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Bijections between sets of invariant ideals, via the ladder technique

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ABSTRACT. We present a new method of establishing a bijective correspondence — in fact, a lattice isomorphism — between action- and coaction-invariant ideals of C^*-algebras and their crossed products by a fixed locally compact group. It is known that such a correspondence exists whenever the group is amenable; our results hold for any locally compact group under a natural form of coaction invariance.

CONTENTS

1. Introduction

One of the most fundamental things to know about a C^*-algebra is its ideal structure. If the C^*-algebra arises as the crossed product $A \rtimes_\alpha G$ of an action (A, G, α), it is natural to compare the ideal structures of A and $A \rtimes_\alpha G$. This is most easily done when we restrict attention to ideals that are related in some way to the action of G. More precisely, we focus on ideals I of A that are α-invariant. Then the crossed product is (isomorphic to) an ideal $I \rtimes_\alpha G$ of $A \rtimes_\alpha G$, and the obvious question is which ideals of $A \rtimes_\alpha G$ arise in this way. It turns out that they are precisely those ideals that are invariant under the dual coaction $\hat{\alpha}$, and there is a bijection between the two sets of invariant ideals. When G is amenable this is an old result of Gootman and Lazar ([3]), and in this paper we prove it in complete generality (Theorem 3.2 (a)). Dually, starting with a coaction (A, δ) of G, the δ-invariant ideals of A correspond to $\hat{\delta}$-invariant ideals of the crossed product $A \rtimes_\delta G$. Again, Gootman–Lazar proved this for amenable G, and in this paper we prove it in general (Theorem 3.2 (b)). Nilsen ([9]) has a

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related pair of results, but their relationship to ours is complicated because she used a different notion of coaction invariance than we do.

Perhaps more important, though, is our method of proof—we introduce what we call the "ladder technique", which leads to quick proofs of the aforementioned bijections, using only crossed-product duality and the Rieffel correspondence between ideals of Morita equivalent C^*-algebras. We expect the ladder technique to be useful in other situations; for example three of us are currently working on an application to ideals of Fell bundles over groupoids.

We begin in Section 2 with a detailed overview of the crossed-product duality theorems we need, the essential definitions and facts regarding invariant ideals, and an identification of the associated Rieffel correspondences. Then in Section 3 we prove our main theorem, which actually comprises four versions: starting with actions we can consider either full or reduced crossed products, and dually we can start with either maximal or normal coactions. Finally, in Section 4 we close with a brief discussion comparing our results to those of Gootman–Lazar and Nilsen. And we point out that one of our theorems was proved, using somewhat more technical methods (unrelated to the ladder technique), in a recent paper two of us wrote with Tron Omland.

2. Preliminaries

Below we will recall suitable versions of the Imai–Takai and Katayama duality theorems for crossed products. But to prepare for this we start with some background on actions and coactions. Further details about action crossed products can be found in [13]. Our main references for coactions and their crossed products are [2, Appendix A], [1], [11], [8], [6], and [9].

Throughout, G is a locally compact group and A is a C^*-algebra. We write L^2 for $L^2(G)$, λ and ρ for the left and right regular representations of G on L^2, respectively, $\mathcal{K}$ for the compact operators $\mathcal{K}(L^2)$, and M (sometimes) for the representation of $C_0(G)$ on L^2 by multiplication operators. We write w_G for the unitary element of $M(C_0(G) \otimes C^*(G)) = C_b^\beta(G, M(C^*(G)))$ given by the norm-bounded strictly continuous function $w_G(s) = s$ for $s \in G$, where the β signifies that we are using the strict topology on $M(C^*(G))$.

We use (A, α) to denote an action α of G on A. A *coaction* (A, δ) of G on A is an injective nondegenerate homomorphism $\delta : A \to M(A \otimes C^*(G))$ satisfying:

(a) $\overline{\operatorname{span}}\, \delta(A)(1 \otimes C^*(G)) = A \otimes C^*(G)$
(b) $(\delta \otimes \mathrm{id})\circ\delta = (\mathrm{id} \otimes \delta_G)\circ\delta$,

where $\otimes$ always denotes the minimal C^*-tensor product and $\delta_G : C^*(G) \to M(C^*(G) \otimes C^*(G))$ is the usual comultiplication of $C^*(G)$, i.e., the integrated form of the strictly continuous unitary homomorphism $s \mapsto s \otimes s$. Here, and similarly throughout the paper (unlike the convention used in [2]), $\delta(A)(1 \otimes C^*(G))$ represents the set of products $\{\delta(a)(1 \otimes z) \mid a \in A, z \in C^*(G)\}$. (Also note that by definition, our coactions are *nondegenerate* in the sense of [2, Definition 2.10].) The (full) crossed product of an action (A, α) is $A \rtimes_\alpha G$, which comes with a universal covariant representation (i_A, i_G) and a dual coaction $\hat{\alpha}$. The reduced crossed product of (A, α) is $A \rtimes_{\alpha,r} G$. The crossed product of a coaction (A, δ) is $A \rtimes_\delta G$, which comes with a universal covariant representation (j_A, j_G) and a dual action $\hat{\delta}$.

For any action (A, α), the *canonical surjection* $\Phi_\alpha : A \rtimes_\alpha G \rtimes_{\hat{\alpha}} G \to A \otimes \mathcal{K}$ is determined by

(a) $\Phi_\alpha \circ j_{A \rtimes_\alpha G} \circ i_A = (\mathrm{id} \otimes M) \circ \alpha^{-1}$
(b) $\Phi_\alpha \circ j_{A \rtimes_\alpha G} \circ i_G = 1 \otimes \lambda$
(c) $\Phi_\alpha \circ j_G = 1 \otimes M$.

In item (a), "α^{-1}" refers to the map that sends $a \in A$ to the element of $M(A \otimes C_0(G))$ determined by the function $s \mapsto \alpha_{s^{-1}}(a)$ (see [11]). For a coaction (A, δ), the *canonical surjection* $\Phi_\delta : A \rtimes_\delta G \rtimes_{\hat{\delta}} G \to A \otimes \mathcal{K}$ is determined by

(a) $\Phi_\delta \circ i_{A \rtimes_\delta G} \circ j_A = (\mathrm{id} \otimes \lambda) \circ \delta$
(b) $\Phi_\delta \circ i_{A \rtimes_\delta G} \circ j_G = 1 \otimes M$
(c) $\Phi_\delta \circ i_G = 1 \otimes \rho$.

An *equivariant homomorphism* $\phi : (A, \delta) \to (B, \epsilon)$ between coactions is a possibly degenerate homomorphism ϕ mapping A into B (not $M(B)$) such that the diagram

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "\phi"'] & \widetilde{M}(A \otimes C^*(G)) \arrow[d, "\phi \otimes \mathrm{id}"] \\ B \arrow[r, "\epsilon"'] & \widetilde{M}(B \otimes C^*(G)) \end{tikzcd}$$

commutes (see [6, Definition 3.2]). Immediately following [6, Corollary 3.13] it is noted that "a routine adaptation of the usual arguments" (i.e., carefully applying the definitions) shows that every $\delta - \epsilon$ equivariant homomorphism $\phi : A \to B$ gives rise to a $\hat{\delta} - \hat{\epsilon}$ equivariant homomorphism

$$\phi \rtimes G = (j_B \circ \phi) \times j_G : A \rtimes_\delta G \to B \rtimes_\epsilon G.$$

A coaction (A, δ) is *maximal* if Φ_δ is an isomorphism, and is *normal* if $j_A : A \to M(A \rtimes_\delta G)$ is injective. Equivalently, δ is normal exactly when Φ_δ factors through an isomorphism of $A \rtimes_\delta G \rtimes_{\hat{\delta},r} G$ onto $A \otimes \mathcal{K}$.

Every dual coaction $(A \rtimes_\alpha G, \hat{\alpha})$ is maximal. Every coaction (A, δ) has a *normalization* (A^n, δ^n), which is unique up to isomorphism, such that A^n is a quotient of A, the coaction δ^n is normal, the quotient map $\psi_\delta : A \to A^n$ is $\delta - \delta^n$ equivariant, and the homomorphism $\psi_\delta \rtimes G : A \rtimes_\delta G \to A^n \rtimes_{\delta^n} G$ is an isomorphism. For a dual coaction $\hat{\alpha}$ on an action crossed product $A \rtimes_\alpha G$, the associated map $\psi_{\hat{\alpha}}$ is the regular representation $A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$. Thus, the dual coaction on the reduced crossed product is the normalization $\hat{\alpha}^n$, and the double-dual action on $A \rtimes_{\alpha,r} G \rtimes_{\hat{\alpha}^n} G$ is $\widehat{\hat{\alpha}^n}$.

Theorem 2.1 below is *Imai–Takai duality*. The original version was stated in [4] for reduced crossed products, and did not use the word 'coaction'. Version (a) below for full crossed products appears in [11, Theorem 7]. We use [2] as a convenient reference.

Theorem 2.1 ([2, Theorem A.67]). *For any action (A, α):*

(a) (*full-crossed-product version*) *The canonical map Φ_α is an $\hat{\hat{\alpha}} - (\alpha \otimes \operatorname{Ad} \rho)$ equivariant isomorphism of $A \rtimes_\alpha G \rtimes_{\hat{\alpha}} G$ onto $A \otimes \mathcal{K}$.*

(b) (*reduced-crossed-product version*) *There is an $\widehat{\hat{\alpha}^n} - (\alpha \otimes \operatorname{Ad} \rho)$ equivariant isomorphism $\Phi_{\alpha,r}$ of $A \rtimes_{\alpha,r} G \rtimes_{\hat{\alpha}^n} G$ onto $A \otimes \mathcal{K}$ such that $\Phi_\alpha = \Phi_{\alpha,r} \circ (\psi_{\hat{\alpha}} \rtimes G)$.*

Theorem 2.2 below is *Katayama duality*, which is the dual version of Theorem 2.1. The original version was stated in [5, Theorem 8] for reduced coactions, but we prefer to work with full coactions.

Theorem 2.2 ([2, Theorem A.69]). *For any coaction (A, δ):*

(a) (*maximal coaction version*) *If δ is maximal, the canonical surjection Φ_δ is a $\hat{\hat{\delta}} - \epsilon$ equivariant isomorphism of $A \rtimes_\delta G \rtimes_{\hat{\delta}} G$ onto $A \otimes \mathcal{K}$.*

(b) (*normal coaction version*) *If δ is normal, there is a $\hat{\hat{\delta}}^n - \epsilon$ equivariant isomorphism $\Phi_{\delta,r}$ of $A \rtimes_\delta G \rtimes_{\hat{\delta},r} G$ onto $A \otimes \mathcal{K}$ such that $\Phi_\delta = \Phi_{\delta,r} \circ \psi_{\hat{\hat{\delta}}}$.*

The coaction $(A \otimes \mathcal{K}, \epsilon)$ associated to (A, δ) in Theorem 2.2 is defined by

$$\epsilon = \operatorname{Ad}\bigl(1 \otimes (M \otimes \mathrm{id})(w_G^*)\bigr) \circ (\mathrm{id} \otimes \Sigma) \circ (\delta \otimes \mathrm{id}),$$

where $\Sigma : C^*(G) \otimes \mathcal{K} \to \mathcal{K} \otimes C^*(G)$ is the flip isomorphism determined on elementary tensors by $a \otimes b \mapsto b \otimes a$. The statement about equivariance in part (a) is actually missing from [2]; it follows from the analogous result for normal coactions in [9, Remark 5] and the equivalence of maximal and normal coactions ([8, Theorem 3.3]).

Given an action (A, α), we say that a (closed, two-sided) ideal I of A is α-*invariant* if α restricts to an action $\alpha|$ on I; that is, if $\alpha_g(I) \subseteq I$ for each $g \in G$. (See [13, Section 3.4] for further discussion.) We write $\mathscr{I}_\alpha(A)$ for the set of α-invariant ideals of A.

For a coaction (A, δ), an ideal I of A is δ-*invariant* if δ restricts to a coaction $\delta|$ on I. By results in Section 2 of [10], this is equivalent to the condition that

$$\overline{\operatorname{span}}\, \delta(I)(1 \otimes C^*(G)) = I \otimes C^*(G). \tag{2.1}$$

(See also the discussion preceding Definition 3.17 in [6].) We write $\mathscr{I}_\delta(A)$ for the set of δ-invariant ideals of A.

Somewhat surprisingly (to us), the fact that crossed products of invariant ideals are *invariant* ideals has not been clearly stated or entirely justified elsewhere in the literature.

Proposition 2.3. (a) *For any action (A, α), for each $I \in \mathscr{I}_\alpha(A)$ the inclusion map $\phi : I \hookrightarrow A$ is $\alpha| - \alpha$ equivariant, and $\phi \rtimes G$ is an isomorphism of*

$I \rtimes_{\alpha|} G$ onto an $\hat\alpha$-invariant ideal $I \rtimes_\alpha G$ of $A \rtimes_\alpha G$. Moreover, the image $I \rtimes_{\alpha,r} G$ of $I \rtimes_\alpha G$ under the regular representation of $A \rtimes_\alpha G$ is an $\hat\alpha^n$-invariant ideal of the reduced crossed product $A \rtimes_{\alpha,r} G$.

(b) *For any coaction (A, δ), for each $I \in \mathscr{I}_\delta(A)$ the inclusion map $\phi : I \hookrightarrow A$ is $\delta| - \delta$ equivariant, and $\phi \rtimes G$ is an isomorphism of $I \rtimes_{\delta|} G$ onto a $\hat\delta$-invariant ideal $I \rtimes_\delta G$ of $A \rtimes_\delta G$.*

Proof. (a) Except for invariance, this is a consequence of [13, Proposition 3.19]. For the $\hat\alpha$-invariance of $I \rtimes_\alpha G$, we have

$$I \rtimes_\alpha G = \phi \rtimes G(I \rtimes_{\alpha|} G) = \overline{\operatorname{span}}\, i_A(\phi(I))i_G(C^*(G)),$$

so that

$$\begin{aligned}\hat\alpha(I \rtimes_\alpha G) &= \overline{\operatorname{span}}\, \hat\alpha\big(i_A(\phi(I))\big)\hat\alpha\big(i_G(C^*(G))\big)\\ &= \overline{\operatorname{span}}\, \big(i_A(\phi(I)) \otimes 1\big)(i_G \otimes \mathrm{id})(\delta_G(C^*(G)))\end{aligned}$$

by definition of $\hat\alpha$. Thus,

$$\begin{aligned}&\overline{\operatorname{span}}\, \hat\alpha(I \rtimes_\alpha G)(1 \otimes C^*(G))\\ &\quad= \overline{\operatorname{span}}\, \big(i_A(\phi(I)) \otimes 1\big)(i_G \otimes \mathrm{id})(\delta_G(C^*(G)))(1 \otimes C^*(G))\\ &\quad= \overline{\operatorname{span}}\, \big(i_A(\phi(I)) \otimes 1\big)(i_G \otimes \mathrm{id})\big(\delta_G(C^*(G))(1 \otimes C^*(G))\big)\\ &\quad= \overline{\operatorname{span}}\, \big(i_A(\phi(I)) \otimes 1\big)(i_G \otimes \mathrm{id})\big(C^*(G) \otimes C^*(G)\big)\\ &\quad= \overline{\operatorname{span}}\, \big(i_A(\phi(I)) \otimes 1\big)\big(i_G(C^*(G)) \otimes C^*(G)\big)\\ &\quad= \overline{\operatorname{span}}\, \big(i_A(\phi(I))i_G(C^*(G))\big) \otimes C^*(G)\\ &\quad= \big(\phi \rtimes G(I \rtimes_{\alpha|} G)\big) \otimes C^*(G)\\ &\quad= (I \rtimes_\alpha G) \otimes C^*(G),\end{aligned}$$

which shows that $I \rtimes_\alpha G$ is $\hat\alpha$-invariant. Invariance of $I \rtimes_{\alpha,r} G$ now follows from $\hat\alpha - \hat\alpha^n$ equivariance of the regular representation.

(b) Some of this — apart from invariance of $I \rtimes_\delta G$ — is addressed in [10, Proposition 2.1], but with a different convention regarding equivariant maps.

For us, the restriction $\delta|$ is a coaction on I by definition of invariance, and the inclusion ϕ is trivially $\delta| - \delta$ equivariant. The verification that the induced $\widehat{\delta|} - \hat\delta$ equivariant homomorphism $\phi \rtimes G : I \rtimes_{\delta|} G \to A \rtimes_\delta G$ is injective follows a standard computation with covariant representations on Hilbert space (see [10, proof of Proposition 2.1], for example).

To see that $I \rtimes_\delta G = \phi \rtimes G(I \rtimes_{\delta|} G)$ is an ideal of $A \rtimes_\delta G$, first note that [6, Lemma 3.8] implies

$$A \rtimes_\delta G = \overline{\operatorname{span}}\, j_A(A)j_G(C_0(G)) = \overline{\operatorname{span}}\, j_G(C_0(G))j_A(A),$$

so that

$$\begin{aligned}(I \rtimes_\delta G)(A \rtimes_\delta G) &= \overline{\operatorname{span}}\, j_A(I)j_G(C_0(G))j_A(A)\\ &= \overline{\operatorname{span}}\, j_A(I)j_A(A)j_G(C_0(G))\\ &= \overline{\operatorname{span}}\, j_A(I)j_G(C_0(G))\\ &= I \rtimes_\delta G,\end{aligned}$$

and similarly for $(A \rtimes_\delta G)(I \rtimes_\delta G)$.

Finally, to see that $I \rtimes_\delta G$ is $\hat{\delta}$-invariant, for any $s \in G$ we compute:

$$\begin{aligned}\hat{\delta}_s\,(I \rtimes_\delta G) &= \hat{\delta}_s\left(\phi \rtimes G(I \rtimes_{\delta|} G)\right)\\ &= \phi \rtimes G\big(\widehat{\delta|}_s(I \rtimes_{\delta|} G)\big)\\ &= \phi \rtimes G(I \rtimes_{\delta|} G)\\ &= I \rtimes_\delta G.\end{aligned}$$ □

For our main result, Theorem 3.2, we refer to the *Rieffel correspondence* (see, for example, [12, Proposition 3.24]). This is the lattice isomorphism between the ideals of A and the ideals of B that arises from a $B-A$ imprimitivity bimodule X by associating each ideal I of A with the ideal J of B given by

$$J = \overline{\operatorname{span}}\,{}_B\langle X \cdot I\,,\, X\rangle.$$

If A and B are equipped with actions or coactions of G and X has a suitably compatible action or coaction (see [2, Definitions 2.5 and 2.10]), then the Rieffel correspondence preserves invariance of ideals, and therefore restricts to a bijection between the appropriately-invariant ideals of A and those of B. Again, since we could not find a precise statement of this fact in the literature, we provide one here.

Lemma 2.4. *Let X be a $B-A$ imprimitivity bimodule, let I be an ideal of A, and let J be the ideal of B associated to I under the Rieffel correspondence.*

(a) *If (A,α) and (B,β) are actions of G and there exists a $\beta-\alpha$ compatible action γ on X, then I is α-invariant if and only if J is β-invariant.*
(b) *If (A,δ) and (B,ϵ) are coactions of G and there exists a nondegenerate $\epsilon-\delta$ compatible coaction ζ on X, then I is δ-invariant if and only if J is ϵ-invariant.*

Proof. Part (a) is straightforward, so we only address part (b). Moreover, by symmetry it suffices to prove only one implication; so suppose I is δ-invariant. Then $\overline{\operatorname{span}}\,(1 \otimes C^*(G))\delta(I) = I \otimes C^*(G)$ (after taking adjoints in (2.1)), and nondegeneracy of ζ means that $\overline{\operatorname{span}}\,(1 \otimes C^*(G))\zeta(X) = X \otimes C^*(G)$. So, writing

M for $M(B \otimes C^*(G))$, we have:

$$\begin{aligned}
\overline{\operatorname{span}}\,(1 \otimes C^*(G))\epsilon(J) &= \overline{\operatorname{span}}\,(1 \otimes C^*(G))\epsilon\big({}_B\langle X \cdot I, X\rangle\big) \\
&= \overline{\operatorname{span}}\,(1 \otimes C^*(G))_B\langle \zeta(X \cdot I), \zeta(X)\rangle \\
&= \overline{\operatorname{span}}\,{}_M\langle (1 \otimes C^*(G))\zeta(X)\delta(I), (1 \otimes C^*(G))\zeta(X)\rangle \\
&= \overline{\operatorname{span}}\,{}_M\langle (X \otimes C^*(G))\delta(I), X \otimes C^*(G), \rangle \\
&= \overline{\operatorname{span}}\,{}_M\langle (X \otimes C^*(G))(1 \otimes C^*(G))\delta(I), X \otimes C^*(G)\rangle \\
&= \overline{\operatorname{span}}\,{}_M\langle (X \otimes C^*(G))(I \otimes C^*(G)), X \otimes C^*(G)\rangle \\
&= \overline{\operatorname{span}}\,{}_B\langle X \cdot I, X\rangle \otimes C^*(G) \\
&= J \otimes C^*(G).
\end{aligned}$$

Thus J is ϵ-invariant. □

The isomorphisms from the duality Theorems 2.1 and 2.2 allow us to make the $A \otimes \mathcal{K} - A$ imprimitivity bimodule $A \otimes L^2$ into an $A \rtimes G \rtimes G - A$ imprimitivity bimodule (for each of the four types of crossed products), and when we do this we can identify the associated ideals explicitly:

Proposition 2.5. (a) *Let (A, α) be an action, and let I be an α-invariant ideal of A. Then the ideal of $A \rtimes_\alpha G \rtimes_{\hat{\alpha}} G$ associated to I by the Rieffel correspondence is $I \rtimes_\alpha G \rtimes_{\hat{\alpha}} G$. The ideal of $A \rtimes_{\alpha,r} G \rtimes_{\hat{\alpha}^n} G$ associated to I by the Rieffel correspondence is $I \rtimes_{\alpha,r} G \rtimes_{\hat{\alpha}^n} G$.*

(b) *Let (A, δ) be a coaction, and let I be a δ-invariant ideal of A. If δ is maximal, then the ideal of $A \rtimes_\delta G \rtimes_{\hat{\delta}} G$ associated to I by the Rieffel correspondence is $I \rtimes_\delta G \rtimes_{\hat{\delta}} G$. If δ is normal, the ideal of $A \rtimes_\delta G \rtimes_{\hat{\delta},r} G$ associated to I by the Rieffel correspondence is $I \rtimes_\delta G \rtimes_{\hat{\delta},r} G$.*

Proof. (a) The key observation is that image of $I \rtimes_\alpha G \rtimes_{\hat{\alpha}} G$ in $A \otimes \mathcal{K}$ under the canonical map Φ_α coincides with the image of $I \rtimes_{\alpha|} G \rtimes_{\widehat{\alpha|}} G$ under the canonical map $\Phi_{\alpha|}$, and this latter image is precisely $I \otimes \mathcal{K}$ by Theorem 2.1 (a). Since the ideal of $A \otimes \mathcal{K}$ associated to I by the Rieffel correspondence is also precisely $I \otimes \mathcal{K}$, it follows that the ideal of $A \rtimes_\alpha G \rtimes_{\hat{\alpha}} G$ associated to I is

$$\Phi_\alpha^{-1}(I \otimes \mathcal{K}) = I \rtimes_\alpha G \rtimes_{\hat{\alpha}} G.$$

The other part of (a), and both parts of (b), are quite similar. □

3. The ladder technique

Our main result, Theorem 3.2 below, rests upon the following basic observation concerning maps between sets. We record it only for convenient reference.

Lemma 3.1. *If*

$$\begin{tikzcd} & W \\ Z \arrow[ur, "h"] & \\ & Y \arrow[ul, "g"'] \arrow[uu, "v"'] \\ X \arrow[uu, "u"] \arrow[ur, "f"] & \end{tikzcd}$$

is a commutative diagram of sets and maps such that u and v are bijections, then f, g, and h are also bijections.

Recall that for any C^*-algebra A, the set $\mathscr{I}(A)$ of all (closed, two-sided) ideals of A is a lattice when ordered by inclusion, with $I \wedge J = I \cap J$ and $I \vee J = I + J$. For any fixed action α or coaction δ of G on A, the intersection and sum of two invariant ideals are themselves invariant, so the sets $\mathscr{I}_\alpha(A)$ and $\mathscr{I}_\delta(A)$ of appropriately invariant ideals of A each form a sublattice of $\mathscr{I}(A)$.

Theorem 3.2. (a) *For any action (A, α), the assignments $I \mapsto I \rtimes_\alpha G$ and $I \mapsto I \rtimes_{\alpha,r} G$ define lattice isomorphisms of $\mathscr{I}_\alpha(A)$ onto $\mathscr{I}_{\hat\alpha}(A \rtimes_\alpha G)$ and $\mathscr{I}_{\hat\alpha^n}(A \rtimes_{\alpha,r} G)$, respectively.*
(b) *For any maximal or normal coaction (A, δ), the assignment $I \mapsto I \rtimes_\delta G$ defines a lattice isomorphism of $\mathscr{I}_\delta(A)$ onto $\mathscr{I}_{\hat\delta}(A \rtimes_\delta G)$.*

Proof. For the first part of (a), consider the *ladder diagram*

$$\begin{tikzcd} & \mathscr{I}_G(A \rtimes_\alpha G \rtimes_{\hat\alpha} G \rtimes_{\hat{\hat\alpha}} G) \\ \mathscr{I}_G(A \rtimes_\alpha G \rtimes_{\hat\alpha} G) \arrow[ur, "h"] & \\ & \mathscr{I}_G(A \rtimes_\alpha G) \arrow[ul, "g"'] \arrow[uu, "v"'] \\ \mathscr{I}_G(A) \arrow[uu, "u"] \arrow[ur, "f"] & \end{tikzcd}$$

(For simplicity here we're writing $\mathscr{I}_G$ for the lattice of appropriately-invariant ideals in each C^*-algebra.) The diagonal maps f, g, and h (the "rungs" of the ladder) are defined by Proposition 2.3; so for example $f(I) = I \rtimes_\alpha G$ for $I \in \mathscr{I}_\alpha(A)$, and $g(J) = J \times_{\hat\alpha} G$ for $J \in \mathscr{I}_{\hat\alpha}(A \rtimes_\alpha G)$. The vertical maps u and v come from the Rieffel correspondence using the imprimitivity bimodules implicit in the duality theorems 2.1 and 2.2; since those bimodules have suitably compatible actions and coactions (respectively), u and v are bijections by

Lemma 2.4, and they make the diagram commute by Proposition 2.5. It follows from Lemma 3.1 that all the maps in the diagram are bijections.

Now u and v are in fact lattice isomorphisms because they are bijective restrictions of lattice isomorphisms. Moreover, it's routine to check that any order-preserving bijection of lattices *whose inverse is also order-preserving* is a lattice isomorphism. Since here it's evident by construction that f and g are order-preserving, we see that $f^{-1} = u^{-1} \circ g$ is too, and it follows that f is a lattice isomorphism.

The other part of (a), and both parts of (b), are quite similar. □

4. Conclusion

We emphasize that our primary intent in this paper is to introduce the "ladder technique"; the bijections in Theorem 3.2, while significant in their own right, are to some extent intended to serve as illustrative examples of the technique. Theorem 3.2 itself is not completely new: for example, Gootman and Lazar proved a version for amenable G. Their results are clearly immediate corollaries of Theorem 3.2:

Theorem 4.1 ([3, Theorems 3.4 & 3.7])**.** *Assume that G is amenable.*

(a) *For any action (A, α) of G, an ideal J of $A \rtimes_\alpha G$ is $\hat{\alpha}$-invariant if and only if it is of the form $I \rtimes_\alpha G$ for an α-invariant ideal I of A. Moreover, I is uniquely determined.*

(b) *For any coaction (A, δ) of G, an ideal J of $A \rtimes_\delta G$ is $\hat{\delta}$-invariant if and only if it is of the form $I \rtimes_\delta G$ for a δ-invariant ideal I of A. Moreover, I is uniquely determined.*

The full-crossed-product part of Theorem 3.2 (a) has also appeared in [7, Theorem 8.2], where it is proved using different techniques (involving Landstad duality).

Nilsen has proved a pair of related results (for arbitrary locally compact G) which are *not* corollaries of Theorem 3.2; nevertheless, it seems relevant to include Nilsen's results here for comparison. One difference is that Nilsen used a different notion of coaction-invariance for ideals: if (A, δ) is a coaction, we say an ideal I of A is *weakly invariant* if δ passes to a coaction on the quotient A/I. This is properly weaker than the notion of invariance used in the current paper (although the two coincide for amenable G). More importantly, Nilsen's bijection was different from ours: she maps an ideal J in a coaction crossed-product $A \rtimes_\delta G$ to its *restriction* $j_A^{-1}(J) \subseteq A$, and similarly for an action and $i_A^{-1}(J)$. Thus, for example, given a (non-normal) coaction δ of G on A, Nilsen makes the (nonzero) kernel of the canonical homomorphism $j_A : A \to M(A \rtimes_\delta G)$ correspond to the zero ideal of $A \rtimes_\delta G$, which is *not* the way the bijection in Theorem 3.2 (b) works.

Theorem 4.2 ([9, Corollaries 3.2 and 3.4]). *Let G be a locally compact group.*

(a) *For any coaction (A, δ) of G, restriction gives a bijection between the $\hat{\hat{\delta}}$-invariant ideals of $A \rtimes_\delta G$ and the weakly δ-invariant ideals of A.*
(b) *For any action (A, α) of G, restriction gives a bijection between the weakly $\hat{\alpha}$-invariant ideals of $A \rtimes_\alpha G$ and the α-invariant ideals of A.*

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