that

$$\|\mathbf{O}_{1,2} h\|_{\mathcal{D}_{\mathrm{harm}}(\Sigma_2)} \leq K \|h\|_{\mathcal{D}_{\mathrm{harm}}(\Sigma_1)}.$$

Proof. Denote by h^{ev} the evaluation of h at a fixed point p (alternatively, it can be chosen to be the integral $\mathscr{H}_k$ for some fixed k, see Definition 3.20). For all c constant on Σ_1 we have

$$\begin{aligned}\|\mathbf{O}_{1,2} h\|^2_{\mathcal{D}_{\mathrm{harm}}(\Sigma_2)} &= \|\mathbf{O}_{1,2}(h+c)\|^2_{\mathcal{D}_{\mathrm{harm}}(\Sigma_2)} \leq \|\mathbf{O}_{1,2}(h+c)\|^2_{H^1_{\mathrm{conf}}(\Sigma_2)} \\ &\leq K^2 \|h+c\|^2_{H^1_{\mathrm{conf}}(\Sigma_1)} \\ &= K^2 \left(\|h\|^2_{\mathcal{D}_{\mathrm{harm}}(\Sigma_1)} + |(h+c)^{\mathrm{ev}}|^2 \right).\end{aligned}$$

The claim follows by choosing c such that $c^{\mathrm{ev}} = -h^{\mathrm{ev}}$. □

Lemma 3.46. *For $k = 1,2$ let Γ_k be a quasicircle in a Riemann surface $\mathscr{R}_k$, and let U_k be collar neighbourhoods of Γ_k. Let $f : U_1 \to U_2$ be a quasiconformal map of an open neighbourhood of $U_1 \cup \Gamma_1$ which takes Γ_1 to Γ_2. Let $h : U_2 \to \mathbb{C}$. Then h has a* CNT *limit of ξ at $p \in \Gamma_2$ if and only if $h \circ f$ has a* CNT *limit of ξ at $f^{-1}(p)$.*

Proof. By conformal invariance of CNT boundary values, it's enough for this to hold for $\Gamma_k = \mathbb{S}^1$ for $k = 1,2$, and a quasiconformal map $f : \mathbb{A}_r \to \mathbb{A}_s$ where $\mathbb{A}_r = \{z : r < |z| < 1\}$ and $\mathbb{A}_2 = \{z : s < |z| < 1\}$. For a proof of this fact see [26]. □

Lemma 3.47. *Let $\mathscr{R}$ be a compact Riemann surface, and $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_m$ be a collection of quasicircles separating $\mathscr{R}$ into components Σ_1 and Σ_2. Let $U_1, \dots, U_m$ be collar neighbourhoods of $\Gamma_1, \dots, \Gamma_n$ in Γ_2. There is a quasiconformal map $f : \mathscr{R} \to \mathscr{R}'$ which is conformal on the complement of the closure of $U_1 \cup \cdots \cup U_m$, such that $f(\Gamma_k)$ is analytic for $k = 1, \dots, m$.*

Proof. This was proven in [26] for a single quasicircle using a sewing argument. The proof extends to a complex of curves without issue. □

With these three lemmas in hand, we may now prove boundedness with respect to the Dirichlet seminorm.

Proof. (of Theorem 3.43). By Lemma 3.47 there is a quasiconformal map $f : \mathscr{R} \to \mathscr{R}'$, which is conformal on Σ_1 and takes each quasicircle Γ_j to an analytic curve Γ_j'. Denote $\Sigma_1' = f(\Sigma_1)$ and $\Sigma_2' = f(\Sigma_2)$.

By quasi-invariance of the Dirichlet norm, there is a fixed K such that for any $h \in \mathcal{D}_{\text{harm}}(\Sigma_1)$ we have

$$\|\mathbf{O}_{\Sigma_1',\Sigma_2'}(h \circ f^{-1}) \circ f\|_{\dot{H}^1(\Sigma_2)} \leq K \|\mathbf{O}_{\Sigma_1',\Sigma_2'}(h \circ f^{-1})\|_{\mathcal{D}_{\text{harm}}(\Sigma_2')}. \tag{3.26}$$

Now analytic curves are WP quasicircles, so by Theorems 3.38 and 3.42, $\mathbf{O}_{\Sigma_1',\Sigma_2'}$ is bounded with respect to H^1_{conf}. Since Σ_1' is connected, by Lemma 3.45 there is a K' such that

$$\begin{aligned} \|\mathbf{O}_{\Sigma_1',\Sigma_2'}(h \circ f^{-1})\|_{\mathcal{D}_{\text{harm}}(\Sigma_2')} &\leq K' \|h \circ f^{-1}\|_{\mathcal{D}_{\text{harm}}(\Sigma_1')} \\ &= K' \|h\|_{\mathcal{D}_{\text{harm}}(\Sigma_1)} \end{aligned} \tag{3.27}$$

where the second equality is just invariance of Dirichlet energy under conformal maps.

Finally, by Lemma 3.46, $\mathbf{O}_{\Sigma_1,\Sigma_2} h$ has the same CNT boundary values as $\mathbf{O}_{\Sigma_1',\Sigma_2'}(h \circ f^{-1}) \circ f$. Let

$$F := \mathbf{O}_{\Sigma_1',\Sigma_2'}(h \circ f^{-1}) \circ f - \mathbf{O}_{\Sigma_1,\Sigma_2} h.$$

Then $F \in H^1(\Sigma_2)$ by Theorem 2.32. Then using $F|_{\partial \Sigma_2} = 0$, the harmonicity of $\mathbf{O}_{\Sigma_1,\Sigma_2} h$ and the Sobolev space Stokes' theorem (see e.g. Theorem 4.3.1 page 133 in [9]; note that we treat $\partial \Sigma_2$ as analytic in the double), which also works for manifolds with several oriented boundary curves, one can show that

$$\int_{\Sigma_2} \partial(\mathbf{O}_{\Sigma_1,\Sigma_2} h) \wedge \overline{\partial F} = 0.$$

This yields that

$$\begin{aligned}\|\mathbf{O}_{\Sigma_1,\Sigma_2}h\|^2_{\mathcal{D}_{\mathrm{harm}}(\Sigma_2)} &\leq \|\mathbf{O}_{\Sigma_1,\Sigma_2}h\|^2_{\dot{H}^1(\Sigma_2)} + \|F\|^2_{\dot{H}^1(\Sigma_2)} \\ &= \|\mathbf{O}_{\Sigma_1,\Sigma_2}h\|^2_{\dot{H}^1(\Sigma_2)} + 2\mathrm{Re}\iint_{\Sigma_2} \partial(\mathbf{O}_{\Sigma_1,\Sigma_2}h) \wedge \overline{\partial F} \\ &\quad + \|F\|^2_{\dot{H}^1(\Sigma_2)} \\ &= \|\mathbf{O}_{\Sigma_1',\Sigma_2'}(h\circ f^{-1})\circ f\|^2_{\dot{H}^1(\Sigma_2)},\end{aligned} \tag{3.28}$$

which is just the manifestation of the Dirichlet principle. Therefore we have

$$\|\mathbf{O}_{\Sigma_1,\Sigma_2}h\|_{\mathcal{D}_{\mathrm{harm}}(\Sigma_2)} \leq \|\mathbf{O}_{\Sigma_1',\Sigma_2'}(h\circ f^{-1})\circ f\|_{\dot{H}^1(\Sigma_2)}. \tag{3.29}$$

The claim follows from (3.26), (3.27), (3.29). □

Definition 3.48. For a Riemann surface Σ, with finitely many connected components, let $\dot{\mathcal{D}}_{\mathrm{harm}}(\Sigma)$ be the equivalence classes of $\mathcal{D}_{\mathrm{harm}}(\Sigma)$ modulo functions which are constant on each connected component of Σ.

It is clear that on $\dot{\mathcal{D}}_{\mathrm{harm}}(\Sigma)$ the Dirichlet seminorm becomes a norm.

Let $\mathscr{R}$ be a compact Riemann surface, separated by quasicircles into Σ_1 and Σ_2. If Σ_1 is connected and c is a constant, then $\mathbf{O}_{\Sigma_1,\Sigma_2}$ is also constant on Σ_2 so the operator

$$\dot{\mathbf{O}}_{\Sigma_1,\Sigma_2} : \dot{\mathcal{D}}_{\mathrm{harm}}(\Sigma_1) \to \dot{\mathcal{D}}_{\mathrm{harm}}(\Sigma_2) \tag{3.30}$$

is well-defined. We have

Corollary 3.49. *Let $\mathscr{R}$ be a compact Riemann surface, separated by quasicircles into Σ_1 and Σ_2. Assume that Σ_1 is connected. Then $\dot{\mathbf{O}}_{\Sigma_1,\Sigma_2}$ is bounded with respect to the Dirichlet norm.*

One further observation must be made. As a set, $\partial\Sigma_1 = \Gamma = \partial\Sigma_2$. By Theorem 3.40 and Theorem 3.34, we now have the following striking result.

Corollary 3.50. *Let $\mathscr{R}$ be a compact Riemann surface and $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ be a family of quasicircles separating $\mathscr{R}$ into Σ_1 and Σ_2. Then*

$$\mathcal{H}(\partial\Sigma_1) = \mathcal{H}(\partial\Sigma_2).$$

We can now define

$$\mathcal{H}(\Gamma) = \mathcal{H}(\partial\Sigma_1) = \mathcal{H}(\partial\Sigma_2).$$

This result requires the fact that Γ consists of quasicircles and does not appear to hold in general. In the case of the Riemann sphere, the authors have shown that it holds with a Dirichlet-bounded identification of the spaces, precisely for quasicircles [25].

References

[1] Ahlfors, Lars V.; Sario, Leo. Riemann surfaces. Princeton Mathematical Series, No. 26. *Princeton University Press, Princeton, N.J.*, 1960. xi+382 pp. MR0114911 (22 #5729), Zbl 0196.33801. 326, 340

[2] Bergman, Stefan; and Schiffer, Menahem M. Kernel functions and conformal mapping. *Compositio Math.* **8**, (1951), 205–249. MR0039812 (12,602c), Zbl 0043.08403

[3] Booss-Bavnbek, B.; Wojciechowski, K. P. Elliptic boundary problems for Dirac operators. Mathematics: Theory and Applications. Birkhäuser Boston, Inc., Boston, MA, 1993. MR1233386 (94h:58168), Zbl 0797.58004 doi: 10.1007/978-1-4612-0337-7 335, 336

[4] Bishop, Chris. J. Weil-Petersson curves, conformal energies, β-numbers, and minimal surfaces. *To appear in Annals of Mathematics.* Available at http://www.math.stonybrook.edu/ bishop/papers/wpbeta.pdf. 356

[5] Brewster, Kevin; Mitrea, Darius; Mitrea, Irina; Mitrea, Marius. Extending Sobolev functions with partially vanishing traces from locally (ε, δ)-domains and applications to mixed boundary problems. *J. Funct. Anal.* **266** (2014), no. 7, 4314–4421. MR3170211, Zbl 1312.46042, doi: 10.1016/j.jfa.2014.02.001. 345

[6] Chipot, Michel. Elements of nonlinear analysis. *Birkhäuser Advanced Texts: Basler Lehrbücher.* [Birkhäuser Advanced Texts: Basel Textbooks] Birkhäuser Verlag, Basel, 2000. MR1801735 (2002c:35001), Zbl 0964.35002, doi: 10.1007/978-3-0348-8428-0. 337

[7] Conway, John. B. Functions of one complex variable II. *Graduate Texts in Mathematics*, 159. Springer-Verlag, New York, 1995. MR1344449 (96i:30001), Zbl 0887.30003, doi: 10.1007/978-1-4612-0817-4. 328, 329

[8] Douglas, J. Solution of the problem of Plateau. *Trans. Amer. Math. Soc.* **33** (1931), no. 1, 263-–321. MR1501590, Zbl 0001.14102, doi: 10.2307/1989472. 335

[9] Evans, Lawrence C.; Gariepy, Ronald F. Measure theory and fine properties of functions. *Studies in Advanced Mathematics.* CRC Press, Boca Raton, FL, 1992. MR1158660 (93f:28001), Zbl 0804.28001, doi: doi.org/10.1201/b18333. 363

[10] Figalli, Alessio. On flows of $H^{3/2}$-vector fields on the circle. *Math. Ann.* **347** (2010), no. 1, 43—57. MR2593282 (2011k:46045), Zbl 1195.46032, doi: 10.1007/s00208-009-0426-5. 358

[11] Gay-Balmaz, François; Ratiu, Tudor S. The geometry of the universal Teichmüller space and the Euler-Weil-Petersson equation. *Adv. Math.* **279** (2015), 717—778. MR3345193, Zbl 1320.32018, doi: 10.1016/j.aim.2015.04.005. 358

[12] Gilkey, Peter B. Invariance theory, the heat equation, and the Atiyah-Singer index theorem. Second edition. *Studies in Advanced Mathematics.* CRC Press, Boca Raton, FL, 1995. ISBN: 0-8493-7874-4. MR1396308 (98b:58156), Zbl 0856.58001. 336

[13] Jonsson, Alf. Besov spaces on closed subsets of $\mathbf{R}^n$. *Trans. Amer. Math. Soc.* **341** no. 1, (1994), 355–370. MR1132434 (94c:46065), Zbl 0803.46035, doi: 10.2307/2154626. 323

[14] Jonsson, Alf; Wallin, Hans. Function spaces on subsets of $\mathbf{R}^n$. *Math. Rep.* **2** no. 1, 1984. MR0820626 (87f:46056), Zbl 0875.46003. 323

[15] Nag, Subhashis. The Complex Analytic Theory of Teichmüller Spaces, Canadian Mathematical Society Series of Monographs and Advanced Texts. *John Wiley & Sons, Inc., New York* 1988. xiv+427 pp. ISBN:0-471-62773-9. MR0927291 (89f:32040), Zbl 0667.30040. 324

[16] Nag, Subhashis; Sullivan, Dennis. Teichmüller theory and the universal period mapping via quantum calculus and the $H^{1/2}$ space on the circle. *Osaka J. Math.* **32** (1995), no. 1, 1–34. MR1323099 (96c:32023), Zbl 0820.30027. 359

[17] Osborn, Howard. The Dirichlet functional I. *J. Math. Anal. Appl.* **1** (1960), 61–112. MR133473 (24 #A3304), Zbl 0099.08103, doi: 10.1016/0022-247X(60)90030-5 323

[18] Pommerenke, Christian. Boundary behaviour of conformal maps. *Grundelehren der mathematischen Wissenschaften* v 299, Springer-Verlag, 1991.MR1217706 (95b:30008), Zbl 0762.30001, doi: 10.1007/978-3-662-02770-7. 343

[19] RADNELL, DAVID; SCHIPPERS, ERIC. Quasisymmetric sewing in rigged Teichmüller space, *Commun. Contemp. Math.* **8** (2006), no. 4, 481–534. MR2258875 (2007f:32015), Zbl 1108.30034, doi: 10.1142/S0219199706002210. 324

[20] RADNELL, DAVID; AND SCHIPPERS, ERIC. Fiber structure and local coordinates for the Teichmüller space of a bordered Riemann surface, *Conform. Geom. Dyn.* **14** (2010), 14–34. MR2593332 (2011b:32018), Zbl 1196.30040, doi: 10.1090/S1088-4173-10-00206-7. 324

[21] RADNELL, DAVID; SCHIPPERS, ERIC; AND STAUBACH, WOLFGANG. *Quasiconformal Teichmüller theory as an analytical foundation for two dimensional conformal field theory*. Lie algebras, vertex operator algebras, and related topics, Contemp. Math., **695**, 205–238, Amer. Math. Soc., Providence, RI, 2017. MR3709713, Zbl 1393.30034, doi: 10.1090/conm/695/14003. 322

[22] RADNELL, DAVID; SCHIPPERS, ERIC; SHIRAZI, MOMAHAMMAD; AND STAUBACH, WOLFGANG. Schiffer operators and calculation of a determinant line in conformal field theory, *New York J. Math.* **27** (2021), 253–271. MR4209534, Zbl https://zbmath.org/1456.30026. 322

[23] RUDIN, WALTER. Functional analysis. Second edition. *International Series in Pure and Applied Mathematics.* McGraw-Hill, Inc., New York, 1991. MR1157815 (92k:46001), Zbl 0867.46001. 337

[24] SCHIPPERS, ERIC; AND STAUBACH, WOLFGANG. Riemann boundary value problem on quasidisks, Faber isomorphism and Grunsky operator. *Complex Anal. Oper. Theory* **12** (2018) no. 2, 325–354. MR3756161, Zbl 1391.30054, doi: 10.1007/s11785-016-0598-4. 322

[25] SCHIPPERS, ERIC; AND STAUBACH, WOLFGANG. Harmonic reflection in quasicircles and well-posedness of a Riemann-Hilbert problem on quasidisks. *J. Math. Anal. Appl.* **448** (2017), no. 2, 864–884.MR3582264, Zbl 1357.31001, doi: 10.1016/j.jmaa.2016.11.041. 324, 344, 364

[26] SCHIPPERS, ERIC; STAUBACH, WOLFGANG. Transmission of harmonic functions through quasicircles on compact Riemann surfaces. *Ann. Acad. Sci. Fenn. Math.* **45** (2020), no. 2, 1111-–1134. MR4112278, Zbl 1461.30051, doi: 10.5186/aasfm.2020.4559. 324, 325, 338, 345, 346, 349, 358, 362, 363

[27] SCHIPPERS, ERIC; STAUBACH, WOLFGANG. Plemelj-Sokhotski isomorphism for quasicircles in Riemann surfaces and the Schiffer operator. *Math. Ann.* **378** (2020), no. 3-4, 1613-–1653. MR4163537, Zbl 1461.30098, doi: 10.1007/s00208-019-01922-4. 322, 324, 348

[28] SCHIPPERS, ERIC; STAUBACH, WOLFGANG. Analysis on quasicircles – A unified approach through transmission and jump problems. *EMS Surv. Math. Sci.* **9** (2022), no.1, 31—97. MR4551458, Zbl 1515.30060, doi: 10.4171/EMSS/53. 322, 324, 348, 352, 361

[29] SCHIPPERS, ERIC; AND STAUBACH, WOLFGANG. A scattering theory of harmonic one-forms on Riemann surfaces. arXiv:2112.00835v1. 321

[30] SCHIPPERS, ERIC; AND STAUBACH, WOLFGANG. Overfare of harmonic one-forms on Riemann surfaces. *New York J. Math.* **30** (2024), 1437–1478. MR4805525, Zbl 07941341. 322, 342

[31] SCHIPPERS, ERIC; AND STAUBACH, WOLFGANG. Scattering theory on Riemann surfaces I: Schiffer operators, cohomology, and index theorems. Preprint. 322, 342, 359

[32] SCHIPPERS, ERIC; AND STAUBACH, WOLFGANG. Scattering theory on Riemann surfaces II: the scattering matrix and generalized period mappings. Preprint. 322, 342, 359

[33] SCHIPPERS, ERIC; AND STAUBACH, WOLFGANG. A Survey of Scattering Theory on Riemann Surfaces with Applications in Global Analysis and Geometry. *Special issue of Vietnam Journal of Mathematics dedicated to Carlos E. Kenig's 70th birthday*, **51** 4 (2023), 911–934. MR4669022, Zbl 1532.14043, doi: 10.1007/s10013-023-00636-0. 321

[34] SCHIPPERS, ERIC; AND SHIRAZI, MOHAMMAD. Faber series for L^2 holomorphic one-forms on Riemann surfaces with boundary. *Comp. Meth. and Func. Theor.* doi: 10.1007/s40315-024-00529-4. arXiv:2303.15677v1 323

[35] SCHIPPERS, ERIC; SHIRAZI, MOHAMMAD; AND STAUBACH, WOLFGANG. Schiffer comparison operators and approximations on Riemann surfaces bordered by quasicircles, *J. Geom.*

Anal. **31** (2021), no. 6, 5877–5908. MR4267630, Zbl 1475.30101, doi: 10.1007/s12220-020-00508-w. 322

[36] TAYLOR, MICHAEL E. Partial differential equations I. Basic theory. Third edition. *Applied Mathematical Sciences*, 115. Springer, New York, 2011. MR4703940, Zbl 1527.35002, doi: 10.1007/978-3-031-33859-5. 346, 352

[37] TRIEBEL, HANS. Theory of function spaces. Reprint of 1983 edition. Also published in 1983 by Birkhäuser Verlag. *Modern Birkhäuser Classics.* Birkhäuser/Springer Basel AG, Basel, 2010. MR3024598, Zbl 1235.46002, doi: 10.1007/978-3-0346-0416-1. 335

[38] VODOP'YANOV, S. K. Mappings of homogeneous groups and embeddings of function spaces, *Siberian Math. Zh.* **30** (1989), 25–41. MR1025287 (90m:46061), Zbl 0731.46019, doi: 10.1007/BF00971258. 359

(Eric Schippers) MACHRAY HALL, DEPT. OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG, MB R3T 2N2, CANADA
eric.schippers@umanitoba.ca

(Wolfgang Staubach) DEPARTMENT OF MATHEMATICS, UPPSALA UNIVERSITY, S-751 06 UPPSALA, SWEDEN
wulf@math.uu.se

This paper is available via http://nyjm.albany.edu/j/2025/31-14.html.