The Fermat cubic and monodromy of lines

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ABSTRACT. In this paper we study properties of the locus of second type lines of a general cubic threefold and fourfold. By analysing the geometry of the Fano scheme of lines of the Fermat cubic fourfold and in particular giving an explicit description of the locus of second type lines, we deduce that the Voisin map is birational over the second type locus. For a general cubic three-fold, by studying properties of the second type locus again, we compute that various natural geometric monodromy groups are the full symmetric group.

CONTENTS

1. Introduction

For a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$, one defines the Fano scheme of lines $F(X) \subset G(2, n+2)$ parametrising lines contained in X. This is a smooth and irreducible variety with rich geometry and history, in particular in the low dimensional cases where, e.g., for $n = 2$ we famously have that $F(X)$ consists of 27 smooth points, for $n = 3$ we have that $\text{Alb}(F(X))$ is isomorphic to the intermediate Jacobian $J(X)$ used by Clemens–Griffiths to conclude that X is not rational, and for $n = 4$ we have that $F(X)$ is an irreducible holomorphic symplectic variety deformation equivalent to $S^{[2]}$, the Hilbert scheme of two points on a K3 surface. Conjectures abound relating the geometry of $F(X)$ and rationality of X.

One may also consider the locus $F_2(X) \subset F(X)$ of lines of second type, roughly consisting of lines with too many tangent 2-planes. The universal families $\mathbb{L}, \mathbb{L}_2$ over $F(X), F_2(X)$ provide useful correspondences, relating the Chow and cohomology theories of X and $F(X), F_2(X)$. The geometry and properties of

the locus $F_2(X)$ (which is non-empty only if $n \geqslant 3$ and is smooth and irreducible if X is general), though intimately tied to the geometry of X, is rather complicated and we refer to [Huy23] for basic results and motivation. When $n = 3$, Fano proved that $F_2(X)$ is a bicanonical curve in the surface $F(X)$, which is smooth and irreducible and of genus 91 if the cubic is general. In the case where $n = 4$, only recently have its invariants become known (see [Huy23, GK23]).

The first aim of this paper is to obtain a description (see Proposition 3.5) of the locus $F_2(X)$ in the case where X is the Fermat cubic fourfold.

THEOREM A. *Let $X = V(\sum_{i=0}^5 x_i^3) \subset \mathbb{P}^5$ be the Fermat cubic. The locus of second type lines $F_2 \subset F$ is isomorphic to the union of 10 copies of the self-product of the Fermat cubic curve and 45 copies of the Fermat cubic surface.*

A salient feature of $F_2(X)$ in the case $n = 4$ is that it is contained in the locus of indeterminacy of the Voisin map

$$\phi : F(X) \dashrightarrow F(X)$$
$$\ell \mapsto \ell'$$

where there exists a $\mathbb{P}^2 \cong \Pi \subset \mathbb{P}^5$ such that $X \cap \Pi = 2\ell + \ell'$. In fact, the indeterminacy locus is equal $F_2(X)$ if X contains no 2-planes, and in this case ϕ is resolved by a single blowup. We obtain the following (see Proposition 3.12) as an application of the above analysis by degenerating to the Fermat cubic fourfold.

THEOREM B. *Let $X \subset \mathbb{P}^5$ be a general cubic fourfold and $E_{F_2} \subset \mathrm{Bl}_{F_2}F$ the exceptional divisor. Then the Voisin map $\phi : E_{F_2} \to F$ is birational onto its image, a divisor $D \subset F$.*

The image of E_{F_2} in the second part of the above statement has received considerable attention in recent years as it provides an explicit uniruled divisor in the hyperkähler fourfold F (see [OSY19, CMP24]).

In a similar vein, either by degenerating to the Fermat if $n = 3, 4$, or more generally for any $n \geqslant 3$ using a different degeneration (cf. [GK24, Remark 3.8]), we present a proof of the fact that the natural morphism $\mathbb{L}_2 \to X$ is birational onto its image, a divisor $W \subset X$. As an application of this and some more refined statements in the case $n = 3$, we compute (see Section 5) that the monodromy groups of various natural finite morphisms are the full symmetric group, where in the following $C_\ell \subset F(X)$ is the locus of lines which meet ℓ.

THEOREM C. *Let $X \subset \mathbb{P}^4$ be a general smooth cubic threefold.*
  (1) The degree 6 morphism $\mathbb{L} \to X$ has monodromy group S_6.
  (2) For $\ell \subset X$ a general line, the monodromy of the natural degree 5 morphism $\pi_\ell : C_\ell \to \ell, [\ell'] \mapsto \ell \cap \ell'$ is S_5.
  (3) For $\ell \subset X$ a general line of second type, the monodromy of the natural degree 4 morphism $\pi_\ell : C_\ell \to \ell, [\ell'] \mapsto \ell \cap \ell'$ is S_4.

2. Preliminaries

Let $n \geqslant 3$ and let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic n-fold, $F \subset G(2, n+2)$ its Fano scheme of lines and $F_2 \subset F$ the locus of lines of second type.

Consider the induced morphism from the family of lines over F and F_2 respectively

$$
\begin{array}{ccc}
\mathbb{L}_2 \lhook\joinrel\longrightarrow \mathbb{L} & \xrightarrow{\;p\;} & X \\
\downarrow & \downarrow{\scriptstyle q} & \\
F_2 \lhook\joinrel\longrightarrow F. &
\end{array}
$$

for $\mathbb{L} = \mathbb{P}(\mathcal{U}|_F) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{U}^{\vee}|_F))$ (i.e., one dimensional subspaces) where $\mathcal{U}$ the universal bundle on the Grassmannian $G(2, n+2)$ and $\mathbb{L}_2$ its restriction to F_2. We denote the scheme-theoretic image of the morphism $\mathbb{L}_2 = q^{-1}(F_2) \to X$ by W. In other words, the locus W is that spanned by second type lines. Recall the following result, essentially going back to Clemens–Griffiths.

PROPOSITION 2.1. *([Huy23, §2.2]) Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface with $n > 2$ and $F, \mathbb{L}, \mathbb{L}_2, W$ as above. Then*

(1) *F_2 is pure $n - 2$-dimensional and reduced, and W is reduced.*
(2) *The morphism $\mathbb{L}_2 \to W$ is generically finite and W is a divisor, in particular of pure codimension one.*
(3) *$\mathbb{L}_2$ is the non-smooth locus of the morphism p.*
(4) *If X is general, F_2 is smooth and irreducible and hence W is irreducible.*

Proof. These are a combination of [Huy23, 2.2.11-2.2.15], noting that the description of F_2 as a degeneracy locus of the right codimension implies that it is Cohen–Macaulay, hence has no embedded points. $\qquad\square$

In low dimensions, we have the following. In the cubic threefold case, the morphism p is generically finite of degree six and is ramified at second type lines, meaning they count with multiplicity at least two out of the six. For cubic fourfolds, the generic fibre $C_x = p^{-1}(x)$ is a $(2, 3)$ curve in $\mathbb{P}^3$ (many of whose properties are known, see [GK24, §3] for a summary), and the locus $\mathbb{L}_2$ consists of the locus of singularities of the C_x.

If $n = 4$ we use special notation for the above, in line with [GK24, §2]. From now on, X will be a cubic fourfold and we denote by $S \subset F$ the locus of second type lines in its Fano scheme. Denote by $\phi : F \dashrightarrow F$ the Voisin map, taking a general line ℓ to its residual intersection on the unique tangent 2-plane to ℓ. This map is not defined on the locus of lines of X contained inside a 2-plane inside X and on the second type locus $S \subset F$. If X is general then it contains no 2-planes and S is smooth and irreducible, and we can resolve this map by a single blowup of F at S and we obtain a lift $\tilde{\phi} : \mathrm{Bl}_S(F) \to F$. In this case, the

fibre $E_\ell \cong \mathbb{P}^1$ of the exceptional divisor $E \subset \widetilde{F}$ over a point $[\ell] \in S$ parametrises pairs (ℓ, Π) where Π is a 2-plane tangent along the second type line ℓ and in fact [Huy23, Remark 2.2.18] gives that $\mathrm{Bl}_S F \cong \widetilde{F}$ where

$$\widetilde{F} = \{(\ell, \Pi) : \Pi \text{ tangent along } \ell\} \subset G(2,6) \times G(3,6).$$

We also constructed in [GK24, Proposition 4.2] a universal family

$$\widetilde{\mathcal{F}} := \{(X, (\ell, \ell', \Pi)) : X \cap \Pi = 2\ell + \ell' \text{ or } \ell, \ell' \subset \Pi \subset X\} \subset |\mathcal{O}_{\mathbb{P}^5}(3)| \times \mathcal{G} \quad (1)$$

where $\mathcal{G} = \{(\ell, \ell', \Pi) : \ell, \ell' \subset \Pi\} \subset G(2,6) \times G(2,6) \times G(3,6)$ (actually, the case $\ell, \ell' \subset \Pi \subset X$ was omitted from [GK24] because the interest there was in general cubics which do not contain any planes). From the discussion above, if X is general, then the fibre $\widetilde{\mathcal{F}}_X$ of the projection $\mathrm{pr}_1 : \widetilde{\mathcal{F}} \to |\mathcal{O}_{\mathbb{P}^5}(3)|$ over X is just $\widetilde{F} \cong \mathrm{Bl}_S(F)$. On the other hand, over special smooth cubics X, for example the Fermat, $\widetilde{\mathcal{F}}_X$ is at least irreducible and birational to F as seen from the following lemma. We will mostly call the above locus $\widetilde{\mathcal{F}}_X$ to avoid confusing it with $\mathrm{Bl}_S F$.

LEMMA 2.2. *Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold. Then $\widetilde{\mathcal{F}}_X$ is irreducible and birational to F the Fano variety of lines on X.*

Proof. Note that $\widetilde{\mathcal{F}}$ is smooth and irreducible from [GK24, Proposition 4.2] and for a general cubic X the fibre $\widetilde{\mathcal{F}}_X$ is irreducible, smooth and birational to F as discussed above. From semi-continuity of fibre dimension on the source [Sta21, Lemma 02FZ], for any smooth cubic X, every irreducible component of $\widetilde{\mathcal{F}}_X$ has dimension at least 4. As F is irreducible and 4-dimensional for any smooth cubic and by definition the fibres of $\pi : \widetilde{\mathcal{F}}_X \to F$ parametrise planes tangent to a line, they are generically one point, whereas for second type lines is 1-dimensional from [Huy23, Corollary 2.2.6]. Hence, if the locus in $\widetilde{\mathcal{F}}_X$ over $S \subset F$ were an irreducible component of $\widetilde{\mathcal{F}}_X$, it would be of dimension three, contradicting the above mentioned semi-continuity. In other words, for every smooth cubic π is birational and $\widetilde{\mathcal{F}}_X$ is irreducible. $\square$

For $Y \subset F$ a subvariety, we denote by $E_Y \subseteq \widetilde{\mathcal{F}}_X$ its inverse image in $\widetilde{\mathcal{F}}_X$. From its very definition, the Voisin map extends to a morphism $\phi = \mathrm{pr}_1 : \widetilde{\mathcal{F}}_X \to F$ for any smooth cubic fourfold, by sending $\ell \mapsto \ell'$ in Equation (1). Note also that $\widetilde{\mathcal{F}}$ admits a morphism to the universal Fano variety of lines $\mathcal{F}$ by projecting to the first factor of $\mathcal{G}$. In particular, we let $\mathcal{E}_S$ be the inverse image of the universal family of second type loci, which as noted, for general $X \in |\mathcal{O}_{\mathbb{P}^5}(3)|$ is just a $\mathbb{P}^1$-bundle over S, namely $\mathcal{E}_S|_X \cong E$.

3. The Fermat cubic and applications

Before we specialise to the case of the Fermat cubic fourfold, we recall some facts about the threefold case (see [CG72, Rou09, GK19]). Let $Y \subset \mathbb{P}^4$ be any smooth cubic threefold. Its Fano surface $F(Y) \subset G(2,5)$ is a smooth irreducible projective general type surface canonically embedded and of degree $K^2_{F(Y)} = 45$

in the Plücker $\mathbb{P}^9$. The locus of second type lines $F_2(Y) \subset F(Y)$ is pure 1-dimensional and of class $2K_{F(Y)} = \mathcal{O}_{\mathbb{P}^9}(2)|_{F(Y)}$.

PROPOSITION 3.1. *Let $Y = V(\sum_{i=0}^{4} x_i^3) \subset \mathbb{P}^4$ be the Fermat threefold. Then the locus of second type lines $F_2(Y)$ consists of the union of 30 smooth elliptic curves, each isomorphic to the Fermat plane cubic. In particular, for*

$$W = p(q^{-1}(F_2(Y))) \subset Y,$$

we have $\mathcal{O}_Y(W) \cong \mathcal{O}_Y(30)$ and that

$$p : q^{-1}(F_2(Y)) \to W$$

is birational when restricted to any of the 30 irreducible components of $q^{-1}(F_2(Y))$.

Proof. This basically follows from the work of Roulleau on the Fermat cubic [Rou09, Proposition 10 and p.395]. First, note that from loc. cit. a smooth elliptic curve in $F(Y)$ corresponds to a cone in X over a plane cubic, with vertex an Eckardt point of X. Letting $H = \mathcal{O}_{\mathbb{P}^9}(1)|_{F(Y)} \cong K_{F(Y)} \in \text{Pic} F(Y)$ be the Plücker polarisation, one proves that each such elliptic curve E has degree $H.E = 3$. Now, loc. cit. constructs 30 distinct elliptic curves E_i in $F(Y)$, each isomorphic to the Fermat plane cubic and their configuration is described in [Rou09, p.395]. As every line through an Eckardt point is of second type (see the beginning of Section 5), we obtain $E_i \subset F_2(Y)$ for all $i = 1, \dots, 30$. Moreover, as $H.[F_2(Y)] = K_{F(Y)}.(2K_{F(Y)}) = 90$, it must be that $F_2(Y) = \sum E_i$ and so W is the union of these 30 cones over plane Fermat cubics, each of which is a hyperplane section of Y, giving $\mathcal{O}_Y(W) \cong \mathcal{O}_Y(30)$. Finally,

$$[\mathcal{O}_Y(1)]^2 p_* q^*[F_2(Y)] = (q_* p^*[\mathcal{O}_Y(1)]^2).[F_2(Y)]$$
$$= H.[F_2(Y)]$$
$$= 90,$$

which forces the restriction of p to any irreducible component $q^{-1}(E_i)$ of the universal second type family to be birational onto its image. $\square$

Consider now, $X = V(\sum_{i=0}^{5} x_i^3) \subset \mathbb{P}^5$ the Fermat cubic fourfold. For $\{i, j, k\} \subset \{0, \dots, 5\}$, consider the hypersurfaces

$$V_{i,j,k} = V(x_i^3 + x_j^3 + x_k^3) \subset X$$
$$V_{i,j} = V(x_i^3 + x_j^3) \subset X.$$

If $\{i, j, k, i', j', k'\} = \{0, \dots, 5\}$ then $V_{i,j,k} = V_{i',j',k'}$, so in this way we obtain 10 distinct hypersurfaces $V_{i,j,k}$ in X. Denote by

$$\begin{aligned} C_{i,j,k} &= V_{i,j,k} \cap V(x_{i'}, x_{j'}, x_{k'}) \\ C'_{i,j,k} &= V_{i',j',k'} \cap V(x_i, x_j, x_k). \end{aligned} \tag{2}$$

Then,

$$V_{i,j,k} = \text{Join}(C_{i,j,k}, C'_{i,j,k}),$$

the join variety of $C_{i,j,k}, C'_{i,j,k}$.

For the $V_{i,j}$, note that there are 15 distinct hypersurfaces $V_{i,j}$ in X. Suppose that $0 \leqslant i' < i'' < j' < j'' \leqslant 5$, with $\{i', i'', j', j'', i, j\} = \{0, \dots, 5\}$ and let

$$G_{i,j} = V(x_{i'}^3 + x_{i''}^3 + x_{j'}^3 + x_{j''}^3, x_i, x_j) \qquad (3)$$

a Fermat cubic surface. We have $V(x_i^3 + x_j^3, x_{i'}, x_{i''}, x_{j'}, x_{j''}) = \{p_{i,j}^{(1)}, p_{i,j}^{(2)}, p_{i,j}^{(3)}\}$ corresponding to the three cube roots of -1. Then

$$V_{i,j} = \mathrm{Join}(G_{i,j}, \{p_{i,j}^{(1)}, p_{i,j}^{(2)}, p_{i,j}^{(3)}\})$$

the join variety of $G_{i,j}$ with the set of points $\{p_{i,j}^{(1)}, p_{i,j}^{(2)}, p_{i,j}^{(3)}\}$.

REMARK 3.2. Actually, the 45 points $p_{i,j}^{(\mu)}$ are all the Eckardt points of the Fermat cubic fourfold. They are Eckardt since there is a 2-dimensional family of lines in X through them, namely the lines joining the point $p_{i,j}^{(\mu)}$ with the points of the surface $G_{i,j}$. Finally, the maximal number of Eckardt points for a cubic fourfold is given as 45 by [CCS97, 3.12].

3.1. The second type locus for the Fermat cubic fourfold. Recall first the following Lemma.

LEMMA 3.3. ([CG72, Definition 6.6]) *A line $\ell \subset X = V(f)$ in a smooth cubic fourfold is of second type if and only if the Gauss map $\Phi : X \to (\mathbb{P}^5)^\vee$, $\Phi(x) = \nabla(f)(x)$ restricted to ℓ is a two-to-one covering of $\mathbb{P}^1$. In other words, for every point $x \in \ell$, there is an antipodal point $x' \in \ell$ at which $\Phi(x) = \Phi(x')$.*

In the case of the Fermat $X \subset \mathbb{P}^5$, $\nabla(f) = \langle x_0^2, \dots, x_5^2 \rangle$ which we denote by ∇X. The locus $W \subset X$ spanned by second type lines is divisorial from Proposition 2.1 for any smooth cubic.

Let $F \subset \mathrm{G}(2, 6)$ be the Fano scheme of lines of the Fermat cubic and $S \subset F$ the second type locus. We denote by $S_{i,j,k} \subset F$ the locus of ruling lines of the variety $\mathrm{Join}(C_{i,j,k}, C'_{i,j,k})$ and by $S_{i,j}^{(\mu)} \subset F$ the locus of ruling lines of the variety $\mathrm{Join}(G_{i,j}, \{p_{i,j}^{(\mu)}\})$, $\mu = 1, 2, 3$. Note that $S_{i,j}^{(\mu)}$ is isomorphic to $G_{i,j}$ of Equation (3), i.e., to the Fermat cubic surface. On the other hand, each $S_{i,j,k}$ is isomorphic to $C_{i,j,k} \times C'_{i,j,k}$ for two Fermat cubic curves, since these two curves are disjoint.

LEMMA 3.4. *We have that $V_{i,j,k}, V_{i,j} \subset W$ and $S_{i,j}^{(\mu)}, S_{i,j,k} \subset S$ for all $\{i, j, k\} \subset \{0, \dots, 5\}$ and $\mu \in \{1, 2, 3\}$.*

Proof. For $V_{i,j,k}$, by symmetry we may assume without loss of generality that $i = 0, j = 1, k = 2$. Consider a general point

$$p = [a_0 : \cdots : a_5] \in V_{0,1,2} = V(x_0^3 + x_1^3 + x_2^3, x_3^3 + x_4^3 + x_5^3) \subset \mathbb{P}^5,$$

in the sense that it satisfies $a_0^3 + a_1^3 + a_2^3 = a_3^3 + a_4^3 + a_5^3 = 0$ so that not all a_i for $i \in \{0, 1, 2\}$ are zero nor $a_i = 0$ for all $i \in \{3, 4, 5\}$. To prove that

$p \in W$, we need to find a second type line ℓ_p containing p. For this, take $p' = [-a_0 : -a_1 : -a_2 : a_3 : a_4 : a_5]$ (necessarily $\neq p$) and let ℓ_p be given parametrically by

$$[(\lambda - t)a_0 : (\lambda - t)a_1 : (\lambda - t)a_2 : (\lambda + t)a_3 : (\lambda + t)a_4 : (\lambda + t)a_5] \subset V_{i,j,k},$$

i.e., the line $\lambda p + t p'$. Indeed ℓ_p is a line of second type as the antipodal (under the Gauss map) to the point $\lambda p + t p'$ is $t p + \lambda p'$, i.e., they have the same image under Φ. The two ramification points of the Gauss map restricted to ℓ_p are $[a_0 : a_1 : a_2 : 0 : 0 : 0]$ and $[0 : 0 : 0 : a_3 : a_4 : a_5]$.

For $V_{i,j}$, again by symmetry we may consider only the case $i = 0, j = 1$. Consider a general point

$$p = [a : \omega a : a_2 : \cdots : a_5] \in V_{0,1} = V(x_0^3 + x_1^3, x_2^3 + \ldots + x_5^3)$$

for ω a cube root of -1 satisfying that neither $a = 0$ nor $a_2 = \ldots = a_5 = 0$. To find a second type line ℓ_p through p, we take $p' = [-a : -\omega a : a_2 : \ldots : a_5]$ and define ℓ_p parametrically by $\lambda p + t p'$. One checks that $\ell_p \subset V_{0,1}$ and that the antipodal point of $\lambda p + t p'$ is $t p + \lambda p'$ and that the ramification points of the Gauss map are $[0 : 0 : a_2 : \ldots : a_5]$ and $[a : \omega a : 0 : \ldots : 0]$. In this case, we end up with three irreducible components as $V_{0,1}$ is reducible. $\quad\square$

PROPOSITION 3.5. *We have*

$$W = \cup_{i,j,k} V_{i,j,k} \bigcup \cup_{i,j} V_{i,j},$$
$$S = \cup_{i,j,k} S_{i,j,k} \bigcup \cup_{i,j,\mu} S_{i,j}^{(\mu)}.$$

Moreover, the morphism $p_{q^{-1}(S)} : q^{-1}(S) \to W$ is generically of degree one when restricted to any one of the $55 = 3 \cdot 15 + 10$ irreducible components of S as above.

Proof. By definition, $W = p(q^{-1}(S))$ and one computes $(p_* q^*[S]).H_X^3 = 225$ from [GK24, Lemma 2.6], so that $\mathcal{O}_X(W) = \mathcal{O}_X(75)$. At this point, it is not clear that the degree of $p : q^{-1}(S) \to W$ is generically one. On the other hand, we have constructed $10 + 15 = 25$ divisors $V_{i,j}, V_{i,j,k}$ above, all of which are contained in W. As their total degree is $3 \cdot 3 \cdot 25 = 225$, this forces their union to be W. Since S is pure 2-dimensional and the universal family over every irreducible component of it maps generically finitely to X (see [Huy23, Lemma 2.2.12]), the above also concludes that

$$S = \cup_{i,j,\mu} S_{i,j}^{(\mu)} \bigcup \cup_{i,j,k} S_{i,j,k}.$$

The degree of the map p on any irreducible component of $q^{-1}(S)$ must also be generically one by degree considerations. $\quad\square$

3.2. The Voisin map for the Fermat cubic fourfold. In this section, we denote by $\widetilde{F} := \widetilde{\mathcal{F}}_X$ for X the Fermat cubic (see Section 2), which is irreducible and birational to F from Lemma 2.2. Note that as S is singular (non-normal even), $\widetilde{F}$ is not isomorphic to the blowup of F at S, but is rather defined in

terms of tangent 2-planes. We recall also that for $Y \subset F$ a subvariety, we denote by $E_Y \subseteq \widetilde{\mathcal{F}}$ its inverse image in $\widetilde{\mathcal{F}}$, so that for example for $S_{i,j}^{(\mu)}, S_{i,j,k}$ the irreducible components of S for the Fermat, defined in the previous section, denote by $E_{S_{i,j}^{(\mu)}}, E_{S_{i,j,k}} \subset E := E_S$ their inverse images in $\widetilde{\mathcal{F}}$.
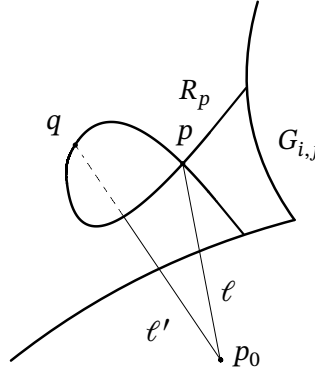

FIGURE 1

LEMMA 3.6. *The morphism ϕ contracts the threefold $E_{S_{i,j}^{(\mu)}}$ onto the surface $S_{i,j}^{(\mu)}$ in F.*

Proof. We refer to Figure 1. Let $[\ell] \in S_{i,j}^{(\mu)}$ be a general point with $\ell = \overline{p_0 p}$, $p_0 = p_{i,j}^{(\mu)}$, $p \in G_{i,j}$. Then $R_p = T_p X \cap G_{i,j}$ is a cubic curve with a nodal singularity at p. As $\text{Join}(p_0, G_{i,j}) \subset T_{p_0} X$ and $\ell \subset T_p X$, we obtain

$$\text{Join}(p_0, R_p) \subset T_p X \cap T_{p_0} X = H \cong \mathbb{P}^3.$$

Then R_p parametrises 2-planes in H which contain ℓ. Indeed, each such plane Π intersects R_p at a unique point q (other that p) and then $\phi(\ell, \Pi) = \overline{p_0 q}$. Therefore

$$\phi(E_{S_{i,j}^{(\mu)}}) \subset S_{i,j}^{(\mu)}.$$

In fact, it is easy to see that equality holds, since the smooth points of the curves R_p for $p \in G_{i,j}$ cover the whole $G_{i,j}$. $\square$

We now discuss the restriction of the map ϕ to $S_{i,j,k}$, where we refer to Figure 2. Without loss of generality, we may assume that $i = 0, j = 1, k = 2$. Let

$$\Pi_1 = V(x_3, x_4, x_5) \cong \mathbb{P}^2_{(x_0, x_1, x_2)}$$
$$\Pi_2 = V(x_0, x_1, x_2) \cong \mathbb{P}^2_{(x_3, x_4, x_5)}.$$

A $p \in \mathbb{P}^5 \backslash \Pi_1 \cup \Pi_2$ can be written as $p = [x : y]$ with $[x] \in \Pi_1$ and $[y] \in \Pi_2$. Geometrically, this means that p is in the unique line joining $[x] \in \Pi_1$ with
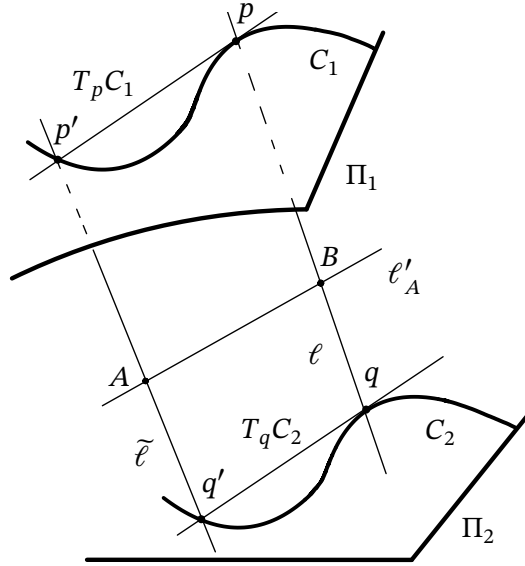
FIGURE 2

$[y] \in \Pi_2$ and also that the ruling lines of $\mathrm{Join}(\Pi_1, \Pi_2)$ do not intersect at points outside Π_1, Π_2. To simplify notation, let

$$
\begin{aligned}
C_1 &= C_{0,1,2} = X \cap \Pi_1 = V(x_0^3 + x_1^3 + x_2^3) \subset \Pi_1 \\
C_2 &= C_{3,4,5} = X \cap \Pi_2 = V(x_3^3 + x_4^3 + x_5^3) \subset \Pi_2.
\end{aligned}
\tag{4}
$$

Note that $p = [x : y] \in \mathrm{Join}(C_1, C_2)$ (outside C_1 and C_2) if and only if $[x] \in C_1$ and $[y] \in C_2$.

Let $p = [x_0 : x_1 : x_2] \in C_1$. We have

$$
\nabla_p X = 3\langle x_0^2, x_1^2, x_2^2, 0, 0, 0 \rangle
$$
$$
\nabla_p C_1 = 3\langle x_0^2, x_1^2, x_2^2 \rangle
$$

giving $T_p X|_{\Pi_1} = T_p C_1$. Similarly, for $q \in C_2$ we have $T_q X|_{\Pi_2} = T_q C_2$.

Let now $\ell = \overline{pq}$, with p, q general points of C_1, resp. C_2, a ruling line of $\mathrm{Join}(C_1, C_2)$ as in Figure 2. The tangent line $T_p C_1$ intersects C_1 at an additional point p' and, similarly $T_q C_2$ intersects C_2 at an additional point q'. Since p, q are general, $p' \neq p$ and $q' \neq q$. We call the line $\widetilde{\ell} = \overline{p'q'}$ the *adjoint* line to ℓ. We now have that $\ell, \overline{pq'}, T_p C_1 \subset T_p X$ and $\ell, \overline{qp'}, T_q C_2 \subset T_q X$. Then $p, q, p', q' \in T_p X \cap T_q X = H$ and since $\ell, \widetilde{\ell}$ are skew lines we have that $H \cong \mathbb{P}^3$ is spanned by the above points (note that $T_p X \neq T_q X$ because the vectors $\nabla_p X, \nabla_q X$ at these two points are different). Then the points of the adjoint line $\widetilde{\ell}$ parametrise 2-planes in H which contain ℓ. For any $A \in \widetilde{\ell}$, we denote by Π_A the plane spanned by (p, q, A) and we have

$$
\phi(\ell, \Pi_A) = \ell'_A,
\tag{5}
$$

with ℓ'_A a line in X which passes through A and intersects ℓ at a point B.

REMARK 3.7. The point

$$p_{i,j}^{(\mu)} \in C_{i,j,k}$$

is a flex point and a ruling line ℓ of $\mathrm{Join}(C_1, C_2)$ which passes through it is necessarily contained in $S_{i,j}^{(\mu)}$. Its image under ϕ is given as in Lemma 3.6.

LEMMA 3.8. *For* $0 \leqslant i, j, i', j', k' \leqslant 5$ *distinct and* $1 \leqslant \mu \leqslant 3$ *we have*

$$\phi(E_{S_{i,j}^{(\mu)}}) \subset \phi(E_{S_{i',j',k'}}).$$

Proof. Without loss of generality, we assume that $(i', j', k') = (0, 1, 2)$ and that $(i, j) = (4, 5)$, $\mu = 1$. Then $p_0 := p_{4,5}^{(1)} = [0 : \cdots : 0 : 1 : -1]$. We will show that

$$\phi(E_{S_{4,5}^{(0)}}) \subset \phi(E_{S_{0,1,2}}).$$

Let $G = G_{4,5}$ and $C_1 = C_{0,1,2} = \Pi_1 \cap G$. As we have seen in Lemma 3.6,

$$\phi(E_{S_{4,5}^{(0)}}) \subset S_{4,5}^{(0)}$$

so it remains to show that $S_{4,5}^{(0)} \subset \phi(E_{S_{0,1,2}})$. For $[\ell' = \overline{p_0 q}] \in S_{4,5}^{(0)}$ we have $q = [b_0 : \cdots : b_3 : 0 : 0] \in G$, and let $q' = [b_0 : b_1 : b_2 : 0 : 0 : 0] \in \Pi_1$ be the point of intersection of ℓ' with Π_1. Let $p \in C_1$ with $\overline{q'p} = T_p C_1 \subset \Pi_1$. Note that if $q = [x : y] \in G$ with $[x] \in T_p C_1$ then $q \in G \cap T_p X = R_p$: indeed, if $p = [a_0 : a_1 : a_2 : 0 : 0 : 0]$ then $\nabla_p X = 3 \langle a_0^2, a_1^2, a_2^2, 0, 0, 0 \rangle$ and therefore a point $q = [b_0 : \cdots : b_3 : 0 : 0] \in G$ lies in R_p if and only if $a_0^2 b_0 + a_1^2 b_1 + a_2^2 b_2 = 0$, i.e., the point $[b_0 : b_1 : b_2 : 0 : 0 : 0]$ lies in $T_p C_1$, therefore $q' \in T_p C_1$. Note here that C_1 is contained in the Hessian of $G \subset \mathbb{P}^3$ (given by $x_0 x_1 x_2 x_3 = 0$) and so R_p has a cuspidal singularity at p. By what we have described in Remark 3.7, if $\ell = \overline{p_0 p}$ then $\phi(\ell, \Pi_q) = \ell'$ with $\Pi_q = \langle p_0 pq \rangle$, which proves the claim. Finally, note that $[\ell] \in S_{0,1,2} \cap S_{4,5}^{(0)}$. $\qquad \square$

LEMMA 3.9. *The morphism* ϕ *is generically one-to-one when restricted to* $E_{S_{i,j,k}}$. *Moreover, the images* $\phi(E_{S_{i,j,k}})$ *are all different from one another for different* $S_{i,j,k}$.

Proof. Given a general $\ell' := \ell'_A$ as in Equation (5) (see also Figure 2), we first observe that in case ℓ' is not one of the lines which pass through p, p', q, q' then the only lines in $S_{i,j,k}$ which intersect ℓ' are the lines ℓ and its adjoint $\tilde{\ell}$: Indeed, if $[x : y] \in \ell'$ then $[x] \in T_p C_1$ and $[y] \in T_q C_2$, so since $T_p C_1$ intersects C_1 only at the points p, p' and $T_q C_2$ intersects C_2 only at the points q, q', the only other possible candidate line with image ℓ' under the map ϕ is the line $\tilde{\ell}$, in which case $T_p C_1$ would also be tangent to C_1 at p', a contradiction since C_1 is a cubic curve.

For the second claim, without loss of generality we show that $\phi(E_{S_{0,1,2}}) \neq \phi(E_{S_{0,2,3}})$. We have seen in Lemma 3.8 that $S_{4,5}^{(0)} \subset \phi(E_{S_{0,1,2}})$. For a general $[\ell'] \in$

$S_{4,5}^{(0)}$ we will show that $[\ell'] \notin \phi(E_{S_{0,2,3}})$. Indeed, $\ell' = [s, -s, ta_2, \cdots, ta_5]$, with $a_2^3 + \ldots + a_5^3 = 0$. Then ℓ' intersects $V_{0,2,3}$ at the points $[s, t]$ satisfying the system $s^3 + (ta_2)^3 + (ta_3)^3 = 0$ and $(-s)^3 + (ta_4)^3 + (ta_5)^3 = 0$. By the above relation, these two equations are equivalent and hence the intersection points correspond to $[s, t]$ with $(s/t)^3 = a_4^3 + a_5^3$. This has, generically, three distinct solutions. But, from what we have seen for the first claim above, in order for $[\ell']$ to be in $\phi(E_{S_{0,1,2}})$ it has to intersect $V_{0,2,3}$ at two points only, or infinitely many. Hence, the general such $[\ell'] \notin \phi(E_{S_{0,2,3}})$, which proves the claim.   □

In other words, the $E_{S_{i,j,k}}$ map generically one-to-one onto their images in F, whereas the $E_{S_{i,j}^{(\mu)}}$ are contracted onto $S_{i,j}^{(\mu)} \subset \phi(E_{S_{i,j,k}})$. Hence, overall, we have a union of divisors

$$\phi(E_S) = \bigcup_{i,j,k} \phi(E_{S_{i,j,k}}).$$

PROPOSITION 3.10. *Let $p \in X$ be a general point on the Fermat cubic fourfold. Then, out of the 1-dimensional family of lines through p, there are precisely 360 distinct lines with the property that each of them is residual to some line of second type under the Voisin map.*

Proof. Let $p = [x : y]$ for $[x] \in \Pi_1$ and $[y] \in \Pi_2$ (with notation as in Equation (4)). Then to each pair of a tangent line from $[x]$ to C_1 (there are 6 such choices) and a tangent line from $[y]$ to C_2 (also 6 choices) we have that $[x : y]$ is contained in a line of $\phi(E_{S_{0,1,2}})$, giving 36 such choices of lines overall. Since we have 10 components $\phi(E_{S_{i,j,k}})$ the total number of lines in $\phi(E_S)$ which pass through p are 360, noting that since $S_{i,j}^{(\mu)}$ is 2-dimensional, the span of such lines in X is 3-dimensional so there are no lines in these loci passing through a general point p.   □

In the following, $H \in \operatorname{Pic} F$ will denote the Plücker polarisation as usual.

COROLLARY 3.11. *For $\phi : \widetilde{F} = \widetilde{\mathcal{F}}_X \to F$ and $E_S \subset \widetilde{F}$ on the Fermat X, we have that $[\phi(E_S)] = 60H$ in $\operatorname{Pic} F$.*

Proof. Proposition 3.10 implies that for $x \in X$ a general point and $C_x = p^{-1}(x)$ the curve of lines through x, we have $[C_x].[q^{-1}(\phi(E_S))] = 360$. Since $\phi_*[E_S] = 60H \in \operatorname{Pic} F$ from [Huy23, Remark 6.4.19] and $[C_x].q^*[60H] = 360$ too from [GK24, Lemma 2.6]. Note this also gives that the degree of the morphism $\phi|_{E_S} : E_S \to \phi(E_S)$ is one on the components $E_{S_{i,j,k}}$ which do not get contracted.   □

To end this section, we address a question raised in [Huy23, Remark 6.4.19].

PROPOSITION 3.12. *Let $X \subset \mathbb{P}^5$ be a general cubic fourfold. The Voisin map $\phi : \widetilde{\mathcal{F}}_X \to F$ is generically one-to-one when restricted to $E_S \subset \widetilde{\mathcal{F}}_X$.*

In plain words, a general line which is residual to a second type line is residual to exactly one second type line.

Proof. Let $\mathcal{E}_S$ be the inverse image in $\widetilde{\mathcal{F}}$ (recall its definition in (1)) of the universal second type locus $\mathcal{S}$. Denote by $\mathcal{D} = \phi(\mathcal{E}_S) \subset \mathcal{F}$ its scheme-theoretic image under the Voisin map, which comes with a morphism to $B = |\mathcal{O}_{\mathbb{P}^5}(3)|_{\text{sm}}$. There is an open $U \subseteq B$, so that for $t \in U$, $\mathcal{D}_t$ is a divisor in $\mathcal{F}_t$, and is reduced as $\mathcal{D}$ is so, and irreducible from Proposition 2.1.(4). Over points in U, $\mathcal{D}_t$ agrees with $\phi(E_{S_t})$. From the principle of conservation of numbers [Ful98, Proposition 10.2], since from Corollary 3.11 we know that $[C_x].q^*[\phi(E_S)] = 360$ on the general point x of the Fermat, the same must be true of $[C_{x,t}].q^*[\mathcal{D}_t]$ for every point $t \in B$. As for t general the class of $\phi(E_{S_t})$ in $\text{Pic}\, F_t$ is $60H$ (see Corollary 3.11) and $[C_x].q^*(60H) = 360$, we obtain that for $t \in U$, $[\phi(E_{S_t})] = \phi_*[E_{S_t}]$ which implies that the degree of ϕ restricted to E_{S_t} is one as required. $\qquad\square$

REMARK 3.13.   (1) As $K_F = 0$ we obtain that for a general cubic fourfold, $K_{\widetilde{F}} = E_S$. In particular the morphism $\phi : \widetilde{F} \to F$ is ramified along E_S. The above, along with the Riemann–Hurwitz formula, shows that above a general branch point of F, there is precisely one ramification point of simple ramification. In fact a very recent preprint [GG24] uses this and techniques similar to those in Section 5 to conclude that the Voisin map has full monodromy group S_{16}.

   (2) Presumably the $[C_x].[q^{-1}(\phi(E_S))] = 360$ points of C_x for $x \in X$ general are related to the 360 intersection points of the 120 tritangent 2-planes a space $(2, 3)$-curve has.

4. Some properties of the second type locus

In previous work of ours ([GK24, Remark 3.8]), we sketched how to obtain the following result using the correspondence between hyperplane sections with two nodes and $(2, 3)$-complete intersections (cf. [CG72, Lemma 6.5]). At the time, we had given a proof in the case of cubic fourfolds but as this is an important property of the second type locus and we will be using it for threefolds too, we include its proof for any $n \geqslant 3$ here.

THEOREM 4.1. *Let $n \geqslant 3$ and let $X \subset \mathbb{P}^{n+1}$ be a general cubic n-fold, F its Fano scheme of lines and $F_2 \subset F \subset G(2, n + 2)$ the locus of lines of second type. Consider the induced morphism from the family of lines*

$$\mathbb{L}_2 \xrightarrow{\;p\;} W \subset X.$$
$$\Big\downarrow{\scriptstyle q}$$
$$F_2$$

Then p is birational onto its image W.

In other words, on a general cubic, a general point through which passes a second type line has exactly one second type line through it.

Proof. First one notes that there is a bijection between pairs (Y, x) where $Y \subset \mathbb{P}^n$ is a cubic $(n-1)$-fold with two A_1 singularities, one of which is x, and $(2, 3)$-complete intersections in $\mathbb{P}^{n-1}$ with one A_1 singularity and no other singular points (see [GK24, Remark 3.8]).

Starting from such a $(2, 3)$-complete intersection in $\mathbb{P}^{n-1}$

$$C_0 \cap Q_0 = V(f_2) \cap V(f_3)$$

and taking its corresponding (Y_0, p_0) in $\mathbb{P}^n$, with equation

$$f = f_2 \cdot x_n + f_3, \quad p_0 = [0 : \cdots : 0 : 1],$$

one constructs a smooth cubic n-fold $X_0 \subset \mathbb{P}^{n+1}$ (cf. the proof of [GK24, Proposition 3.5]) with equation

$$g = f + x_n^2 x_{n+1}$$

containing the point $x_0 = [0 : \cdots : 0 : 1 : 0]$ whose tangent hyperplane meets X_0 at the pair (Y_0, p_0). The span of lines through x_0 is a cone over $C_0 \cap Q_0$. As for any smooth cubic X the singular locus of $p : \mathbb{L} \to X$ is precisely $\mathbb{L}_2$ (see Proposition 2.1), the above constructs one point in X_0 with a second type line through it corresponding to the unique singularity of $p^{-1}(x_0) = C_0 \cap Q_0$. As having a unique A_1 singularity is an open condition, this must hold generically for (the necessarily singular) fibres over points of $p(\mathbb{L}_2) = W_0 \subset X_0$. This implies that for X_0, the map $\mathbb{L}_2 \to W_0$ is birational.

To conclude for the general cubic n-fold, we argue as follows. Consider the universal smooth family $\mathcal{X} \to |\mathcal{O}_{\mathbb{P}^{n+1}}(3)|_{sm}$ and the universal locus $\mathcal{W} := p(q^{-1}(\mathcal{F}_2))$ over $|\mathcal{O}_{\mathbb{P}^{n+1}}(3)|_{sm}$, where $\mathcal{F}_2$ is the universal second type locus, known to be irreducible from [Huy23, Proof of Proposition 2.2.13]. From Proposition 2.1, as the locus of second type lines $F_{2,t}$ of any smooth cubic $\mathcal{X}_t$ is pure $(n-2)$-dimensional, we obtain that the $\mathbb{P}^1$-bundle $q^{-1}(F_{2,t})$ is pure $(n-1)$-dimensional, and the same proposition gives that $\mathcal{W}_t$ is a divisor in $\mathcal{X}_t$ for any $t \in |\mathcal{O}_{\mathbb{P}^{n+1}}(3)|_{sm}$. Hence, the morphism $q^{-1}(F_{2,t}) \to \mathcal{W}_t$ is generically finite. If t is general then $F_{2,t}, \mathcal{W}_t$ are irreducible and reduced. As over some points, namely the generic point of W_0 in the special cubic X_0 above we know that the fibre in $\mathbb{L}$ has a single A_1 singularity, we obtain the same for the generic cubic. In particular, the morphism $p : q^{-1}(F_{2,t}) \to \mathcal{W}_t$ is birational for a general t as required. $\square$

REMARK 4.2. In [GK19, Fact 3.2.(2)] we erroneously stated that in the case of a cubic threefold $X \subset \mathbb{P}^4$, [CG72, 10.18] claims that $\mathbb{L} \to X$ has one single ramification point of ramification index two over a generic point of the branch locus. What they do claim (and prove) is that for a generic point $x \in X$ through which there passes a line of second type, this line will count with multiplicity two as one of the six lines through x. In particular, there could, a priori, still be multiple ramification points, i.e., other second types lines through x. The above proposition indeed confirms the stronger claim. Either way, this does not affect any results in [GK19] as in the proof we only required what the weaker statement about multiplicities pertains to.

One can derive another proof of Theorem 4.1 for a general cubic threefold or fourfold by degenerating to the Fermat (instead of X_0 used in the above proof) and using Propositions 3.1, 3.5 respectively. In these cases the class of W in the Picard group of the cubic is $\mathcal{O}(30)$ and $\mathcal{O}(75)$ respectively.

We now extend the proof of Theorem 4.1 to obtain the following refinement which will be used in the next section.

LEMMA 4.3. *Let $X \subset \mathbb{P}^4$ be a general cubic 3-fold and $\ell \in F_2$ a general second type line. Then there is a point on ℓ through which there pass precisely one further second type line and two more of first type.*

Proof. The same correspondence as in the proof of Theorem 4.1 can be extended to

1. (Y, x) where $Y \subset \mathbb{P}^n$ is a cubic $(n-1)$-fold with three non-collinear ordinary double points, one of which is x,
2. $(2, 3)$-complete intersections in $\mathbb{P}^{n-1}$ with two ordinary double points.

We recall the construction (see also [AV08, Lemma 1.2] for the fourfold case). The $(2, 3)$-complete intersection $C \cap Q \subset \mathbb{P}^{n-1}$ can be blown up to obtain a variety $\widetilde{Y} \to \mathbb{P}^{n-1}$ which has two singularities above the singular points of $C \cap Q$. One now contracts the strict transform $\widetilde{Q}$ of Q to obtain Y, which is also singular at the image of $\widetilde{Q}$, the marked point.

The same method as we sketched in the proof of Theorem 4.1 gives that starting with a $(2, 3)$-intersection as above, we obtain a smooth cubic $X \subset \mathbb{P}^{n+1}$ with a point x_0 whose tangent hyperplane meets X at a (Y, x_0) as in the correspondence. In particular, the two other singular points of Y will correspond to two distinct second type lines through x_0, while every other line through x_0 will be of first type.

Applying the above in the $n = 3$ case, we obtain a special smooth cubic, call it $X_0 \subset \mathbb{P}^4$ which has a second type line ℓ_0 and a point $p_0 \in \ell_0$ so that there are precisely four lines in X_0 passing through p_0, two of second type, and two of first.

Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold. For a line $\ell \in F_2$, consider $C_\ell \subset F$ the locus of lines meeting ℓ. We have a degree four morphism $C_\ell \to \ell$ taking a line ℓ' meeting ℓ to the point $\ell \cap \ell'$. Consider the universal family

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ \pi\ } & \mathbb{L}_2 = q^{-1}(\mathcal{F}_2) \\
\downarrow & \swarrow_{\mathrm{pr}_2} & \\
\mathcal{F}_2 & &
\end{array}
$$

where for $\ell \in \mathcal{F}_2$, the fibre in $\mathcal{C}$ over ℓ is C_ℓ, and $\pi_\ell : \mathcal{C} \to \mathbb{P}^1_\ell$ is the one described above. For any ℓ, π_ℓ is ramified at the points of intersection of C_ℓ with F_2 in F, in particular at a non-empty locus. Furthermore, there exist $p_0 \in \ell_0, \ell'_0 \subset X_0$ both second type lines constructed in our special X_0 above where the ramification of $\pi_{\ell_0} : C_{\ell_0} \to \ell_0$ at the point $[\ell'_0] \in C_\ell$ is simple. This

ramification type must hold generically over $\mathcal{F}_2$, for example since the ramification index $e_{\ell_t} = \text{length}\left((\Omega_\pi)_{[\ell_t]}\right) + 1$ is upper semicontinuous (see [Liu02, Exercise 7.1.6] and the upper semicontinuity of the rank) over the branch locus of π, which is a divisor from purity and smoothness of $\mathcal{F}_2$. $\square$

5. Monodromy for cubic 3-folds

In this section, let $X \subset \mathbb{P}^4$ be a smooth cubic hypersurface, and denote by $F, F_2, \mathbb{L}, \mathbb{L}_2$ as in Section 2. We refer to [CG72] (cf. [Huy23, Remark 5.1.6] for the following properties. In the threefold case, the morphism $p : \mathbb{L} \to X$ is generically finite of degree six, i.e., there are six lines through a general point of X. There are more than 6 lines through a point if and only if the point is an Eckardt point, in which case there are infinitely many lines through that point, all of second type. A cubic threefold can contain at most 30 Eckardt points (in fact this is the case for the Fermat, see Proposition 3.1), but the generic one contains none, so in this case $\mathbb{L} \to X$ is finite. For a line $\ell \in F$, we denote by $C_\ell \subset F$ the locus of lines meeting ℓ - more precisely, C_ℓ is the image in F under q of the closure of $p^{-1}(\ell) \setminus q^{-1}([\ell])$.

Given a generically finite dominant morphism $f : X \to Y$ of degree d between irreducible varieties (necessarily of the same dimension), we obtain a degree d extension of function fields $k(X)/k(Y)$, and taking the Galois closure $K/k(Y)$ of this extension, we denote by $\text{Gal}_f = \text{Gal}(K/k(Y))$. This agrees with the usual monodromy group (see [Har79, SY21]) Mon_f, which is defined as the image in S_d of the group of deck transformations of the unramified (i.e., topological) cover $X \setminus \text{ram}(f) \to U$, where $U = Y \setminus \text{branch}(f)$ is the largest dense open in Y over which f is étale.

Recall the following classical results for $f : X \to Y$ and U as above and assume furthermore that X is locally analytically irreducible (e.g., smooth).

LEMMA 5.1. *([Har79, p698]) If there exists a fibre of f with a unique point of ramification index two and all other points unramified, then $\text{Mon}_f \subseteq S_d$ contains a transposition.*

Let $X_U^{(s)}$ be the complement of the big diagonal in the fibre product of X s-times with itself over U. In other words, over a point of U, $X_U^{(s)}$ consists of s distinct points in the fibre.

LEMMA 5.2. *([SY21, Proposition 2]) $X_U^{(s)}$ is irreducible if and only if Mon_f is an s-transitive subgroup of S_d.*

We note the well-known fact that a subgroup of S_d which contains a transposition and that is 2-transitive must be the whole S_d.

PROPOSITION 5.3. *If X is a general cubic threefold, the Galois group G of $\mathbb{L} \to X$ is S_6.*

Proof. Since $\mathbb{L} = \mathbb{P}(\mathcal{U}|_F)$ is the projectivised universal bundle, it is irreducible and hence G is a transitive subgroup of S_6 from Lemma 5.2. If G is 2-transitive

and contains a transposition, it must be the whole S_6, so we now show each of these facts.

First, we show that G contains a transposition. From Theorem 4.1, a general second type line $\ell \subset X$ counts with multiplicity two (out of the six) at a general point on it, and there are four other distinct type one lines through that point, i.e., there is a simply ramified fibre of $\mathbb{L} \to X$, since from 2.1 the ramification locus of $\mathbb{L} \to X$ is precisely the universal second type locus. Now, Lemma 5.1 implies that G contains a transposition.

Finally, we prove that G is 2-transitive. Note that over a point $[\ell]$, the morphism $\mathbb{L} \times_X \mathbb{L} \to F$ has fibre C_ℓ the curve of lines meeting ℓ. For any X, the general C_ℓ is a curve, and is irreducible as the class of C_ℓ is ample. Restricting to the open $U \subset X$ over which $\mathbb{L} \to X$ is étale (note this also removes any Eckardt points, above which the fibre is infinite), we obtain a flat morphism $\mathbb{L}_U^{(2)} \to F|_U$, since it a composition of the finite (hence flat) morphism $\mathbb{L}_U^{(2)} \to \mathbb{L}_U$ and the flat morphism $\mathbb{L}|_U \to F|_U$. As the base and the general fibre are irreducible and the morphism is open, so too is the total space $\mathbb{L}_U^{(2)}$ irreducible, from an elementary lemma [Sta21, Tag 004Z]. We conclude from Lemma 5.2.   □

Consider next the natural morphism

$$\pi_\ell : C_\ell \to \ell \cong \mathbb{P}^1$$
$$[\ell'] \mapsto \ell \cap \ell'.$$

As for any smooth cubic and general line $\ell \subset X$ the variety C_ℓ is a curve, and there are six lines through a point, the degree of this morphism is five. Analogously, if ℓ is a second type line, as it counts with multiplicity at least two as one of the six, we obtain an induced degree 4 morphism $\pi_\ell : C_\ell \to \ell$.

PROPOSITION 5.4. *Let X be a general cubic threefold. If ℓ is a general line in X, then* $\mathrm{Mon}_{\pi_\ell} \cong S_5$. *If ℓ a general line of second type, then* $\mathrm{Mon}_{\pi_\ell} \cong S_4$.

Proof. A general ℓ of first type will meet a general line of second type ℓ' at a general point p of ℓ' (more specifically, outside the finite locus of points through which there pass more second type lines). This means from Theorem 4.1 that $C_\ell \to \ell$ is simply ramified at $[\ell']$ over p and there are no further ramification points, hence the Galois group has a transposition from Lemma 5.1. Transitivity in S_5 follows since C_ℓ is irreducible for general ℓ on any X. Since S_5 is the only subgroup of S_5 which has a transposition and is transitive, we conclude the result.

To prove the second claim, note that Lemma 4.3 gives that there is always a transposition in the monodromy group. Again, as C_ℓ is irreducible from [GK19, Lemma 3.3] and S_4 is the only transitive subgroup of S_4 which contains a transposition, we obtain the result.   □

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