New York Journal of Mathematics

New York J. Math. **31** (2025) 438–464.

Hyperbolic L-space knots and their upsilon invariants

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ABSTRACT. For a knot in the 3–sphere, the Upsilon invariant is a piecewise linear function defined on the interval $[0, 2]$. It is known that for an L–space knot, the Upsilon invariant is determined only by the Alexander polynomial. We exhibit infinitely many pairs of hyperbolic L–space knots such that two knots of each pair have distinct Alexander polynomials, so they are not concordant, but share the same Upsilon invariant. Conversely, we examine the restorability of the Alexander polynomial of an L–space knot from the Upsilon invariant through the Legendre–Fenchel transformation.

CONTENTS

1. Introduction

For a knot K in the 3–sphere S^3, Ozsváth, Stipsicz and Szabó [30] defined the Upsilon invariant $\Upsilon_K(t)$, which is a piecewise linear real-valued function defined on the interval $[0, 2]$. This invariant is additive under connected sum of knots, and the sign changes for the mirror image of a knot. Also, it gives a lower bound for the genus, the concordance genus and the four genus. Although it is originally defined through some modified knot Floer complex, Livingston [23] later gives an alternative interpretation on the full knot Floer complex CFK^∞.

As the most important feature, the Upsilon invariant is a concordance invariant, so it is not apparently strong enough to distinguish knots, although it

Received November 22, 2022.

2010 *Mathematics Subject Classification.* 57M25, 57M27.

Key words and phrases. Upsilon invariant, L–space knot.

The author has been supported by JSPS KAKENHI Grant Number 20K03587.

ISSN 1076-9803/2025

has been used to establish various powerful results about independent elements in the knot concordance group [11, 16, 30, 37]. For a smoothly slice knot, the Upsilon invariant is the zero function. It depends only on the signature for an alternating knot or a quasi-alternating knot [30]. Also, it is determined by the τ–invariant for concordance genus one knots [11].

In this paper, we concentrate on L–space knots, which are recognized to form an important class of knots in recent research. A knot is called an *L–space knot* if it admits a positive surgery yielding an L–space. Positive torus knots are typical examples of L–space knots. Note that any non-trivial L–space knot is prime [19] and non-slice [27]. For an L–space knot, the Upsilon invariant is determined only by the Alexander polynomial [30, Theorem 6.2].

There is another interesting route to lead to the Upsilon invariant of an L–space knot. The Alexander polynomial gives the formal semigroup [36], in turn, the gap function [6]. These notions have the same information as the Alexander polynomial. Then the Upsilon invariant is obtained as the Legendre–Fenchel transform of the gap function [5].

In general, the gap function for an L–space knot is not convex, so the further Legendre–Fenchel transformation on the Upsilon invariant does not return the original gap function. Thus there is a possibility that distinct gap functions, equivalently Alexander polynomials, correspond to the same Upsilon invariant. In other words, it is expected to exist non-concordant L–space knots with the same Upsilon invariant. Among torus knots, there is no duplication of Upsilon invariant. Our main result shows that this is possible among hyperbolic L–space knots.

Theorem 1.1. *There exist infinitely many pairs of hyperbolic L–space knots K_1 and K_2 such that they have distinct Alexander polynomials but share the same non-zero Upsilon invariant.*

We remark that the Alexander polynomial is a concordance invariant for L–space knots [19]. Thus two hyperbolic L–space knots in our pair are not concordant. In the literature, there are plenty of examples of non-concordant knots sharing the same Upsilon invariant [1, 11, 17, 37, 38, 39, 41]. However, these are not L–space knots.

Since the Upsilon invariant is determined only by the Alexander polynomial for an L–space knot, any pair of L–space knots sharing the same Alexander polynomial have the same Upsilon invariant. For example, the hyperbolic L–space knot `t09847` in the SnapPy census has the same Alexander polynomial as the $(2,7)$–cable of $T(2,5)$, which is an L–space knot. There are infinitely many such pairs consisting of a hyperbolic L–space knot and an iterated torus L–space knot (found in [3]).

However, we checked Dunfield's list of 632 hyperbolic L–space knots ([2, 3]), and confirmed that there is no duplication among their Alexander polynomials and that there is no one sharing the same Alexander polynomial as a torus knot. This leads us to pose a question.

Question 1.2. (1) *Do there exist hyperbolic L–space knots which have the same Alexander polynomial? Do there exist hyperbolic L–space knots which are concordant?*
(2) *Does there exist a hyperbolic L–space knot which is concordant to a torus knot?*

In general, it is rare that the Alexander polynomial of an L–space knot is restorable from the Upsilon invariant. The reason is the fact that the gap function, which has the same information as the Alexander polynomial, is not convex, and the Upsilon invariant depends only on the convex hull of the gap function. In fact, our knots in Theorem 1.1 are designed so that they have distinct Alexander polynomials, but their gap functions share the same convex hull, so the same Upsilon invariant.

On the other hand, the gap function may be restorable from its convex hull. This means that the Alexander polynomial may also be restorable from the Upsilon invariant through the Legendre–Fenchel transformation. We can give infinitely many such gap functions, equivalently Alexander polynomials, but there lies a hard question, called a geography question, whether such gap function can be realized by an L-space knot or not.

In this paper, we can give only two hyperbolic L–space knots whose Alexander polynomials are restorable from the Upsilon invariants.

Theorem 1.3. *Let K be the hyperbolic L–space knot* `t09847` *or* `v2871` *in the SnapPy census. Then the Alexander polynomial $\Delta_K(t)$ of K is restorable from the Upsilon invariant $\Upsilon_K(t)$. That is, the equation $\Upsilon_K(t) = \Upsilon_{K'}(t)$ implies $\Delta_K(t) = \Delta_{K'}(t)$ (up to units) for any other L–space knot K'.*

In Section 2, we give a pair of knots K_1 and K_2, which yields an infinite family of pairs of L–space knots. In Section 3, we calculate their Alexander polynomials and the formal semigroups, which are sufficient to prove that the knots are hyperbolic. Section 4 gives the gap functions and their convex hulls, and confirm that they correspond to the same Upsilon invariant. Section 5 shows that the knots admit L–space surgery through the Montesinos trick, which completes the proof of Theorem 1.1. In the last section, we investigate the restorability of Alexander polynomial from the Upsilon invariant, and prove Theorem 1.3.

2. The pairs of hyperbolic L-space knots

For any integer integer $n \geq 1$, the surgery diagrams illustrated in Figure 1 define our knots K_1 and K_2, where the surgery coefficient on C_1 is $-1/n$ and that on C_2 is $-1/2$. The images of K after these surgeries in (1) and (2) of Figure 1 give K_1 and K_2, respectively. (The link with orientations is placed in a strongly invertible position, and the axis is depicted there for later use.)

Hence, our knots are the closures of 4–braids

$$(\sigma_2\sigma_1\sigma_3\sigma_2)(\sigma_1\sigma_2\sigma_3)^{4n}\sigma_2^{-1}(\sigma_2\sigma_3)^6 \quad \text{and} \quad (\sigma_2\sigma_1\sigma_3\sigma_2)(\sigma_1\sigma_2\sigma_3)^{4n}\sigma_3^{-1}(\sigma_2\sigma_3)^6,$$

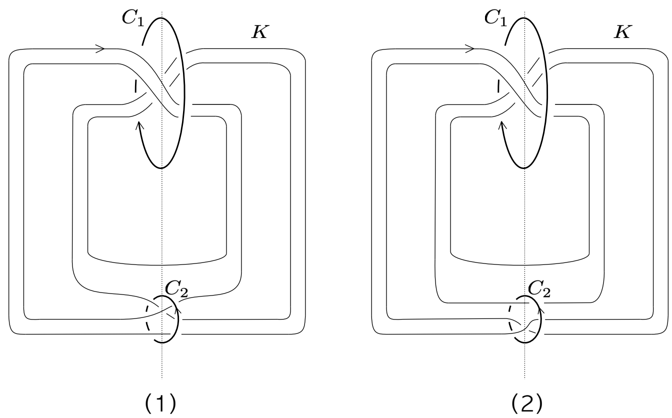


FIGURE 1. The knots K_1 and K_2 are the images of K after performing $(-1/n)$–surgery on C_1 and $(-1/2)$–surgery on C_2.

where σ_i is the standard generator of the 4–strand braid group. When $n = 1$, K_1 is `m240`, and K_2 is `t10496` in the SnapPy census [8].

Theorem 1.1 immediately follows from the next.

Theorem 2.1. *For each integer $n \geq 1$, the knots K_1 and K_2 defined above satisfy the following.*

(1) *They are hyperbolic.*
(2) $(16n + 21)$*–surgery on K_1 and* $(16n + 20)$*–surgery on K_2 yield L–spaces.*
(3) *Their Alexander polynomials are distinct.*
(4) *They share the same Upsilon invariant.*

Proof. This follows from Lemmas 3.6, 5.1, 5.2, Theorems 3.1, 3.2, and Corollary 4.3. (To see that the Alexander polynomials of K_1 and K_2 are distinct, it may be easier to compare their formal semigroups. From Propositions 3.3 and 3.4, we have that $4n + 7 \in \mathcal{S}_{K_1}$, but $4n + 7 \notin \mathcal{S}_{K_2}$.) □

Each diagram in Figure 1 has a single negative crossing, but it can be cancelled obviously with some positive crossing. Hence both knots are represented as the closures of positive braids, which implies that they are fibered [32]. Then it is straightforward to calculate their genera $g(K_i)$, and we see that $g(K_1) = g(K_2) = 6n + 6$.

Also, if once we know that K_i is an L–space knot, then r–surgery on K_i gives an L–space if and only if $r \geq 2g(K_i) - 1 = 12n + 11$ by [14, 29]. Our choices of surgery coefficients in Theorem 2.1(2) come from the manageability in the process of the Montesinos trick in Section 5.

3. Alexander polynomials

We calculate the Alexander polynomials of K_1 and K_2. Since K_1 and K_2 are obtained from K by performing some surgeries on C_1 and C_2, we mimic the technique of [3].

Theorem 3.1. *The Alexander polynomial of K_1 is given as*

$$\Delta_{K_1}(t) = \sum_{i=0}^{n}(t^{8n+12+4i} - t^{8n+11+4i}) + (t^{8n+9} - t^{8n+8}) + \sum_{i=0}^{n}(t^{4n+6+4i} - t^{4n+4+4i})$$
$$+ (t^{4n+3} - t^{4n+1}) + \sum_{i=0}^{n-1}(t^{4+4i} - t^{1+4i}) + 1.$$

Proof. Let $L = K \cup C_1 \cup C_2$ be the oriented link illustrated in Figure 1(1). Its multivariable Alexander polynomial is

$$\Delta_L(x,y,z) = x^6y^3z^2 + x^5y^2z - x^3y^3z^2 + x^3y^2z^2 - x^3y^2z$$
$$- x^2y^2z^2 + x^4y + x^3yz - x^3y + x^3 - xyz - 1,$$

where the variables x, y, z correspond to the (oriented) meridians of K, C_1, C_2, respectively. (We used [8, 18] for the calculation.)

Perform $(-1/n)$–surgery on C_1 and $(-1/2)$–surgery on C_2. Let C_1^n and C_2^n be the images of C_1 and C_2, respectively, after these surgeries. Then the link $K \cup C_1 \cup C_2$ changes to $K_1 \cup C_1^n \cup C_2^n$. These two links have homeomorphic exteriors, and the homeomorphism induces the isomorphism on their homology groups which relates the Alexander polynomials of two links [12, 25].

Let μ_K, μ_{C_1} and μ_{C_2} be the homology classes of meridians of K, C_1, C_2, respectively. We assume that each meridian has linking number one with the corresponding component. Furthermore, let λ_K, λ_{C_1} and λ_{C_2} be the homology classes of their oriented longitudes. We see that $\lambda_{C_1} = 4\mu_K$ and $\lambda_{C_2} = 3\mu_K$.

Let μ_{K_1}, $\mu_{C_1^n}$ and $\mu_{C_2^n}$ be the homology classes of meridians of K_1, C_1^n and C_2^n. Then we have that $\mu_{K_1} = \mu_K$, $\mu_{C_1^n} = -\mu_{C_1} + n\lambda_{C_1}$, $\mu_{C_2^n} = -\mu_{C_2} + 2\lambda_{C_2}$. Hence

$$\mu_K = \mu_{K_1}, \quad \mu_{C_1} = -\mu_{C_1^n} + 4n\mu_{K_1}, \quad \mu_{C_2} = -\mu_{C_2^n} + 6\mu_{K_1}.$$

Thus we have the relation between the Alexander polynomials as

$$\Delta_{K_1 \cup C_1^n \cup C_2^n}(x,y,z) = \Delta_L(x, x^{4n}y^{-1}, x^6z^{-1}). \tag{3.1}$$

Since $\mathrm{lk}(K_1, C_2^n) = \mathrm{lk}(K, C_2) = 3$ and $\mathrm{lk}(C_1^n, C_2^n) = \mathrm{lk}(C_1, C_2) = 0$, the Torres condition [35] gives

$$\Delta_{K_1 \cup C_1^n \cup C_2^n}(x,y,1) = (x^3y^0 - 1)\Delta_{K_1 \cup C_1^n}(x,y)$$
$$= (x^3 - 1)\Delta_{K_1 \cup C_1^n}(x,y).$$

Furthermore, since $\mathrm{lk}(K_1, C_1^n) = \mathrm{lk}(K, C_1) = 4$,

$$\Delta_{K_1 \cup C_1^n}(x,1) = \frac{x^4 - 1}{x - 1}\Delta_{K_1}(x).$$

Thus

$$\Delta_{K_1}(x) = \frac{x-1}{x^4-1}\Delta_{K_1 \cup C_1^n}(x,1) = \frac{x-1}{(x^4-1)(x^3-1)}\Delta_{K_1 \cup C_1^n \cup C_2^n}(x,1,1).$$

Then the relation (3.1) gives

$$\begin{aligned}
\Delta_{K_1}(t) &= \frac{t-1}{(t^4-1)(t^3-1)}\Delta_L(t, t^{4n}, t^6) \\
&= \frac{t-1}{(t^4-1)(t^3-1)}(t^{12n+18} - t^{12n+15} + t^{8n+15} - t^{8n+14} + t^{8n+11} - t^{8n+9} \\
&\quad + t^{4n+9} - t^{4n+7} + t^{4n+4} - t^{4n+3} + t^3 - 1) \\
&= \frac{1}{t^3+t^2+t+1} \cdot \frac{1}{t^3-1}(t^{12n+15}(t^3-1) + t^{8n+9}(t^6-1) - t^{8n+11}(t^3-1) \\
&\quad + t^{4n+3}(t^6-1) - t^{4n+4}(t^3-1) + (t^3-1)) \\
&= \frac{1}{t^3+t^2+t+1}(t^{12n+15} - t^{8n+11} - t^{4n+4} + 1 + (t^{8n+9} + t^{4n+3})(t^3+1)).
\end{aligned}$$

We put

$$\begin{aligned}
A_1 &= \sum_{i=0}^{n}(t^{8n+12+4i} - t^{8n+11+4i}), \\
A_2 &= t^{8n+9} - t^{8n+8}, \\
A_3 &= \sum_{i=0}^{n}(t^{4n+6+4i} - t^{4n+4+4i}), \\
A_4 &= t^{4n+3} - t^{4n+1}, \\
A_5 &= \sum_{i=0}^{n-1}(t^{4+4i} - t^{1+4i}).
\end{aligned}$$

Then a direct calculation shows

$$\begin{aligned}
(t^3+t^2+t+1)A_1 &= t^{12n+15} - t^{8n+11}, \\
(t^3+t^2+t+1)A_2 &= t^{8n+12} - t^{8n+8}, \\
(t^3+t^2+t+1)A_3 &= (t^{8n+9} - t^{4n+5}) + (t^{8n+8} - t^{4n+4}), \\
(t^3+t^2+t+1)A_4 &= (t^{4n+6} - t^{4n+2}) + (t^{4n+5} - t^{4n+1}), \\
(t^3+t^2+t+1)A_5 &= (t^{4n+3} - t^3) + (t^{4n+2} - t^2) + (t^{4n+1} - t).
\end{aligned}$$

Thus

$$\begin{aligned}
(t^3+t^2+t+1)(A_1+A_2+A_3+A_4+A_5+1) &= t^{12n+15} - t^{8n+11} - t^{4n+4} + 1 \\
&\quad + t^{8n+12} + t^{8n+9} + t^{4n+6} + t^{4n+3}.
\end{aligned}$$

We have the conclusion $\Delta_{K_1}(t) = A_1 + A_2 + A_3 + A_4 + A_5 + 1$ as desired. □

Theorem 3.2. *The Alexander polynomial of K_2 is given as*

$$\Delta_{K_2}(t) = \sum_{i=0}^{n}(t^{8n+12+4i} - t^{8n+11+4i}) + (t^{8n+9} - t^{8n+8}) + \sum_{i=0}^{2n-1}(t^{4n+8+2i} - t^{4n+7+2i}) + (t^{4n+6} - t^{4n+4}) + (t^{4n+3} - t^{4n+1}) + \sum_{i=0}^{n-1}(t^{4+4i} - t^{1+4i}) + 1.$$

Proof. The argument is very similar to the proof of Theorem 3.1, so we omit the details.

Let $L = K \cup C_1 \cup C_2$ be the oriented link illustrated in Figure 1(2). Its multivariable Alexander polynomial is

$$\Delta_L(x, y, z) = x^6y^3z^2 - x^3y^3z^2 + x^4y^2z + x^5yz + x^3y^2z^2 - x^3y^2z - x^4yz - x^2y^2z^2 + x^4y + x^2y^2z + x^3yz - x^3y - xy^2z - x^2yz + x^3 - 1.$$

where x, y, z correspond to the meridians of K, C_1, C_2, respectively.

Then

$$\begin{aligned}\Delta_{K_2}(t) &= \frac{t-1}{(t^4-1)(t^3-1)}\Delta_L(t, t^{4n}, t^6)\\ &= \frac{t-1}{(t^4-1)(t^3-1)}(t^{12n+18} - t^{12n+15} + t^{8n+10} + t^{4n+11} + t^{8n+15}\\ &\quad - t^{8n+9} - t^{4n+10} - t^{8n+14} + t^{4n+4} + t^{8n+8} + t^{4n+9} - t^{4n+3}\\ &\quad - t^{8n+7} - t^{4n+8} + t^3 - 1)\\ &= \frac{1}{t^3+t^2+t+1}\cdot\frac{1}{t^3-1}((t^{12n+15} + t^{8n+7} + t^{4n+8} + 1)(t^3-1)\\ &\quad + (t^{8n+9} - t^{4n+4} - t^{8n+8} + t^{4n+3})(t^6-1))\\ &= \frac{1}{t^3+t^2+t+1}(t^{12n+15} + t^{8n+7} + t^{4n+8} + 1\\ &\qquad + (t^{8n+9} - t^{4n+4} - t^{8n+8} + t^{4n+3})(t^3+1)).\end{aligned}$$

Again, we put

$$\begin{aligned}B_1 &= \sum_{i=0}^{n}(t^{8n+12+4i} - t^{8n+11+4i}),\\ B_2 &= t^{8n+9} - t^{8n+8},\\ B_3 &= \sum_{i=0}^{2n-1}(t^{4n+8+2i} - t^{4n+7+2i}),\\ B_4 &= t^{4n+6} - t^{4n+4},\\ B_5 &= t^{4n+3} - t^{4n+1},\\ B_6 &= \sum_{i=0}^{n-1}(t^{4+4i} - t^{1+4i}).\end{aligned}$$

A direct calculation shows

$$\begin{aligned}
(t^3+t^2+t+1)B_1 &= t^{12n+15}-t^{8n+11},\\
(t^3+t^2+t+1)B_2 &= t^{8n+12}-t^{8n+8},\\
(t^3+t^2+t+1)B_3 &= (t^{8n+9}-t^{4n+9})+(t^{8n+7}-t^{4n+7}),\\
(t^3+t^2+t+1)B_4 &= (t^{4n+9}-t^{4n+5})+(t^{4n+8}-t^{4n+4}),\\
(t^3+t^2+t+1)B_5 &= (t^{4n+6}-t^{4n+2})+(t^{4n+5}-t^{4n+1}),\\
(t^3+t^2+t+1)B_6 &= (t^{4n+3}-t^{3})+(t^{4n+2}-t^{2})+(t^{4n+1}-t).
\end{aligned}$$

This shows that

$$\begin{aligned}
(t^3+t^2+t+1)(B_1+B_2+B_3+B_4+B_5+B_6+1) &= t^{12n+15}+t^{8n+7}+t^{4n+8}+1\\
&\quad+(t^{8n+9}-t^{4n+4}-t^{8n+8}+t^{4n+3})(t^3+1).
\end{aligned}$$

Thus $\Delta_{K_2}(t)=B_1+B_2+B_3+B_4+B_5+B_6+1$ as desired. □

We recall the notion of formal semigroup for an L–space knot [36]. Let K be an L–space knot in the 3–sphere. Then the Alexander polynomial of K has a form of

$$\Delta_K(t)=1-t^{a_1}+t^{a_2}+\cdots-t^{a_{k-1}}+t^{a_k}, \tag{3.2}$$

where $1=a_1<a_2<\cdots<a_k=2g(K)$, and $g(K)$ is the genus of K [27]. We expand the Alexander function into a formal power series as

$$\frac{\Delta_K(t)}{1-t}=\sum_{s\in\mathcal{S}_K}t^s. \tag{3.3}$$

(This is called the Milnor torsion in [10].) The set $\mathcal{S}_K$ is a subset of non-negative integers, called the *formal semigroup* of K. For example, for a torus knot $T(p,q)$ $(1<p<q)$, its formal semigroup is known to be the actual semigroup of rank two,

$$\langle p,q\rangle=\{ap+bq\mid a,b\geq 0\}$$

(see [6, 36]). If an L–space knot is an iterated torus knot, then its formal semigroup is also a semigroup [36], but in general, the formal semigroup of a hyperbolic L–space knot is hardly a semigroup [3, 34].

Let $\mathbb{Z}_{\geq m}=\{i\in\mathbb{Z}\mid i\geq m\}$ and $\mathbb{Z}_{<0}=\{i\in\mathbb{Z}\mid i<0\}$.

Proposition 3.3. *The formal semigroup of K_1 is given as*

$$\begin{aligned}
\mathcal{S}_{K_1}=\{0,4,8,\dots,4n\}&\cup\{4n+3\}\\
&\cup\{4n+6,4n+7,4n+10,4n+11,\dots,8n+6,8n+7\}\cup\{8n+9,8n+10\}\\
&\cup\{8n+12,8n+13,8n+14,8n+16,8n+17,8n+18,\\
&\quad\dots,12n+8,12n+9,12n+10\}\cup\mathbb{Z}_{\geq 12n+12}.
\end{aligned}$$

Proof. We use $A_1,A_2,\dots,A_5$ in the proof of Theorem 3.1. For

$$\frac{\Delta_{K_1}}{1-t}=\frac{A_1}{1-t}+\frac{A_2}{1-t}+\frac{A_3}{1-t}+\frac{A_4}{1-t}+\frac{A_5}{1-t}+\frac{1}{1-t},$$

we expand each term as follows;

$$\frac{A_1}{1-t} = -\sum_{i=0}^{n} t^{8n+11+4i},$$
$$\frac{A_2}{1-t} = -t^{8n+8},$$
$$\frac{A_3}{1-t} = -\sum_{i=0}^{n} (t^{4n+5+4i} + t^{4n+4+4i}),$$
$$\frac{A_4}{1-t} = -t^{4n+2} - t^{4n+1},$$
$$\frac{A_5}{1-t} = -\sum_{i=0}^{n-1} (t^{3+4i} + t^{2+4i} + t^{1+4i}),$$
$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots .$$

The conclusion immediately follows from these. □

Proposition 3.4. *The formal semigroup of K_2 is given as*

$$\begin{aligned}\mathcal{S}_{K_2} = \{0, 4, 8, \dots, 4n\} &\cup \{4n+3\}\\ &\cup \{4n+6, 4n+8, 4n+10, \dots, 8n+4\} \cup \{8n+6, 8n+7, 8n+9, 8n+10\}\\ &\cup \{8n+12, 8n+13, 8n+14, 8n+16, 8n+17, 8n+18,\\ &\quad \dots, 12n+8, 12n+9, 12n+10\} \cup \mathbb{Z}_{\geq 12n+12}.\end{aligned}$$

Proof. The argument is similar to the proof of Proposition 3.3. For $B_1, B_2, \dots, B_6$ in the proof of Theorem 3.2, we expand

$$\frac{B_1}{1-t} = -\sum_{i=0}^{n} t^{8n+11+4i},$$
$$\frac{B_2}{1-t} = -t^{8n+8},$$
$$\frac{B_3}{1-t} = -\sum_{i=0}^{2n-1} t^{4n+7+2i},$$
$$\frac{B_4}{1-t} = -t^{4n+4} - t^{4n+5},$$
$$\frac{B_5}{1-t} = -t^{4n+1} - t^{4n+2},$$
$$\frac{B_6}{1-t} = -\sum_{i=0}^{n-1} (t^{3+4i} + t^{2+4i} + t^{1+4i}).$$

Then the conclusion follows from these again. □

Corollary 3.5. *For $i = 1, 2$, the formal semigroup of K_i is not a semigroup.*

Proof. By Propositions 3.3 and 3.4, we see that $4 \in \mathcal{S}_{K_i}$ but $4n+4 \notin \mathcal{S}_{K_i}$. Hence $\mathcal{S}_{K_i}$ is not closed under the addition, so is not a semigroup. □

Lemma 3.6. *Both of K_1 and K_2 are hyperbolic.*

Proof. By Corollary 3.5, the formal semigroup of K_i is not a semigroup. Hence K_i is not a torus knot, because the formal semigroup of a torus knot is a semigroup.

Assume for a contradiction that K_i is a satellite knot. Since K_i is the closure of a 4–braid, its bridge number is at most four. By [31], it is equal to four. Moreover, the companion is a 2–bridge knot and the pattern knot has wrapping number two. We know that both of the companion and the pattern knot are L–space knots and the pattern is braided by [4, 16]. Thus the companion is a 2–bridge torus knot [27], and K_i is its 2–cable. By [36], the formal semigroup of an iterated torus L–space knot is a semigroup, which contradicts Corollary 3.5. We have thus shown that K_i is hyperbolic. □

4. Upsilon invariants

In this section, we verify that the Upsilon invariants of K_1 and K_2 are the same. We will not calculate the Upsilon invariants. Instead, we determine the gap functions defined later. For an L–space knot, the Upsilon invariant is the Legendre–Fenchel transform of the gap function [5]. Hence if the gap functions of K_1 and K_2 share the same convex hull, then their Upsilon invariants also coincide.

First, we quickly review the Legendre–Fenchel transformation.

For a function $f : \mathbb{R} \to \mathbb{R}$, the Legendre–Fenchel transform $f^* : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is defined as

$$f^*(t) = \sup_{x \in \mathbb{R}} \{tx - f(x)\}.$$

The domain of f^* is the set $\{t \mid f^*(t) < \infty\}$.

The Legendre transform is defined only for differentiable convex functions, but the Legendre–Fenchel transform can be defined even for non-convex functions with non-differentiable points. The transform f^* is always a convex function. Hence, if f is not convex, then the double Legendre–Fenchel transform f^{**} does not return f. In this case, f^{**} gives the convex hull of the function f. Thus we see that f^* depends only on the convex hull of f.

Next, we recall the notion of gap function introduced in [6].

Let K be an L–space knot with formal semigroup $\mathcal{S}_K$. Then $\mathcal{G}_K = \mathbb{Z} - \mathcal{S}_K$ is called the *gap set*. In fact, $\mathcal{G}_K = \mathbb{Z}_{<0} \cup \{a_1, a_2, \dots, a_g\}$, where $g = g(K)$, and $0 < a_1 < a_2 < \cdots < a_g$. The part $a_1, a_2, \dots, a_g$ is called the gap sequence. Then it is easy to restore the Alexander polynomial as

$$\Delta_K(t) = 1 + (t-1)(t^{a_1} + t^{a_2} + \cdots + t^{a_g}).$$

From the gap set $\mathcal{G}_K$, we define the function $I : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ by

$$I(m) = \#\{i \in \mathcal{G}_K \mid i \geq m\},$$

and let $J(m) = I(m+g)$. Then we extend $J(m)$ linearly to obtain a piecewise linear function on $\mathbb{R}$. That is, for $k \in \mathbb{Z}$, if $J(k) = J(k+1)$, then $J(x) = J(k)$ on $[k, k+1]$, and if $J(k+1) = J(k) - 1$, then $J(k+x) = J(k) - x$ for $0 \le x \le 1$. Borodzik and Hedden [5] showed that the Upsilon invariant of K is the Legendre–Fenchel transform of the function $2J(-m)$. We call this function $2J(-m)$ the *gap function* of K.

Example 4.1. Let K be the $(-2,3,7)$–pretzel knot. It admits a lens space surgery, so is an L–space knot. Also, it has genus 5. The Alexander polynomial $\Delta_K(t)$ is $1 - t + t^3 - t^4 + t^5 - t^6 + t^7 - t^9 + t^{10}$. Then $\mathcal{S}_K = \{0,3,5,7,8\} \cup \mathbb{Z}_{\ge 10}$, and $\mathcal{G}_K = \mathbb{Z}_{<0} \cup \{1,2,4,6,9\}$. Tables 1 and 2 show the values of $I(m)$ and the gap function $2J(-m)$.

m	≥ 10	9	8	7	6	5	4	3	2	1	0	-1	-2	...
$I(m)$	0	1	1	1	2	2	3	3	4	5	5	6	7	...

TABLE 1. $I(m)$ for the $(-2,3,7)$–pretzel knot.

m	≤ -5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	...
$2J(-m)$	0	2	2	2	4	4	6	6	8	10	10	12	14	...

TABLE 2. The gap function $2J(-m)$ for the $(-2,3,7)$–pretzel knot.

Figure 2 shows the graph of the gap function $2J(-m)$ and its convex hull (broken line). Here, the convex hull $f(x)$ of the gap function is given by

$$f(x) = \begin{cases} 0 & \text{for } x \le -5, \\ \frac{2}{3}(x+5) & \text{for } -5 \le x \le -2, \\ x+4 & \text{for } -2 \le x \le 2, \\ \frac{4}{3}(x-5)+10 & \text{for } 2 \le x \le 5, \\ 2x & \text{for } 5 \le x. \end{cases}$$

Then the Legendre–Fenchel transformation gives the Upsilon invariant

$$\Upsilon_K(t) = \begin{cases} -5t & \text{for } 0 \le t \le \frac{2}{3}, \\ -2t-2 & \text{for } \frac{2}{3} \le t \le 1, \\ 2t-6 & \text{for } 1 \le t \le \frac{4}{3}, \\ 5t-10 & \text{for } \frac{4}{3} \le t \le 2. \end{cases}$$

In general, the gap function of an L–space knot has a specific property.

- The slope of each segment of the graph is 0 or 2.

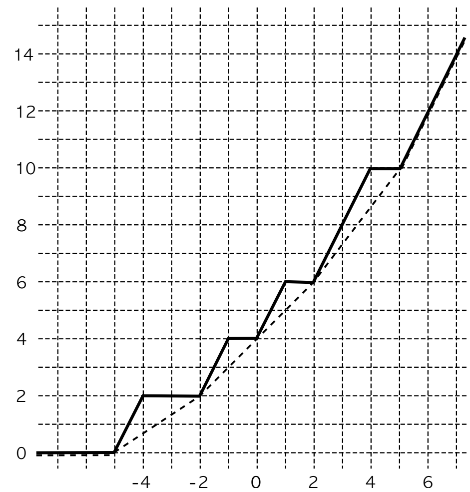


FIGURE 2. The graph of the gap function $2J(-m)$ for the $(-2,3,7)$–pretzel knot and its convex hull (broken line).

Although this observation is easy to see, we will use it essentially in Section 6 with further investigation.

Now, we calculate the gap functions of K_1 and K_2.

From Proposition 3.3, the gap set $\mathcal{G}_{K_1}$ is

$$\begin{aligned}&\mathbb{Z}_{<0} \cup \{1,2,3,5,6,7,\dots,4n-3,4n-2,4n-1\} \cup \{4n+1,4n+2\}\\ &\quad \cup \{4n+4,4n+5,4n+8,4n+9,4n+12,4n+13,\dots,8n+4,8n+5\}\\ &\quad \cup \{8n+8\} \cup \{8n+11,8n+15,8n+19,\dots,12n+7,12n+11\}.\end{aligned}$$

Hence the values of $I(m)$ is given as in Table 3. When m is an integer not in the table, $I(m)$ takes the same value as the nearest m' with $m' > m$. For example, $I(m) = I(12n+11) = 1$ for $m = 12n+10, 12n+9, 12n+8$.

m	$\geq 12n+12$	$12n+11$	$12n+7$	...	$8n+11$	$8n+8$
$I(m)$	0	1	2	...	$n+1$	$n+2$

$8n+5$	$8n+4$	$8n+1$	$8n$	...	$4n+5$	$4n+4$	$4n+2$	$4n+1$
$n+3$	$n+4$	$n+5$	$n+6$	...	$3n+3$	$3n+4$	$3n+5$	$3n+6$

$4n-1$	$4n-2$	$4n-3$	...	3	2	1	-1	-2	...
$3n+7$	$3n+8$	$3n+9$	...	$6n+4$	$6n+5$	$6n+6$	$6n+7$	$6n+8$	...

TABLE 3. The function $I(m)$ for K_1.

Let $J(m) = I(m+g)$ with $g = 6n+6$. Then the gap function $2J(-m)$ takes the values as in Table 4.

Figure 3 shows the graph of the gap function $2J(-m)$ of K_1 when $n = 1$.

m	$\le -6n-6$	$-6n-5$	$-6n-1$	...	$-2n-5$	$-2n-2$
$2J(-m)$	0	2	4	...	$2n+2$	$2n+4$

$-2n+1$	$-2n+2$	$-2n+5$	$-2n+6$	...	$2n+1$	$2n+2$	$2n+4$	$2n+5$
$2n+6$	$2n+8$	$2n+10$	$2n+12$	...	$6n+6$	$6n+8$	$6n+10$	$6n+12$

$2n+7$	$2n+8$	$2n+9$	...	$6n+3$	$6n+4$	$6n+5$	$6n+7$	$6n+8$	...
$6n+14$	$6n+16$	$6n+18$	...	$12n+8$	$12n+10$	$12n+12$	$12n+14$	$12n+16$	...

TABLE 4. The gap function $2J(-m)$ for K_1.

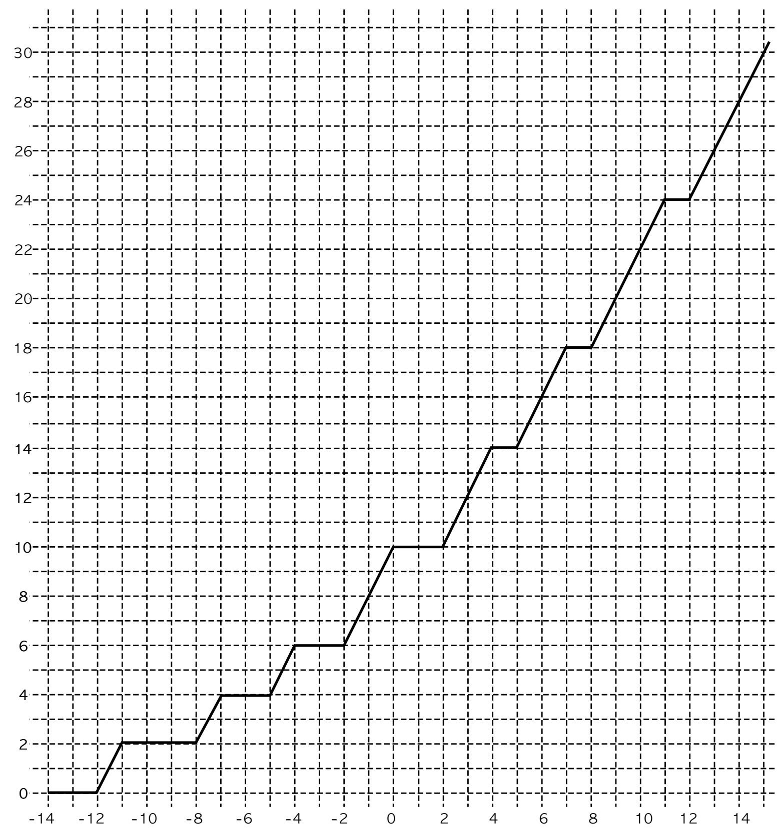


FIGURE 3. The graph of the gap function of K_1 with $n = 1$.

Similarly, the gap set $\mathcal{G}_{K_2}$ is

$$\begin{aligned}&\mathbb{Z}_{<0} \cup \{1, 2, 3, 5, 6, 7, \dots, 4n-3, 4n-2, 4n-1\} \cup \{4n+1, 4n+2\}\\ &\quad \cup \{4n+4, 4n+5\} \cup \{4n+7, 4n+9, \dots, 8n+3, 8n+5\}\\ &\quad \cup \{8n+8, 8n+11\} \cup \{8n+15, 8n+19, \dots, 12n+7, 12n+11\}\end{aligned}$$

from Proposition 3.4.

The values of $I(m)$ and the gap function $2J(-m)$ are given as in Tables 5 and 6.

m	$\geq 12n+12$	$12n+11$	$12n+7$	...	$8n+15$	$8n+11$	$8n+8$
$I(m)$	0	1	2	...	n	$n+1$	$n+2$

$8n+5$	$8n+3$	$8n+1$	...	$4n+7$	$4n+5$	$4n+4$	$4n+2$	$4n+1$
$n+3$	$n+4$	$n+5$	...	$3n+2$	$3n+3$	$3n+4$	$3n+5$	$3n+6$

$4n-1$	$4n-2$	$4n-3$	...	3	2	1	-1	-2	...
$3n+7$	$3n+8$	$3n+9$	...	$6n+4$	$6n+5$	$6n+6$	$6n+7$	$6n+8$	...

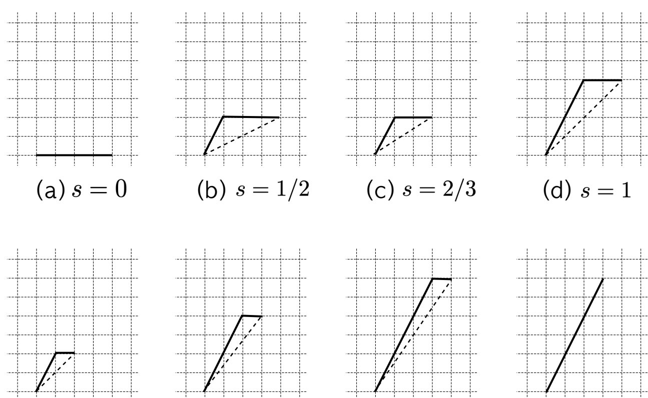
TABLE 5. The function $I(m)$ for K_2.

m	$\leq -6n-6$	$-6n-5$	$-6n-1$	...	$-2n-9$	$-2n-5$	$-2n-2$
$2J(-m)$	0	2	4	...	$2n$	$2n+2$	$2n+4$

$-2n+1$	$-2n+3$	$-2n+5$	...	$2n-1$	$2n+1$	$2n+2$	$2n+4$	$2n+5$
$2n+6$	$2n+8$	$2n+10$	...	$6n+4$	$6n+6$	$6n+8$	$6n+10$	$6n+12$

$2n+7$	$2n+8$	$2n+9$	...	$6n+3$	$6n+4$	$6n+5$	$6n+7$	$6n+8$	...
$6n+14$	$6n+16$	$6n+18$	...	$12n+8$	$12n+10$	$12n+12$	$12n+14$	$12n+16$	...

TABLE 6. The gap function $2J(-m)$ for K_2.



(a) $s=0$ (b) $s=1/2$ (c) $s=2/3$ (d) $s=1$

(e) $s=1$ (f) $s=4/3$ (g) $s=3/2$ (h) $s=2$

FIGURE 4. The parts of the graph of a gap function. The broken lines show the parts of convex hull with slope s.

Lemma 4.2. *For $i = 1, 2$, the convex hull $f(x)$ of the gap function $2J(-m)$ for K_i is given by*

$$f(x) = \begin{cases} 0 & \text{for } x \le -6n-6, \\ \frac{1}{2}(x+6n+6) & \text{for } -6n-6 \le x \le -2n-6, \\ \frac{2}{3}(x+2n+6)+2n & \text{for } -2n-6 \le x \le -2n, \\ x+4n+4 & \text{for } -2n \le x \le 2n, \\ \frac{4}{3}(x-2n)+6n+4 & \text{for } 2n \le x \le 2n+6, \\ \frac{3}{2}(x-2n-6)+6n+12 & \text{for } 2n+6 \le x \le 6n+6, \\ 2x & \text{for } 6n+6 \le x. \end{cases}$$

Proof. Consider the gap function of K_1. Let f be the convex hull. From Table 4, it is obvious that $f(x) = 0$ for $x \le -6n-6$ and $f(x) = 2x$ for $x \ge 6n+6$.

On the interval $[-6n-6, -6n-2]$, the gap function has the branch as shown in Figure 4(b). It repeats on the intervals $[-6n-2, -6n+2], \dots, [-2n-10, -2n-6]$. Thus $f(x) = \frac{1}{2}(x+6n+6)$ on $[-6n-6, -2n-6]$.

On $[-2n-6, -2n-3]$ and $[-2n-3, -2n]$, the branch is of Figure 4(c). Hence $f(x) = \frac{2}{3}(x+2n+6)+2n$ on $[-2n-6, -2n]$.

Similarly, the branch of Figure 4(d) repeats on the intervals $[-2n, 2n+4], [-2n+4, -2n+6], \dots, [2n-4, 2n]$. This gives $f(x) = x+4n+4$ on $[-2n, 2n]$.

On $[2n, 2n+3]$ and $[2n+3, 2n+6]$, the branch of Figure 4(f) appears. Thus $f(x) = \frac{4}{3}(x-2n)+6n+4$ on $[2n, 2n+6]$.

Finally, the branch of Figure 4(g) repeats on $[2n+6, 2n+10], \dots, [6n+2, 6n+6]$. Then $f(x) = \frac{3}{2}(x-2n-6)+6n+12$ on $[2n+6, 6n+6]$. We have thus shown that the convex hull $f(x)$ is given as claimed for K_1.

Next, consider the gap function of K_2. For $x \le -2n$, the situation is the same as K_1.

On $[-2n, -2n+2]$, the branch of Figure 4(e) appears. This branch repeats on $[-2n+2, -2n+4], \dots, [2n-2, 2n]$. However, the convex hull is the same as K_1.

For the remaining range $x \ge 2n$, the gap function is the same as one of K_1. In conclusion, the gap functions of K_1 and K_2 are distinct only on $[-2n, 2n]$, but their convex hulls coincide there. □

Corollary 4.3. *The Upsilon invariants of K_1 and K_2 coincide.*

Proof. By Lemmas 5.1 and 5.2, K_1 and K_2 are L–space knots. Hence, their Upsilon invariants are the Legendre–Fenchel transforms of the gap function $2J(-m)$. In fact, it depends only on the convex hull of the gap function. By Lemma 4.2, K_1 and K_2 have the same convex hull for their gap functions. Thus the conclusion follows. □

5. The Montesinos trick

In this section, we verify that K_1 and K_2 admit positive Dehn surgeries yielding L–spaces by using the Montesinos trick [24]. For a surgery diagram on a strongly invertible link, the Montesinos trick describes the resulting closed 3–manifold as the double branched cover of another knot or link obtained from tangle replacements corresponding to the surgery coefficients on some link obtained from the quotient of the original strongly invertible link under the strong involution (see also [26, 40]).

In Figure 1(1) and (2), each link $K \cup C_1 \cup C_2$ is placed in a strongly invertible position, where the dotted line indicates the axis of the involution.

Lemma 5.1. *For K_1, $(16n+21)$–surgery yields an L–space.*

Proof. Assign the surgery coefficient 3 on K in Figure 1(1). After performing $(-1/n)$–surgery on C_1 and $(-1/2)$–surgery on C_2, our knot K_1 has surgery coefficient $16n+21$.

First, take the quotient of the surgery diagram of Figure 1(1) under the involution around the axis. The neighborhoods of K, C_1 and C_2 yield 3–balls as shown in Figure 5. Since each 3–ball meets the image of the axis in two arcs, it gives a 2–string tangle, where a rational tangle replacement associated to the surgery coefficient is performed.

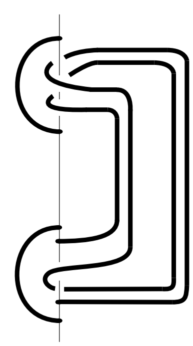


FIGURE 5. The quotient of the surgery diagram of K_1. The neighborhoods of K, C_1 and C_2 yield 3–balls.

The left of Figure 6 shows the knot obtained from the tangle replacements. In the diagram of Figure 1, we should remark that the component K has writhe 3. Hence the tangle replacement corresponding to the quotient of K is realized by the 0–tangle (depicted as the dotted circle).

Then Figures 6, 7, and 8 show the deformation of the knot. Finally, we obtain the Montesinos knot $M(-3/7, -1/3, -1/n)$. Thus the double branched cover is the Seifert fibered manifold $M = M(0; -3/7, -1/3, -1/n)$. We use the notation of [22]. That is, $M(e_0; r_1, r_2, r_3)$ is obtained by e_0–surgery on the unknot with three meridians having $(-/r_i)$–surgery on the i-th one. Then $-M = M(0; 3/7, 1/3, 1/n)$. By the criterion of [21, 22], M is an L–space. □

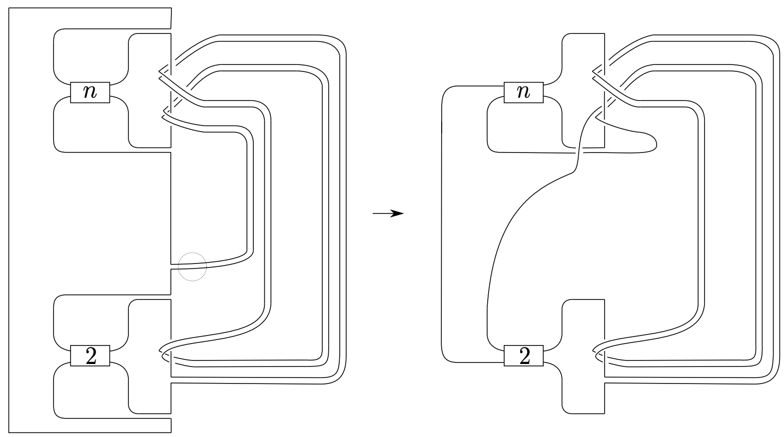


FIGURE 6. The deformation for K_1. Each rectangle box contains horizontal right-handed half-twists with indicated number.

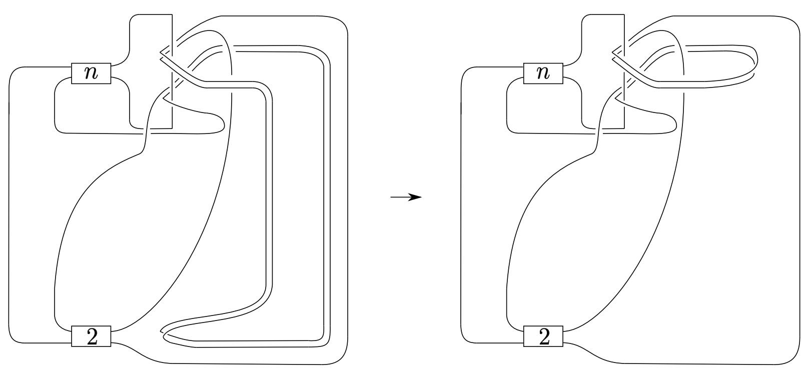


FIGURE 7. The deformation for K_1 (continued from Figure 6).

Lemma 5.2. *For K_2, $(16n + 20)$–surgery yields an L–space.*

Proof. Assign the surgery coefficient 2 on K in Figure 1(2). After performing $(-1/n)$–surgery on C_1 and $(-1/2)$–surgery on C_2, K_2 has surgery coefficient $16n + 20$.

The process is similar to that for K_1. We should remark that the tangle replacement to the quotient of K is realized by (-1)–tangle as depicted in the dotted circle in Figure 9 (left), because K has writhe 3 in the diagram.

Let ℓ be the link as illustrated in the right of Figure 9. We need to verify that the double branched cover of ℓ is an L–space.

For the crossing of ℓ encircled in Figure 9 (right), we perform two resolutions as shown in Figure 10. Let ℓ_∞ and ℓ_0 be the resulting knots. It is straightforward to calculate $\det \ell = 16n + 20$, $\det \ell_\infty = 9$ and $\det \ell_0 = 16n + 11$ from the checkerboard colorings on the diagrams of Figures 9, 12 and 13. Thus the

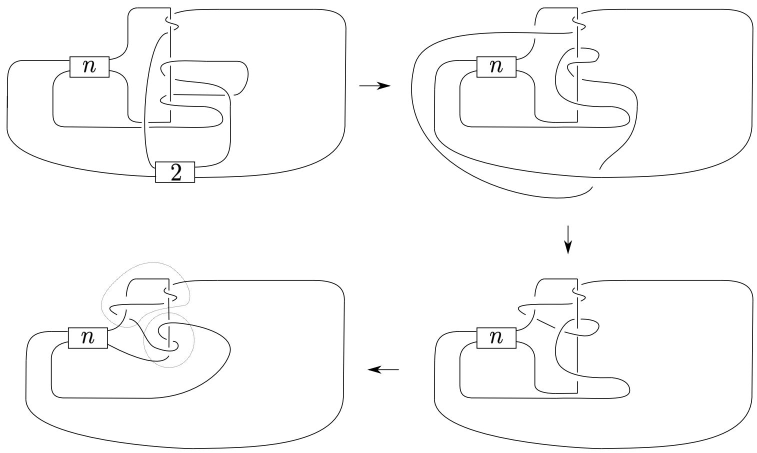


FIGURE 8. The deformation for K_1 (continued from Figure 7). The left bottom is the Montesinos knot $M(-3/7,-1/3,-1/n)$.

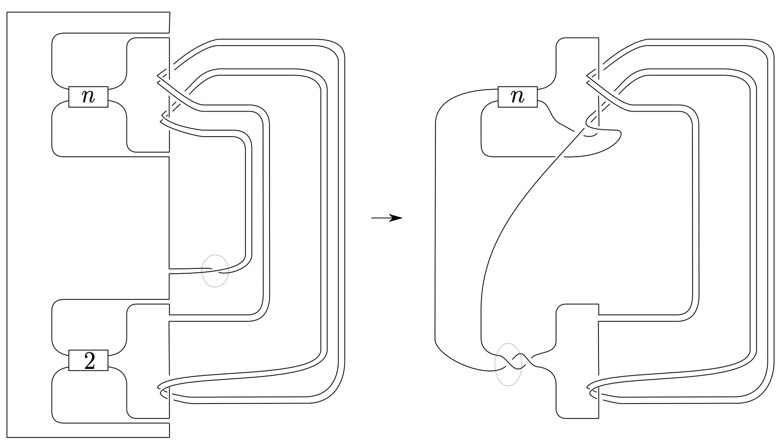


FIGURE 9. The deformation for K_2. Let ℓ be the right link.

equation $\det \ell = \det \ell_\infty + \det \ell_0$ holds. This implies that if the double branched covers of ℓ_∞ and ℓ_0 are L–spaces, then so is the double branched cover of ℓ ([7, 27, 28]).

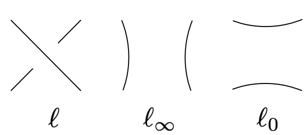


FIGURE 10. Two resolutions.

Claim 5.3. *The knot ℓ_∞ is the $(-3, 3, n-1)$–pretzel knot. Its double branched cover is an L–space.*

Proof. Figures 11 and 12 show that the knot ℓ_∞ is the $(-3, 3, n-1)$–pretzel knot.

If $n = 1$, then ℓ_∞ is the connected sum of torus knots $T(2, 3)$ and $T(2, -3)$. The double branched cover is the connected sum of lens spaces $L(3, 1)\#L(3, -1)$, which is an L–space. If $n = 2$, then ℓ_∞ is the 2–bridge knot $S(4/9)$, so the double cover is a lens space. Hence we assume $n > 2$.

Since ℓ_∞ is the Montesinos knot $M(0; 1/3, -1/3, -1/(n-1))$, its double branched cover M is the Seifert fibered manifold $M(0; 1/3, -1/3, -1/(n-1))$. Then $-M = M(-1; 2/3, 1/3, 1/(n-1))$.

We use the criterion of [22, Theorem 1.1]. It claims that a Seifert fibered manifold $M(e_0; r_1, r_2, r_3)$ (with $1 \geq r_1 \geq r_2 \geq r_3 \geq 0$) is an L–space if and only if either M or $-M$ does not carry a positive transverse contact structure. Moreover, it is proved in [21, Theorem 1.3] (see also [22, p.359]) that such a Seifert fibered manifold M carries no positive transverse contact structure if and only if either $e_0 \geq 0$, or $e_0 = -1$ and there are no coprime integers $m > a > 0$ such that $mr_1 < a < m(1 - r_2)$ and $mr_3 < 1$.

If $n > 3$, then set $r_1 = 2/3$, $r_2 = 1/3$ and $r_3 = 1/(n-1)$. Then $1 \geq r_1 \geq r_2 \geq r_3 \geq 0$. Clearly, there are no integers m and a such that $mr_1 < a < m(1 - r_2)$. Hence $-M$ is an L–space.

Finally, assume $n = 3$. Set $r_1 = 2/3$, $r_2 = 1/2$ and $r_3 = 1/3$. Then $1 \geq r_1 \geq r_2 \geq r_3$. If $1/m > r_3 = 1/3$, then $m < 3$. For $m = 2$ and $a = 1$, $a/m > r_1 = 2/3$ does not hold. Thus there are no coprime integers $m > a > 0$ as desired, which implies that $-M$ is an L–space. □

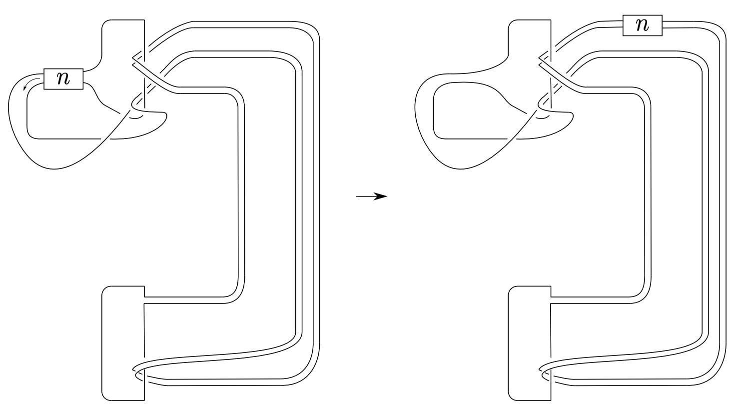


FIGURE 11. A deformation of the knot ℓ_∞.

Claim 5.4. *The double branched cover of ℓ_0 is an L–space.*

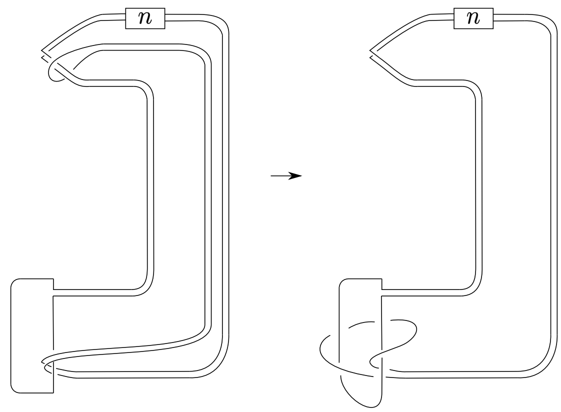


FIGURE 12. A deformation of the knot ℓ_{∞} (continued from Figure 11). The right is the $(-3, 3, n-1)$–pretzel knot.

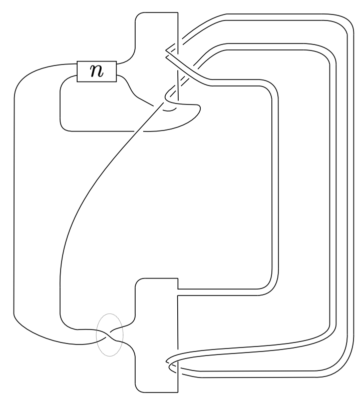


FIGURE 13. The knot ℓ_0.

Proof. For the crossing encircled in Figure 13, we further perform the resolutions, which yield $\ell_{0\infty}$ and ℓ_{00}. Clearly, $\ell_{0\infty} = \ell_{\infty}$. Hence $\det \ell_{0\infty} = 9$.

We can confirm that ℓ_{00} is the connected sum of the Hopf link and a Montesinos knot as shown in Figures 14 and 15. From the diagram of Figure 15, we see that $\det \ell_{00} = 16n + 2$. Recall that $\det \ell_0 = 16n + 11$. Hence the equation $\det \ell_0 = \det \ell_{0\infty} + \det \ell_{00}$ holds.

From Claim 5.3, the double branched cover of $\ell_{0\infty}$ is an L–space. It remains to show that the double branched cover of ℓ_{00} is an L–space.

The double branched cover of the Montesinos knot

$$M = M(1/2, -1/3, n/(2n+1))$$

is the Seifert fibered manifold $M(0; 1/2, -1/3, n/(2n+1))$. Since M is homeomorphic to $M(-1; 1/2, 2/3, n/(2n+1))$, set $r_1 = 2/3$, $r_2 = 1/2$ and $r_3 =$

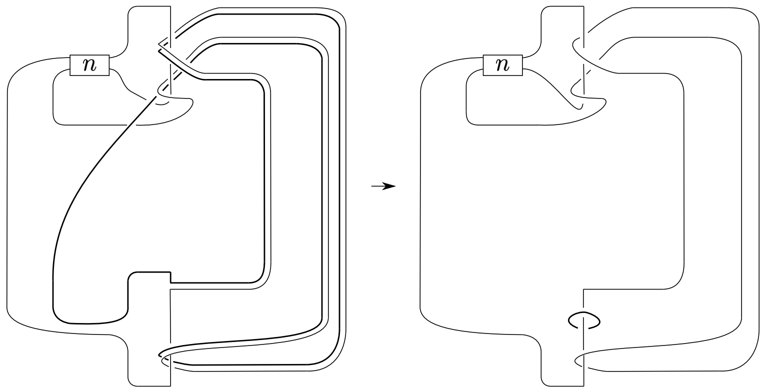


FIGURE 14. The knot ℓ_{00} has the Hopf link as its connected summand.

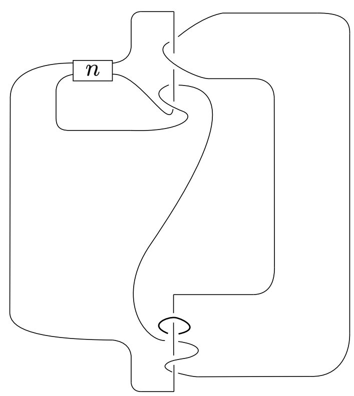


FIGURE 15. The knot ℓ_{00} is the connected sum of the Hopf link and the Montesinos knot $M(1/2, -1/3, n/(2n+1))$.

$n/(2n+1)$. Then $1 \geq r_1 \geq r_2 \geq r_3 \geq 0$. We apply the criterion of [22] again. If $1/m > r_3 = n/(2n+1) \geq 1/3$, then $m < 3$. Hence there are no coprime integers $m > a > 0$ such that $a/m > r_1 = 2/3$. Thus M is an L–space.

The double branched cover of ℓ_{00} is the connected sum of a lens space $L(2,1)$ and M. Since the sum of L–spaces is an L–space [27], we have the conclusion. □

By Claims 5.3 and 5.4, we obtain that the double branched cover of ℓ is an L–space. □

6. Restorability of Alexander polynomials

In this section, we investigate the restorability of Alexander polynomial of an L–space knot from the Upsilon invariant.

As easy examples, we examine two torus knots.

Example 6.1. (1) Let $K = T(3,4)$. Then $\Delta_K(t) = 1 - t + t^3 - t^5 + t^6$, so $\mathcal{S}_K = \{0,3,4\} \cup \mathbb{Z}_{\geq 6}$ and $\mathcal{G}_K = \mathbb{Z}_{<0} \cup \{1,2,5\}$. It is easy to calculate $\Upsilon_K(t)$ as

$$\Upsilon_K(t) = \begin{cases} -3t & \text{for } 0 \leq t \leq \frac{2}{3}, \\ -2 & \text{for } \frac{2}{3} \leq t \leq \frac{4}{3}, \\ 3t - 6 & \text{for } \frac{4}{3} \leq t \leq 2. \end{cases}$$

The Legendre–Fenchel transformation on $\Upsilon_K(t)$ gives a function

$$f(x) = \begin{cases} 0 & \text{for } x \leq -3, \\ \frac{2}{3}(x+3) & \text{for } -3 \leq x \leq 0, \\ \frac{4}{3}x + 2 & \text{for } 0 \leq x \leq 3, \\ 2x & \text{for } 3 \leq x. \end{cases}$$

Of course, this is the convex hull of the gap function of K. Figure 16 shows the graphs of gap function of K and f.

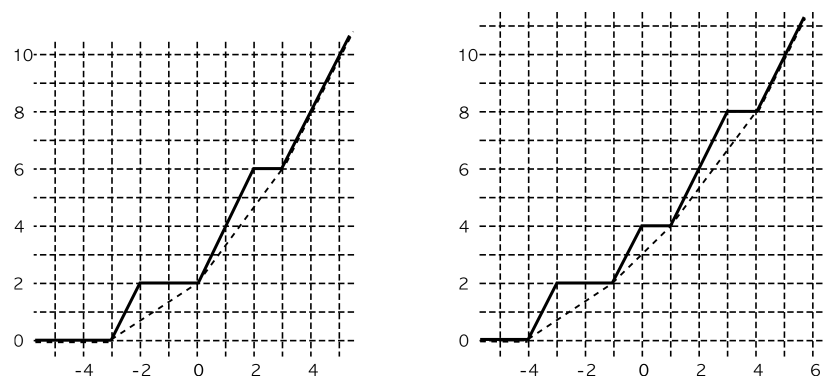


FIGURE 16. The graphs of the gap functions and their convex hulls (broken line) of $T(3,4)$ (left) and $T(3,5)$ (right).

We consider the possibility of another gap function G whose convex hull is f. First, it forces $G(-3) = 0$, $G(0) = 2$ and $G(3) = 6$. Recall that each segment of the graph of a gap function has slope 0 or 2 as mentioned in Section 4. Hence $G(-2) = 2$. Since a gap function is increasing, $G(-1) = 2$. Similarly, it is necessary that $G(1) = 4$ and $G(2) = 6$. Thus G coincides with the gap function of K.

This means that if another L–space knot K' has the same Upsilon invariant as K, then $\Delta_{K'}(t) = \Delta_K(t)$, because a gap function uniquely determines the Alexander polynomial.

(2) Let $K = T(3,5)$. We have $\Delta_K(t) = 1 - t + t^3 - t^4 + t^5 - t^7 + t^8$, so $\mathcal{S}_K = \{0,3,5,6\} \cup \mathbb{Z}_{\geq 8}$, and $\mathcal{G}_K = \mathbb{Z}_{<0} \cup \{1,2,4,7\}$. Then $\Upsilon_K(t)$ is given as

$$\Upsilon_K(t) = \begin{cases} -4t & \text{for } 0 \leq t \leq \frac{2}{3}, \\ -t-2 & \text{for } \frac{2}{3} \leq t \leq 1, \\ t-4 & \text{for } 1 \leq t \leq \frac{4}{3}, \\ 4t-8 & \text{for } \frac{4}{3} \leq t \leq 2. \end{cases}$$

Figure 16 shows the graphs of gap function of K and the convex hull, which is the Legendre–Fenchel transform of $\Upsilon_K(t)$. As in (1), the convex hull uniquely restores the gap function.

In general, it is rare that the convex hull uniquely restores a gap function. In Example 4.1, we determined the gap function and its convex hull of the $(-2,3,7)$–pretzel knot (see Figure 2). It is possible that another gap function G takes the same values on integers except $G(0) = 6$, keeping the same convex hull. This new gap function corresponds to the Alexander polynomial $\Delta(t) = 1 - t + t^3 - t^5 + t^7 - t^9 + t^{10}$. This polynomial satisfies the condition of [20], but there is no hyperbolic L–space knot in Dunfield's list whose Alexander polynomial is $\Delta(t)$. It seems to be a hard question whether there exists a hyperbolic L–space knot with $\Delta(t)$. Of course, there exists a hyperbolic knot whose Alexander polynomial is $\Delta(t)$ by [13, 33]. Also, $\Delta(t)$ is the Alexander polynomial of the $(2,3)$–cable of $T(2,5)$, which is not an L–space knot [15].

If we put off the realizability of the Alexander polynomial or the gap function by a hyperbolic L–space knot, then we can easily design many Alexander polynomials which are restorable from convex hulls.

It is a classical result that any polynomial $\Delta(t)$ satisfying $\Delta(1) = 1$ and $\Delta(t^{-1}) \doteq \Delta(t)$ is realized by a knot in the 3–sphere as its Alexander polynomial. (Here, $\doteq$ shows the equality up to units $\pm t^i$ in the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$.) Furthermore, we assume that $\Delta(t)$ has the form of (3.2). Formally, we define the formal semigroup $\mathcal{S}$ by (3.3), and in turn, its gap set and the gap function.

Proposition 6.2. *Let $m \geq 3$ be an integer, and let $\Delta(t) = 1 - t + t^m - t^{m+1} + t^{m+2} - t^{2m+1} + t^{2m+2}$. Then its gap function, defined formally, is uniquely determined from the convex hull.*

Again, the polynomial $\Delta(t)$ in Proposition 6.2 satisfies the condition of [20], but it is open whether $\Delta(t)$ is realized by a hyperbolic L–space knot or not. (When $m = 3$, $\Delta(t)$ is the Alexander polynomial of $T(3,5)$.)

Proof. By (3.3), the formal semigroup is $\mathcal{S} = \{0, m\} \cup \{m+2, m+3, \ldots, 2m\} \cup \mathbb{Z}_{2m+2}$, so the gap set is $\mathcal{G} = \mathbb{Z}_{<0} \cup \{1, 2, \ldots, m-1\} \cup \{m+1, 2m+1\}$. Set $g = m+1$. Then we can calculate the gap function $2J(-m)$ as in Table 7.

m	$\leq -m-1$	$-m$	0	2	3	$\dots$	m	$m+2$	$m+3$	$\dots$
$2J(-m)$	0	2	4	6	8	$\dots$	$2m+2$	$2m+4$	$2m+6$	$\dots$

TABLE 7. The gap function $2J(-m)$.

Let f be the convex hull. Then it is given by

$$f(x) = \begin{cases} 0 & \text{for } x \leq -m-1, \\ \frac{2}{m}(x+m+1) & \text{for } -m-1 \leq x \leq -1, \\ x+3 & \text{for } -1 \leq x \leq 1, \\ \frac{2m-2}{m}(x-1)+4 & \text{for } 1 \leq x \leq m+1, \\ 2x & \text{for } m+1 \leq x. \end{cases}$$

Since each segment of the graph of any gap function has slope 0 or 2, there is no other gap function whose convex hull is f. □

Finally, we prove Theorem 1.3. For reader's convenience, we record the braid words for the knots `t09847` and `v2871`. Both are the closures of 4–braids, whose words are almost the same:

$$(\sigma_2\sigma_1\sigma_3\sigma_2)^3(\sigma_2\sigma_1^2\sigma_2)\sigma_1 \quad \text{and} \quad (\sigma_2\sigma_1\sigma_3\sigma_2)^3(\sigma_2\sigma_1^2\sigma_2)\sigma_1^3.$$

Proof of Theorem 1.3. In the SnapPy census, the hyperbolic knots `t09847` and `v2871` are known to be L–space knots [9] (see also [3]).

Let K be the hyperbolic knot `t09847`. The Alexander polynomial is $\Delta_K(t) = 1 - t + t^4 - t^5 + t^7 - t^9 + t^{10} - t^{13} + t^{14}$, so the formal semigroup is $\mathcal{S}_K = \{0, 4, 7, 8, 10, 11, 12\} \cup \mathbb{Z}_{\geq 14}$.

Figure 17 shows the graph of the gap function and its convex hull (we omit the details). It consists of branches of types (a), (b), (c), (f), (g) and (h) of Figure 4 from the left. Then there is no other gap function with the same convex hull.

Next, let K be the hyperbolic knot `v2871`. The Alexander polynomial is $1 - t + t^4 - t^5 + t^7 - t^8 + t^9 - t^{11} + t^{12} - t^{15} + t^{16}$, so the formal semigroup is $\{0, 4, 7, 9, 10, 12, 13, 14\} \cup \mathbb{Z}_{\geq 16}$ and the gap set is $\mathbb{Z}_{<0} \cup \{1, 2, 3, 5, 6, 8, 11, 15\}$. Figure 18 shows the graph of the gap function and its convex hull. In this case, the graph consists of branches of types (a), (b), (c), (e), (f), (g) and (h) of Figure 4 from the left. Again, there is no other gap function with the same convex hull. □

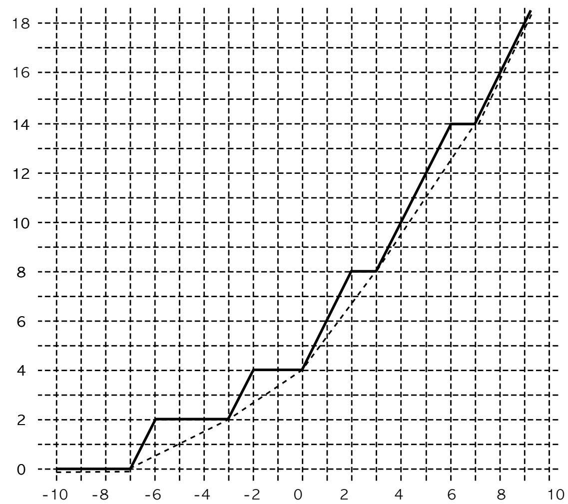


FIGURE 17. The graph of the gap function and its convex hull (broken line) of `t09847`.

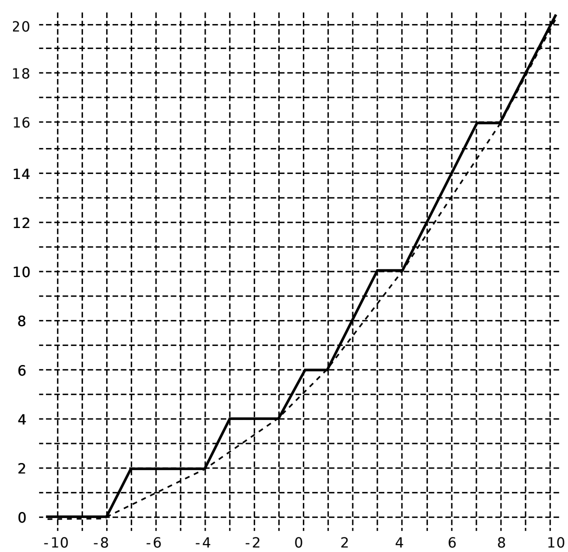


FIGURE 18. The graph of the gap function and its convex hull (broken line) of `v2871`.

Acknowledgement

The author would like to thank Kouki Sato for valuable communication and the referee for careful reading and suggestions.

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This paper is available via http://nyjm.albany.edu/j/2025/31-16.html.