

Jørgensen’s inequality for bicomplex Möbius groups

Zekun Li and Binlin Dai

ABSTRACT. In this paper, we investigate Möbius groups theory in the framework of bicomplex numbers, which are pairs of complex numbers making up a commutative ring with zero-divisors. We shall generalize classical Jørgensen’s inequality to bicomplex analysis and obtain an analogue of Jørgensen type inequality for discrete non-elementary Möbius groups of bicomplex setting generated by two elements.

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1. Introduction

Throughout this paper, we will adopt the same notations and definitions as in [7, 8, 22, 28] such as $\mathrm{PU}(2, 1)$, $\mathrm{PSL}(2, \mathbb{C})$, elementary groups, discrete groups, bicomplex Möbius transformations and so on. For example, the collection of all complex Möbius transformations for which $ad - bc$ takes the value 1 forms a group which can be identified with $\mathrm{PSL}(2, \mathbb{C})$.

In 1976, Jørgensen[28] obtained the following Jørgensen’ inequality, which is important in the study of the geometry of discrete groups.

Theorem 1.1. *Suppose that $f, g \in \mathrm{SL}(2, \mathbb{C})$ generate a group. If*

$$|\mathrm{tr}^2(f) - 4| + |\mathrm{tr}(fgf^{-1}g^{-1}) - 2| < 1. \quad (1)$$

Then the group $\langle f, g \rangle$ is either elementary or not discrete.

It is well known that to study Jørgensen’ inequality of Möbius transformation is very important and there has been an active research in this area. Gehring and Martin[24] discussed quantified forms of Jørgensen’ inequality for the norms

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$m(f)$, $d(f)$, $\varrho(f)$ and obtained the well known fact that $\text{axis}(f)$ and $\text{axis}(g)$ are disjoint whenever $\langle f, g \rangle$ is nonelementary. Alaqad, Gong and Martin [2] used certain shifted Chebyshev polynomials and trace identities to determine new families of Jørgensen type inequalities. In 1989, Jørgensen's inequality was generalized by Martin [39] to nonelementary discrete groups of Möbius transformations of any dimension $n > 2$. Waterman[44] proved a generalization of Jørgensen's inequality to Möbius transformations in high dimension by using Clifford algebras. In 1998, Basmajian[7] generalized Jørgensen's inequality to the two generator subgroup of $\text{PU}(2, 1)$ when the generators are loxodromic or boundary elliptic. Friedland and Hersonsky [23] extended Jørgensen's inequality to certain discrete nonelementary subgroups $\langle A, B \rangle$ of the symplectic group. In [14, 17, 29, 30, 40], Jørgensen's inequality was built in complex hyperbolic space. See [1, 16, 31, 42, 43, 44, 45, 46] etc. for more discussions about Jørgensen's inequality.

In recent years, more and more people try to generalize the classical results of complex analysis to bicomplex analysis. Luna-Elizarrará obtained some classical bicomplex analysis results, for example: bicomplex Riemann mapping theorem [36, Theorem 8.6.2], bicomplex Weierstrass' theorem [36, Theorem 10.2.3], bicomplex Abel's theorem [36, Theorem 10.4.1], bicomplex Cauchy integral theorem [36, Theorem 11.1.1], bicomplex Borel-Pompeiu formula [36, Theorem 11.2.2], bicomplex Cauchy integral representation [36, Theorem 11.2.3], bicomplex residue theorem [37] and bicomplex Laurent theorem [38]. Agarwal discussed bicomplex Maxwell's equations [3] and bicomplex Mittag-Leffler function [4]. Banerjee studied bicomplex Fourier transform [9], bicomplex Laplace transform [10], bicomplex indefinite inner product modules [11] and bicomplex harmonic and isotonic oscillators [12]. In addition, Emanuello [21] studied the Möbius transformation in $\mathbb{R}^{1,1}$ and discussed its conformality, transitivity, and fixed points. Golberg [25] defined the $\mathbb{D}$ -Möbius transformation in $\mathbb{D}^n$ and came to the conclusion that $\mathbb{D}$ -Möbius transformation can be expressed as a product of the six elementary Möbius transformations. In 2022, Dai[15] et al. studied a conjugacy classification according to the number of fixed points in $\text{SL}(2, \mathbb{BC})$, and detailedly prove that the cross-ratio is invariant under hyperbolic Möbius transformations. See [5, 6, 13, 33, 34] etc. for more details of bicomplex analysis.

In this paper, motivated by these developments, we shall give an analogue of Jørgensen type inequality for discrete non-elementary Möbius groups of bicomplex setting generated by two elements. Our main result is the following theorem.

Theorem 1.2. *Let $f, g \in \text{Aut}(\overline{\mathbb{BC}})$ generate a group. If*

$$|\text{tr}^2(f) - 4|_{\mathbf{k}} + |\text{tr}(fgf^{-1}g^{-1}) - 2|_{\mathbf{k}} < 1, \quad (2)$$

then the group $\langle f, g \rangle$ is either elementary or not discrete.

The structure of the paper is as follows. Section 2 preliminaries present the basic and necessary concepts and results about the bicomplex numbers. In Section 3, we give the necessary background material for bicomplex Möbius transformations: Section 3.1 describes the fixed point of bicomplex Möbius transformations; Section 3.2 give necessary condition for a bicomplex Möbius group to be non-elementary. In Section 3.3, we present the geometric representations of bicomplex Möbius transformations. In Section 4 we obtain the proof of Theorem 1.2.

2. Preliminaries

2.1. Bicomplex and hyperbolic numbers. The commutative ring $\mathbb{BC}$ of bicomplex numbers is defined as

$$\mathbb{BC} := \{Z = z_1 + \mathbf{j}z_2 \mid z_1, z_2 \in \mathbb{C}(\mathbf{i})\}.$$

Here $\mathbb{C}(\mathbf{i})$ and $\mathbb{C}(\mathbf{j})$ are different commutative imaginary units, that is

$$\mathbf{i} \neq \mathbf{j}, \quad \mathbf{ij} = \mathbf{ji} = \mathbf{k}, \quad \mathbf{i}^2 = \mathbf{j}^2 = -1,$$

and $\mathbb{C}(\mathbf{i})$ is the set of complex numbers with imaginary unit $\mathbf{i}$. If $z_1 = x_1$ is a real number and $z_2 = y_2\mathbf{i}$ is a purely imaginary complex number, then $Z = x_1 + y_2\mathbf{k}$ is an element of the ring of hyperbolic numbers $\mathbb{D}$.

$$\mathbb{D} := \{x + y\mathbf{k} \mid x, y \in \mathbb{R}\}$$

Both the ring $\mathbb{BC}$ and the ring $\mathbb{D}$ have zero-divisors. In the case of bicomplex numbers, the set of zero-divisors is

$$\mathfrak{S} := \{Z \mid Z \neq 0, z_1^2 + z_2^2 = 0\}$$

and by definition

$$\mathfrak{S}_0 := \mathfrak{S} \cup \{0\}.$$

There are two special zero-divisors $\mathbf{e} := \frac{1+\mathbf{k}}{2}$ and $\mathbf{e}^\dagger := \frac{1-\mathbf{k}}{2}$, which have the properties:

$$\begin{aligned} \mathbf{e} + \mathbf{e}^\dagger &= 1; \quad \mathbf{e} - \mathbf{e}^\dagger = \mathbf{k}; \\ \mathbf{ee}^\dagger &= 0; \quad \mathbf{ee} = \mathbf{e}; \quad \text{and} \quad \mathbf{e}^\dagger\mathbf{e}^\dagger = \mathbf{e}^\dagger. \end{aligned} \tag{3}$$

Consider the sets:

$$\mathbb{BC}_{\mathbf{e}} = \{\beta_1\mathbf{e} \mid \beta_1 \in \mathbb{C}(\mathbf{i})\} \quad \text{and} \quad \mathbb{BC}_{\mathbf{e}^\dagger} = \{\beta_2\mathbf{e}^\dagger \mid \beta_2 \in \mathbb{C}(\mathbf{i})\}.$$

Obviously, the set of zero-divisors in $\mathbb{BC}$ is given by

$$\mathfrak{S}_0 = \mathbb{BC}_{\mathbf{e}} \cup \mathbb{BC}_{\mathbf{e}^\dagger}.$$

Each bicomplex number $Z = z_1 + \mathbf{j}z_2$ can be written as

$$Z = (z_1 - z_2\mathbf{i})\mathbf{e} + (z_1 + z_2\mathbf{i})\mathbf{e}^\dagger = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \quad \text{where } \beta_1, \beta_2 \in \mathbb{C}(\mathbf{i}).$$

The following conjugations were introduced on bicomplex numbers [4, 6, 33, 34].

(i) $\overline{Z} := \overline{z_1} + \overline{z_2}\mathbf{j}$ (the bar-conjugation);

(ii) $Z^\dagger := z_1 - z_2 \mathbf{j}$ (the $\dagger$ -conjugation);

(iii) $Z^* := (\bar{Z})^\dagger = \bar{z}_1 - \bar{z}_2 \mathbf{j}$ (the $*$ -conjugation).

Besides, the above conjugations suggest to consider the three moduli for bicomplex numbers:

(i) $|Z|_{\mathbf{i}}^2 := Z \cdot Z^\dagger = z_1^2 + z_2^2 \in \mathbb{C}(\mathbf{i})$;

(ii) $|Z|_{\mathbf{j}}^2 := Z \cdot \bar{Z} = \eta_1^2 + \eta_2^2 \in \mathbb{C}(\mathbf{j})$, where $Z = \eta_1 + \eta_2 \mathbf{i} = (x_1 + x_2 \mathbf{j}) + (y_1 + y_2 \mathbf{j}) \mathbf{i}$;

(iii) $|Z|_{\mathbf{k}}^2 := Z \cdot Z^* = |\beta_1|^2 \mathbf{e} + |\beta_2|^2 \mathbf{e}^\dagger \in \mathbb{D}^+$, where the set $\mathbb{D}^+$ can be described as $\mathbb{D}^+ = \{v\mathbf{e} + \mu\mathbf{e}^\dagger \mid v, \mu \geq 0\}$.

In [6, 36, 38], the $\mathbf{k}$ -modulus $|\cdot|_{\mathbf{k}}$ is defined by

$$|Z|_{\mathbf{k}} = |\beta_1| \mathbf{e} + |\beta_2| \mathbf{e}^\dagger.$$

The following partial order on hyperbolic numbers, introduced by Luna-Elizarrarás in [35, 36, 37].

Given hyperbolic numbers $\mathfrak{z}$ and $\mathfrak{w}$, we say that $\mathfrak{z} \leq \mathfrak{w}$ if $\mathfrak{w} - \mathfrak{z} \in \mathbb{D}^+$. When $\mathfrak{w} - \mathfrak{z} \in \mathbb{D}^+ \setminus \{0\}$, we write $\mathfrak{z} < \mathfrak{w}$.

The properties of $\mathbf{k}$ -modulus are as follows:

(i) $|Z|_{\mathbf{k}} = 0$ if and only if $Z = 0$.

(ii) it satisfies the multiplicative property:

$$|ZW|_{\mathbf{k}} = |Z|_{\mathbf{k}} \cdot |W|_{\mathbf{k}}$$

(iii) the $\mathbf{k}$ -modulus satisfies the triangle inequality:

$$|Z + W|_{\mathbf{k}} \leq |Z|_{\mathbf{k}} + |W|_{\mathbf{k}}.$$

2.2. $\mathbf{k}$ -modulus and convergence. We consider a convergence on bicomplex numbers in the following form.

Definition 2.1. [38, Definition 2.1] Let $\{Z_n\}_{n=1}^\infty$ be a sequence of bicomplex numbers, we say that Z_n converges in the hyperbolic modulus to $Z_0 \in \mathbb{BC}$ if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|Z_n - Z_0|_{\mathbf{k}} < \varepsilon$$

for any $n \geq N$.

In this case we say that Z_0 is the limit of the sequence which we write as

$$\lim_{n \rightarrow \infty} Z_n = Z_0.$$

Definition 2.2. [38, Proposition 2.2] Let $\{Z_n\}_{n=1}^\infty$ be a sequence of bicomplex numbers, then Z_n converges to Z_0 if and only if the complex sequence $\{\beta_{1,n}\}_{n=1}^\infty$ converges to $\beta_{1,0}$ and the complex sequence $\{\beta_{2,n}\}_{n=1}^\infty$ converges to $\beta_{2,0}$ where $Z_n = \beta_{1,n} \mathbf{e} + \beta_{2,n} \mathbf{e}^\dagger$ and $Z_0 = \beta_{1,0} \mathbf{e} + \beta_{2,0} \mathbf{e}^\dagger$. This type of convergence can be introduced for functions.

Remark 2.3. If the complex sequence $\{\beta_{1,n}\}_{n=1}^\infty$ converges to ∞ and the complex sequence $\{\beta_{2,n}\}_{n=1}^\infty$ converges to $\beta_{2,0}$, then $\lim_{n \rightarrow \infty} Z_n = \infty \mathbf{e} + \beta_{2,0} \mathbf{e}^\dagger$. If the complex

sequence $\{\beta_{1,n}\}_{n=1}^{\infty}$ converges to $\beta_{1,0}$ and the complex sequence $\{\beta_{2,n}\}_{n=1}^{\infty}$ converges to ∞ , then $\lim_{n \rightarrow \infty} Z_n = \beta_{1,0}\mathbf{e} + \infty\mathbf{e}^\dagger$. If the complex sequence $\{\beta_{1,n}\}_{n=1}^{\infty}$ converges to ∞ and the complex sequence $\{\beta_{2,n}\}_{n=1}^{\infty}$ converges to ∞ , then $\lim_{n \rightarrow \infty} Z_n = \infty\mathbf{e} + \infty\mathbf{e}^\dagger$.

2.3. Sets of extended bicomplex numbers. The extended set of bicomplex numbers was introduced in [38]. The extended set of bicomplex numbers is

$$\overline{\mathbb{BC}} := \overline{\mathbb{C}(\mathbf{i})}\mathbf{e} + \overline{\mathbb{C}(\mathbf{i})}\mathbf{e}^\dagger$$

where $\overline{\mathbb{C}(\mathbf{i})}$ is the well-known compactification of the $\mathbb{C}(\mathbf{i})$ -complex plane.

Remark 2.4. Similar definition as $\overline{\mathbb{C}(\mathbf{i})} = \mathbb{C}(\mathbf{i}) \cup \{\infty\}$, we have

$$\overline{\mathbb{BC}} = \mathbb{BC} \cup \{\mathbb{C}(\mathbf{i})\mathbf{e} + \infty\mathbf{e}^\dagger\} \cup \{\infty\mathbf{e} + \mathbb{C}(\mathbf{i})\mathbf{e}^\dagger\} \cup \{\infty\mathbf{e} + \infty\mathbf{e}^\dagger\}.$$

3. Bicomplex Möbius transformation

It is well known that the Möbius transformation plays an important role in the study of complex analysis and there are many further discussions. Emanuello [21] studied the Möbius transformation in $\mathbb{R}^{1,1}$ and discussed its conformality, transitivity, and fixed points. Golberg [25] defined the $\mathbb{D}$ -Möbius transformation in $\mathbb{D}^n$ and came to the conclusion that $\mathbb{D}$ -Möbius transformation can be expressed as a product of the six elementary Möbius transformations.

The bicomplex Möbius transformation [15, 18, 22] $f(Z)$ is denoted by :

$$f(Z) = \frac{AZ + B}{CZ + D} \quad (4)$$

where $A, B, C, D \in \mathbb{BC}$ and $AD - BC \notin \mathfrak{S}_0$. Let $Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger$ and

$$\begin{aligned} A &= a_1\mathbf{e} + a_2\mathbf{e}^\dagger & B &= b_1\mathbf{e} + b_2\mathbf{e}^\dagger \\ C &= c_1\mathbf{e} + c_2\mathbf{e}^\dagger & D &= d_1\mathbf{e} + d_2\mathbf{e}^\dagger. \end{aligned}$$

We have

$$f(Z) = \frac{a_1\beta_1 + b_1}{c_1\beta_1 + d_1}\mathbf{e} + \frac{a_2\beta_2 + b_2}{c_2\beta_2 + d_2}\mathbf{e}^\dagger = f_1(\beta_1)\mathbf{e} + f_2(\beta_2)\mathbf{e}^\dagger, \quad (5)$$

where complex Möbius transformations f_1 and f_2 are called components of f . In [15, 22], $\text{GL}(2, \mathbb{BC})$ was defined by

$$\text{GL}(2, \mathbb{BC}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, B, C, D \in \mathbb{BC}, AD - BC \notin \mathfrak{S}_0 \right\}.$$

The collection of all bicomplex Möbius transformation for which $AD - BC$ takes the value 1 forms a group which can be identified with $\text{PSL}(2, \mathbb{BC})$.

We can represent $f(Z)$ via a matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}(2, \mathbb{BC}). \quad (6)$$

Definition 3.1. [22, Corollary 6.2.6] *The group of bicomplex Möbius transformations acting in $\overline{\mathbb{BC}}$ is called the General Möbius group and is denoted by $\text{Aut}(\overline{\mathbb{BC}})$.*

Remark 3.2. [22, Corollary 6.2.6] *The group of $\text{Aut}(\overline{\mathbb{BC}})$ is the collection of bicomplex Möbius transformations*

$$f(Z) = \frac{AZ + B}{CZ + D}.$$

Lemma 3.3. [32, Theorem 3.1, Theorem 3.3] *Let $\mathcal{A} = \mathcal{A}_1\mathbf{e} + \mathcal{A}_2\mathbf{e}^\dagger$ be any bicomplex matrix in $\text{GL}(2, \mathbb{BC})$. Then*

$$\det \mathcal{A} = \det \mathcal{A}_1\mathbf{e} + \det \mathcal{A}_2\mathbf{e}^\dagger \quad \det \mathcal{A}^{-1} = \det \mathcal{A}_1^{-1}\mathbf{e} + \det \mathcal{A}_2^{-1}\mathbf{e}^\dagger \quad (7)$$

$\det \mathcal{A} = AD - BC$ is called the determinant of $\mathcal{A}$.

By Lemma 3.3, for any $\mathcal{A} \in \text{PSL}(2, \mathbb{BC})$, we have

$$\det \mathcal{A}_1 = \det \mathcal{A}_2 = 1.$$

In order to introduce the following important properties, we need to generalize the trace of the matrix to $\text{GL}(2, \mathbb{BC})$.

Definition 3.4. [15, 21] *Let $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}(2, \mathbb{BC})$, the trace of $\mathcal{A}$ is defined by*

$$\text{tr} \mathcal{A} = A + D. \quad (8)$$

Naturally, $\text{tr}^2(f)$ is defined by

$$\text{tr}^2(f) = \frac{\text{tr}^2(\mathcal{A})}{\det(\mathcal{A})}, \quad \mathcal{A} \in \text{GL}(2, \mathbb{BC}), \quad (9)$$

where $\mathcal{A}$ is any matrix which projects to f . According to (3) and (5), for any $\mathcal{A} \in \text{PSL}(2, \mathbb{BC})$, we have

$$\begin{aligned} \text{tr}^2(f) &= [(a_1 + d_1)\mathbf{e} + (a_2 + d_2)\mathbf{e}^\dagger]^2 \\ &= (a_1 + d_1)^2\mathbf{e} + (a_2 + d_2)^2\mathbf{e}^\dagger \\ &= \text{tr}^2(f_1)\mathbf{e} + \text{tr}^2(f_2)\mathbf{e}^\dagger. \end{aligned}$$

The transitivity of the bicomplex Möbius transformations of $\text{Aut}(\overline{\mathbb{BC}})$ are only one transitive. It is not two transitive in general.

Lemma 3.5. [21, 22] *The bicomplex Möbius transformations are transitive, but not 2-transitive.*

To see that bicomplex Möbius group is not two-transitive, we give a proof by counterexample([15, Chapter 4]). We choose four different points

$$Z_1 = \mathbf{ie}, Z_2 = 2\mathbf{ie}, Z_3 = \mathbf{ie}^\dagger \text{ and } Z_4 = 3\mathbf{i}$$

in $\mathbb{BC}$ to form two sets $\{Z_1, Z_2\}$ and $\{Z_3, Z_4\}$. We choose f that satisfies

$$f(Z_1) = Z_3 \quad f(Z_2) = Z_4,$$

the component f_2 of f has to satisfy both $f_2(0) = \mathbf{i}$ and $f_2(0) = \mathbf{j}$. It is a contradiction.

Based on [15, Definition 2] and [22, Proposition 6.2.5], we have the following definition.

Definition 3.6. Suppose that f and g are two different bicomplex Möbius transformations in $\text{Aut}(\overline{\mathbb{BC}})$, f and g are conjugate, denoted as

$$f \sim g, \quad (10)$$

if and only if there exists bicomplex Möbius transformations h in $\text{Aut}(\overline{\mathbb{BC}})$ such that

$$f = hgh^{-1}.$$

Remark 3.7. A bicomplex Möbius transformation $f(Z) = f_1(\beta_1)\mathbf{e} + f_2(\beta_2)\mathbf{e}^\dagger$ is conjugate to a bicomplex Möbius transformation $g(Z) = g_1(\beta_1)\mathbf{e} + g_2(\beta_2)\mathbf{e}^\dagger$ if and only if there exists bicomplex Möbius transformations $h(Z) = h_1(\beta_1)\mathbf{e} + h_2(\beta_2)\mathbf{e}^\dagger$ such that $g(Z) = hfh^{-1}(Z)$. For any bicomplex Möbius transformation $h(Z) = h_1(\beta_1)\mathbf{e} + h_2(\beta_2)\mathbf{e}^\dagger$, by (5), we have

$$hfh^{-1}(Z) = h_1f_1h_1^{-1}(\beta_1)\mathbf{e} + h_2f_2h_2^{-1}(\beta_2)\mathbf{e}^\dagger. \quad (11)$$

The following lemmas, though rather trivial, are useful.

Lemma 3.8. [15, Theorem 2] Let $f(Z) = f_1(\beta_1)\mathbf{e} + f_2(\beta_2)\mathbf{e}^\dagger$ and $g(Z) = g_1(\beta_1)\mathbf{e} + g_2(\beta_2)\mathbf{e}^\dagger$ in $\text{Aut}(\overline{\mathbb{BC}})$, $f \sim g$ if and only if both $f_1 \sim g_1$ and $f_2 \sim g_2$.

Lemma 3.9. [15, Theorem 3] Suppose that f and g are two different bicomplex Möbius transformations in $\text{Aut}(\overline{\mathbb{BC}})$ where neither component is the identity. Then $f \sim g$ if and only if $\text{tr}^2(f) = \text{tr}^2(g)$.

Remark 3.10. It is not hard to conclude that the trace of bicomplex Möbius transformations is invariant under conjugate transformations.

3.1. Fixed Points.

Definition 3.11. [18, Definition 14] Let f be any bicomplex Möbius transformations in $\text{Aut}(\overline{\mathbb{BC}})$, then $Z \in \overline{\mathbb{BC}}$ is called a fixed point of f if and only if $f(Z) = Z$.

Example 3.12. (i) the mapping $Z + 1$ has one fixed point $\infty\mathbf{e} + \infty\mathbf{e}^\dagger$.
(ii) the mapping $\frac{1}{\beta_1}\mathbf{e} + (\beta_2 + 1)\mathbf{e}^\dagger$ has two fixed points, $\mathbf{e} + \infty\mathbf{e}^\dagger$ and $-\mathbf{e} + \infty\mathbf{e}^\dagger$.
(iii) the mapping $1/Z$ has four fixed points, $\pm 1, \pm \mathbf{k}$.

Remark 3.13. If a point $Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \in \overline{\mathbb{BC}}$ satisfies $f(Z) = Z$, Z is called a fixed point of f . Further,

$$f(Z) = f_1(\beta_1)\mathbf{e} + f_2(\beta_2)\mathbf{e}^\dagger = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger.$$

So Z is a fixed point of f if and only if

$$f_1(\beta_1) = \beta_1, f_2(\beta_2) = \beta_2.$$

Pogorui [41] studied the roots of polynomials with bicomplex coefficients, by Pogorui's bicomplex polynomial theory, Chinmay [18] obtained the fixed points of bicomplex Möbius transformations.

Lemma 3.14. [18, Theorem 4] *Let f be a bicomplex Möbius transformation where neither component is the identity. Then f has either one fixed point, two fixed points or four fixed points.*

The following lemma is crucial for us.

Lemma 3.15. [22, Corollary, 6.3.2] *Let $f(Z) = f_1(\beta_1)\mathbf{e} + f_2(\beta_2)\mathbf{e}^\dagger$ be a bicomplex Möbius transformation where neither component is identity. Then*

- (i) $f(Z)$ has precisely one fixed point if and only if $f(Z)$ is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
- (ii) $f(Z)$ has precisely two fixed points if and only if $f(Z)$ is conjugate to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{e} + \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}\mathbf{e}^\dagger \quad \text{or} \quad \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}\mathbf{e} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{e}^\dagger,$$

where $\lambda \neq 1$.

- (iii) $f(Z)$ has precisely four fixed points if and only if $f(Z)$ is conjugate to

$$\begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}\mathbf{e} + \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}\mathbf{e}^\dagger,$$

where $u \neq 1, \lambda \neq 1$.

3.2. The elementary group.

Definition 3.16. [25, Definition,2] *The hyperbolic-valued function*

$$\|\cdot\|_{\mathbb{D}} : \text{GL}(2, \mathbb{BC}) \rightarrow \mathbb{D}^+,$$

defined as follows:

$$\|\mathcal{A}\|_{\mathbb{D}} := \|\mathcal{A}_1\|\mathbf{e} + \|\mathcal{A}_2\|\mathbf{e}^\dagger, \quad (12)$$

is called a hyperbolic-norm in $\text{GL}(2, \mathbb{BC})$.

The norm $\|\mathcal{A}_i\|$ ($i = 1, 2$) is given by

$$\|\mathcal{A}_i\| = (|a_i|^2 + |b_i|^2 + |c_i|^2 + |d_i|^2)^{1/2} (i = 1, 2).$$

A subgroup G of $\text{GL}(2, \mathbb{C})$ is discrete if one point of G is isolated:

$$\inf \{\|X - I\| : X \in G, X \neq I\} > 0.$$

Similar definition as [8, Chapter 2.3], a subgroup G of $\text{GL}(2, \mathbb{BC})$ is discrete if one point of G is isolated:

Definition 3.17. *A subgroup G of $\text{GL}(2, \mathbb{BC})$ is discrete if one point of G is isolated.*

$$\inf \{\|X - I\|_{\mathbb{D}} : X \in G, X \neq I\} > 0, \quad (13)$$

In terms of sequences, G is discrete if and only if $\mathcal{A}_n \rightarrow I$ and $\mathcal{A}_n \in G$ implies that $\mathcal{A}_n = I$ for almost all n .

Let $X = X_1 \mathbf{e} + X_2 \mathbf{e}^\dagger$, by (12), we have

$$\|X - I\|_{\mathbb{D}} = \|X_1 - I\| \mathbf{e} + \|X_2 - I\| \mathbf{e}^\dagger. \quad (14)$$

So

$$\begin{aligned} & \inf \{\|X - I\|_{\mathbb{D}} : X \in G, X \neq I\} \\ &= \inf \{\|X_1 - I\| : X_1 \in \text{GL}(2, \mathbb{C}), X_1 \neq I\} \mathbf{e} \\ &+ \inf \{\|X_2 - I\| : X_2 \in \text{GL}(2, \mathbb{C}), X_2 \neq I\} \mathbf{e}^\dagger. \end{aligned}$$

It is well known that elementary groups have been defined in [8, Definition 5.1.1]. We shall generalize Beardon's definition to bicomplex numbers and obtain the following definition.

Definition 3.18. A subgroup G of $\text{GL}(2, \mathbb{BC})$ is called elementary if there exists a finite orbit in $\overline{\mathbb{BC}}$.

Remark 3.19. A bicomplex Möbius transformation $f(Z)$ is said to be finite order if there exists $n \in \mathbb{N}$ such that $f(Z)^n = I$.

Let G be an elementary group and suppose that the finite orbit is $\{Z_1, Z_2, \dots, Z_n\}$. If f is in G then there exist an integer m such that f^m fixes Z_j , $j = 1, 2, \dots, n$. With this available we can now classify the elementary groups into four types.

Type 1 : suppose that $n = 1$ and Z_1 is in $\overline{\mathbb{BC}}$.

In this case, G is conjugate to a group, every element of which fixes $\infty \mathbf{e} + \infty \mathbf{e}^\dagger$. According to Lemma 3.5, the bicomplex transformations are transitive. So the conclusion is obvious.

Type 2 : suppose that $n = 2$ and Z_1, Z_2 are in $\overline{\mathbb{BC}}$.

In this case, G is conjugate to a group, every element of which leaves $\{0, \infty \mathbf{e}\}$ or $\{0, \infty \mathbf{e}^\dagger\}$ invariant.

There exist $f \in G$ such that $f(Z_1) = Z_1$ and $f(Z_2) = Z_2$. By Lemma 3.15, we have f_1 has two fixed points or f_2 has two fixed points. Without loss of generality, suppose that f_1 has one fixed point and f_2 has two fixed points. Let $Z_1 = \beta_{1,1} \mathbf{e} + \beta_{2,1} \mathbf{e}^\dagger$, $Z_2 = \beta_{1,1} \mathbf{e} + \beta_{2,2} \mathbf{e}^\dagger$, for any $g \in G$.

(i) $g(Z_1) = Z_1$ and $g(Z_2) = Z_2$.

$$g(Z_1) = g(\beta_{1,1} \mathbf{e} + \beta_{2,1} \mathbf{e}^\dagger) = g_1(\beta_{1,1}) \mathbf{e} + g_2(\beta_{2,1}) \mathbf{e}^\dagger = \beta_{1,1} \mathbf{e} + \beta_{2,1} \mathbf{e}^\dagger = Z_1$$

$$g(Z_2) = g(\beta_{1,1} \mathbf{e} + \beta_{2,2} \mathbf{e}^\dagger) = g_1(\beta_{1,1}) \mathbf{e} + g_2(\beta_{2,2}) \mathbf{e}^\dagger = \beta_{1,1} \mathbf{e} + \beta_{2,2} \mathbf{e}^\dagger = Z_2.$$

By [8, Chapter 4.3], there exists complex Möbius transformations

$$h_1(\beta_{1,1}) = 0, h_2(\beta_{2,1}) = 0, h_2(\beta_{2,2}) = \infty,$$

such that

$$h_1 g_1 h_1^{-1}(0) = 0, h_2 g_2 h_2^{-1}(0) = 0, h_2 g_2 h_2^{-1}(\infty) = \infty.$$

Let $h(Z) = h_1(\beta_1) \mathbf{e} + h_2(\beta_2) \mathbf{e}^\dagger$, we have

$$\begin{aligned} hgh^{-1}(0) &= h_1g_1h_1^{-1}(0)\mathbf{e} + h_2g_2h_2^{-1}(0)\mathbf{e}^\dagger = 0 \\ hgh^{-1}(\infty\mathbf{e}^\dagger) &= h_1g_1h_1^{-1}(\infty)\mathbf{e} + h_2g_2h_2^{-1}(\infty)\mathbf{e}^\dagger = \infty\mathbf{e}^\dagger. \end{aligned}$$

(ii) $g(Z_1) = Z_2$ and $g(Z_2) = Z_1$.

$$\begin{aligned} g(Z_1) &= f(\beta_{1,1}\mathbf{e} + \beta_{2,1}\mathbf{e}^\dagger) = g_1(\beta_{1,1})\mathbf{e} + g_2(\beta_{2,1})\mathbf{e}^\dagger = \beta_{1,1}\mathbf{e} + \beta_{2,2}\mathbf{e}^\dagger = Z_2 \\ g(Z_2) &= g(\beta_{1,1}\mathbf{e} + \beta_{2,2}\mathbf{e}^\dagger) = g_1(\beta_{1,1})\mathbf{e} + g_2(\beta_{2,2})\mathbf{e}^\dagger = \beta_{1,1}\mathbf{e} + \beta_{2,1}\mathbf{e}^\dagger = Z_1. \end{aligned}$$

By [8, Chapter 4.3], there exists complex Möbius transformations

$$h_1(\beta_{1,1}) = 0, h_2(\beta_{2,1}) = 0, h_2(\beta_{2,2}) = \infty,$$

such that

$$h_1g_1h_1^{-1}(0) = 0, h_2g_2h_2^{-1}(0) = \infty, h_2g_2h_2^{-1}(\infty) = 0.$$

Let $h(Z) = h_1(\beta_1)\mathbf{e} + h_2(\beta_2)\mathbf{e}^\dagger$, we have

$$\begin{aligned} hgh^{-1}(0) &= h_1g_1h_1^{-1}(0)\mathbf{e} + h_2g_2h_2^{-1}(0)\mathbf{e}^\dagger = \infty\mathbf{e}^\dagger \\ hgh^{-1}(\infty\mathbf{e}) &= h_1g_1h_1^{-1}(\infty)\mathbf{e} + h_2g_2h_2^{-1}(\infty)\mathbf{e}^\dagger = 0. \end{aligned}$$

This shows that G is conjugate to a group, every element of which leaves $\{0, \infty\mathbf{e}^\dagger\}$ invariant.

Similarly, if f_1 has two fixed points and f_2 has one fixed point, we have G is conjugate to a group, every element of which leaves $\{0, \infty\mathbf{e}\}$ invariant.

Type 3 : suppose that $n = 4$ and Z_1, Z_2, Z_3, Z_4 are in $\overline{\mathbb{BC}}$.

In this case, G is conjugate to a group, every element of which leaves

$$\{0, \infty\mathbf{e}, \infty\mathbf{e}^\dagger, \infty\mathbf{e} + \infty\mathbf{e}^\dagger\}$$

invariant.

There exist $f \in G$ such that $f(Z_i) = Z_i, i = 1, 2, 3, 4$. By Lemma 3.15, we have f_1 has two fixed points and f_2 has two fixed points. Let $Z_1 = \beta_{1,1}\mathbf{e} + \beta_{2,1}\mathbf{e}^\dagger, Z_2 = \beta_{1,1}\mathbf{e} + \beta_{2,2}\mathbf{e}^\dagger, Z_3 = \beta_{1,2}\mathbf{e} + \beta_{2,1}\mathbf{e}^\dagger, Z_4 = \beta_{1,2}\mathbf{e} + \beta_{2,2}\mathbf{e}^\dagger$.

By [8, Chapter 4.3], there exists complex Möbius transformations

$$\begin{aligned} h_1(\beta_{1,1}) &= 0, & h_1(\beta_{1,2}) &= \infty \\ h_2(\beta_{2,1}) &= 0, & h_2(\beta_{2,2}) &= \infty, \end{aligned}$$

such that

$$\begin{aligned} h_1f_1h_1^{-1}(0) &= 0, & h_1f_1h_1^{-1}(\infty) &= \infty \\ h_2f_2h_2^{-1}(0) &= 0, & h_2f_2h_2^{-1}(\infty) &= \infty. \end{aligned}$$

Let $h(Z) = h_1(\beta_1)\mathbf{e} + h_2(\beta_2)\mathbf{e}^\dagger$, we have

$$\begin{aligned} hfh^{-1}(0) &= h_1f_1h_1^{-1}(0)\mathbf{e} + h_2f_2h_2^{-1}(0)\mathbf{e}^\dagger = 0 \\ hfh^{-1}(\infty\mathbf{e}) &= h_1f_1h_1^{-1}(\infty)\mathbf{e} + h_2f_2h_2^{-1}(0)\mathbf{e}^\dagger = \infty\mathbf{e} \\ hfh^{-1}(\infty\mathbf{e}^\dagger) &= h_1f_1h_1^{-1}(0)\mathbf{e} + h_2f_2h_2^{-1}(\infty)\mathbf{e}^\dagger = \infty\mathbf{e}^\dagger \\ hfh^{-1}(\infty\mathbf{e} + \infty\mathbf{e}^\dagger) &= h_1f_1h_1^{-1}(\infty)\mathbf{e} + h_2f_2h_2^{-1}(\infty)\mathbf{e}^\dagger = \infty\mathbf{e} + \infty\mathbf{e}^\dagger \end{aligned}$$

Similar discussion as type 2, for any $f \in G$, there exists h such that hfh^{-1} leaves $\{0, \infty\mathbf{e}, \infty\mathbf{e}^\dagger, \infty\mathbf{e} + \infty\mathbf{e}^\dagger\}$ invariant.

Type 4: suppose that $n = 3$ or $n \geq 5$.

If $n = 3$, by Lemma 3.14, f^m has three fixed points and so is identity: thus each non-trivial element of G is finite order.

Similarly, for $n \geq 5$, each non-trivial element of G is finite order.

Finally, we shall study the elementary bicomplex Möbius groups and give necessary condition for a group to be non-elementary. Let $\text{fix}(f)$ be the set of fixed points of f , we have

Theorem 3.20. *Let $f(Z) = f_1(\beta_1)\mathbf{e} + f_2(\beta_2)\mathbf{e}^\dagger$ be a bicomplex Möbius transformation and f_1, f_2 are not of order two. Define the map $\Theta(g) = gfg^{-1}$. If for some n , we have $\Theta^n(g) = f$, then $\langle f, g \rangle$ is elementary.*

Proof. Defined $g_0 = g = g_{0,1}\mathbf{e} + g_{0,2}\mathbf{e}^\dagger$ and $g_n = \Theta^n(g)$ so for $m \geq 0$,

$$g_{m+1} = g_m f(g_m)^{-1} = g_{m,1}f_1(g_{m,1})^{-1}\mathbf{e} + g_{m,2}f_2(g_{m,2})^{-1}\mathbf{e}^\dagger. \quad (15)$$

Case 1. f has exactly one fixed point. According to Lemma 3.15, let

$$f(Z) = (\beta_1 + 1)\mathbf{e} + (\beta_2 + 1)\mathbf{e}^\dagger.$$

Since $g_1, g_2, \dots, g_n$ are conjugate to f , then g_i has a unique fixed point ($i = 1, 2, \dots, n$). Since

$$g_{n+1}(\infty\mathbf{e} + \infty\mathbf{e}^\dagger) = \infty\mathbf{e} + \infty\mathbf{e}^\dagger \quad (16)$$

and

$$g_{n+1}(g_n(\infty\mathbf{e} + \infty\mathbf{e}^\dagger)) = \infty\mathbf{e} + \infty\mathbf{e}^\dagger, \quad (17)$$

we have

$$g_n(\infty\mathbf{e} + \infty\mathbf{e}^\dagger) = \infty\mathbf{e} + \infty\mathbf{e}^\dagger. \quad (18)$$

It follows that $g_1, g_2, \dots, g_n$ each fix $\infty\mathbf{e} + \infty\mathbf{e}^\dagger$. Thus $\langle f, g \rangle$ is elementary.

Case 2. f has exactly two fixed points. We may assume that

$$f(Z) = (\beta_1 + 1)\mathbf{e} + k\beta_2\mathbf{e}^\dagger (k \neq 0, 1).$$

So $g_1, g_2, \dots, g_n$ each have exactly two fixed points. Since

$$g_{n+1}(\infty\mathbf{e} + \infty\mathbf{e}^\dagger) = \infty\mathbf{e} + \infty\mathbf{e}^\dagger \text{ and } g_{n+1}(\infty\mathbf{e}) = \infty\mathbf{e}.$$

Then

$$\text{fix}(g_{n+1}) = \{\infty\mathbf{e} + \infty\mathbf{e}^\dagger, \infty\mathbf{e}\}.$$

Thus

$$\{\infty\mathbf{e} + \infty\mathbf{e}^\dagger, \infty\mathbf{e}\} = \{g_{n,1}(\infty)\mathbf{e} + g_{n,2}(\infty)\mathbf{e}^\dagger, g_{n,1}(\infty)\mathbf{e} + g_{n,2}(0)\mathbf{e}^\dagger\}.$$

If

$$\begin{aligned} g_{n,1}(\infty)\mathbf{e} + g_{n,2}(\infty)\mathbf{e}^\dagger &= \infty\mathbf{e} \\ g_{n,1}(\infty)\mathbf{e} + g_{n,2}(0)\mathbf{e}^\dagger &= \infty\mathbf{e} + \infty\mathbf{e}^\dagger, \end{aligned}$$

we have

$$g_{n,1}(\infty) = \infty, g_{n,2}(\infty) = 0, g_{n,2}(0) = \infty.$$

Since

$$\begin{aligned} \text{fix}(g_{n,2}) &= \{g_{n-1,2}(0), g_{n-1,2}(\infty)\} \\ g_{n-1,2}(0) &\neq 0, \infty \text{ and } g_{n-1,2}(\infty) \neq 0, \infty. \end{aligned}$$

We have $g_{n,2}$, and hence f_2 , is of order two. It is a contradiction.

Thus

$$\begin{aligned} g_{n,1}(\infty)\mathbf{e} + g_{n,2}(\infty)\mathbf{e}^\dagger &= \infty\mathbf{e} + \infty\mathbf{e}^\dagger \\ g_{n,1}(\infty)\mathbf{e} + g_{n,2}(0)\mathbf{e}^\dagger &= \infty\mathbf{e}. \end{aligned}$$

It follows that $g_1, g_2, \dots, g_n$ each fix $\infty\mathbf{e}$ and $\infty\mathbf{e} + \infty\mathbf{e}^\dagger$. So g leave $\{\infty\mathbf{e} + \infty\mathbf{e}^\dagger, \infty\mathbf{e}\}$ invariant, $\langle f, g \rangle$ is elementary.

Case 3. f has exactly four fixed points. We may assume that

$$f(Z) = \lambda\beta_1\mathbf{e} + k\beta_2\mathbf{e}^\dagger (\lambda, k \neq 0, 1).$$

So $g_1, g_2, \dots, g_n$ each have four fixed points. Now suppose that

$$g_{n+1} = f.$$

Then

$$\begin{aligned} \text{fix}(g_{n+1}) &= \{0, \infty\mathbf{e}, \infty\mathbf{e}^\dagger, \infty\mathbf{e} + \infty\mathbf{e}^\dagger\} \\ &= \{g_{n,1}(0)\mathbf{e} + g_{n,2}(0)\mathbf{e}^\dagger, g_{n,1}(0)\mathbf{e} + g_{n,2}(\infty)\mathbf{e}^\dagger, \\ &\quad g_{n,1}(\infty)\mathbf{e} + g_{n,2}(0)\mathbf{e}^\dagger, g_{n,1}(\infty)\mathbf{e} + g_{n,2}(\infty)\mathbf{e}^\dagger\}. \end{aligned}$$

Similar discussions as case 2, we have $\langle f, g \rangle$ is elementary. $\square$

3.3. Geometric representations of bicomplex Möbius transformations.

The hyperbolic Möbius transformations were introduced in [25]. In the case dimension two, the six types of maps defined by Luna-Elizarrarás et al. are as follows:

1. The reflection in $\Gamma(\hat{h}, v)$

$$F_{\Gamma(\hat{h}, v)}(Z) = F_{\Gamma_1(\hat{h}_1, v_1)}(\beta_1)\mathbf{e} + F_{\Gamma_2(\hat{h}_2, v_2)}(\beta_2)\mathbf{e}^\dagger, F_{\Gamma(\hat{h}, v)}(\infty_{\mathbb{BC}}) = \infty_{\mathbb{BC}},$$

where hyperbolic straight line $\Gamma(\hat{h}, v) = \Gamma_1(\hat{h}_1, v_1)\mathbf{e} + \Gamma_2(\hat{h}_2, v_2)\mathbf{e}^\dagger$ and $F_{\Gamma_i(\hat{h}_i, v_i)}(\beta_i)$ is a reflection in complex straight line $\Gamma_i(\hat{h}_i, v_i)$ ($i = 1, 2$).

2. The reflection in $\mathbb{S}(Z_0, R)$

$$F_{\mathbb{S}(Z_0, R)}(Z) = F_{\mathbb{S}_1(\beta_{1,0}, r_1)}(\beta_1)\mathbf{e} + F_{\mathbb{S}_2(\beta_{2,0}, r_2)}(\beta_2)\mathbf{e}^\dagger,$$

where bicomplex circumference $\mathbb{S}(Z_0, R) = \mathbb{S}_1(\beta_{1,0}, r_1)\mathbf{e} + \mathbb{S}_2(\beta_{2,0}, r_2)\mathbf{e}^\dagger$ and $F_{\mathbb{S}_i(\beta_{i,0}, r_i)}(\beta_i)$ is a reflection in complex circumference $\mathbb{S}_i(\beta_{i,0}, r_i)$ ($i = 1, 2$).

3. Translations

$$F_W(Z) = Z + W = (\beta_1 + w_1)\mathbf{e} + (\beta_2 + w_2)\mathbf{e}^\dagger, F_W(\infty_{\mathbb{BC}}) = \infty_{\mathbb{BC}},$$

where $W = w_1\mathbf{e} + w_2\mathbf{e}^\dagger$, $\infty_{\mathbb{BC}} = \infty\mathbf{e} + \infty\mathbf{e}^\dagger$.

4. Stretching

$$F_g(Z) = gZ = (g_1\beta_1)\mathbf{e} + (g_2\beta_2)\mathbf{e}^\dagger, F_g(\infty_{\mathbb{BC}}) = \infty_{\mathbb{BC}},$$

where $g = g_1\mathbf{e} + g_2\mathbf{e}^\dagger > 0$, $\infty_{\mathbb{BC}} = \infty\mathbf{e} + \infty\mathbf{e}^\dagger$.

5. Orthogonal

$$\|F_{ort}(Z)\|_{\mathbb{D}^2} = \|Z\|_{\mathbb{D}^2}, F_{ort}(\infty_{\mathbb{BC}}) = \infty_{\mathbb{BC}},$$

where $F_{ort}(Z) = T_1(\beta_1)\mathbf{e} + T_2(\beta_2)\mathbf{e}^\dagger$. T_1 and T_2 are $\mathbb{R}$ -linear orthogonal mappings

6. Any mix of 1 and 2. For example,

$$F_{mix}(Z) = F_{\mathbb{S}_1(\beta_{1,0}, r_1)}(\beta_1)\mathbf{e} + F_{\Gamma_2(h_2, v_2)}(\beta_2)\mathbf{e}^\dagger.$$

Based on [15, Definition 3.1] and [25, Definition 11], we have the following definition.

Definition 3.21. A map $F : \overline{\mathbb{D}^2} \rightarrow \overline{\mathbb{D}^2}$ is called a hyperbolic Möbius transformation if it is a superposition of a finite number of the above six types of maps.

Chen and Dai[15] described the relation between the hyperbolic Möbius transformation and bicomplex Möbius transformation. They obtained the following crucial results.

Lemma 3.22. [15, Lemma 3.2] The translation $F_W(Z)$ is a superposition of two reflections in bicomplex circumference, and the stretching $F_g(Z)$ is a superposition of two reflections in hyperbolic straight line.

Theorem 3.23. [15, Theorem 3.3] The map $F(Z)$ is a superposition of an even number of reflections of the form $F_{\Gamma(h,v)}(Z)$, $F_{\mathbb{S}(Z_0, R)}(Z)$ and $F_{mix}(Z)$ if and only if $F(Z)$ is a bicomplex Möbius transformation.

4. The proof of bicomplex Jørgensen' inequality.

Proof of Theorem 1.2. Suppose that $\langle f, g \rangle$ is discrete nonelementary group. Select matrices $\mathcal{A}$ and $\mathcal{B}$ representing f and g respectively in $\text{PSL}(2, \mathbb{BC})$ and define

$$\mathcal{B}_0 = \mathcal{B}, \quad \mathcal{B}_{n+1} = \mathcal{B}_n \mathcal{A} \mathcal{B}_n^{-1}. \quad (19)$$

Case 1. f has one fixed point in $\overline{\mathbb{BC}}$.

According to Lemma 3.9 and Lemma 3.15, we may assume that

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{e} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{e}^\dagger \\ \mathcal{B} &= \begin{pmatrix} a_{0,1} & b_{0,1} \\ c_{0,1} & d_{0,1} \end{pmatrix} \mathbf{e} + \begin{pmatrix} a_{0,2} & b_{0,2} \\ c_{0,2} & d_{0,2} \end{pmatrix} \mathbf{e}^\dagger, \end{aligned}$$

where $c_{0,1} \neq 0$ or $c_{0,2} \neq 0$ (else $\langle f, g \rangle$ is elementary). Without loss of generality, suppose that $c_{0,1} \neq 0$ and $c_{0,2} = 0$. Since

$$|\mathrm{tr}^2(f) - 4|_{\mathbf{k}} + |\mathrm{tr}(fgf^{-1}g^{-1}) - 2|_{\mathbf{k}} < 1 \quad (20)$$

we have

$$\begin{aligned} & \left(|\mathrm{tr}^2(f_1) - 4| + |\mathrm{tr}(f_1 g_1 f_1^{-1} g_1^{-1}) - 2| \right) \mathbf{e} \\ & + \left(|\mathrm{tr}^2(f_2) - 4| + |\mathrm{tr}(f_2 g_2 f_2^{-1} g_2^{-1}) - 2| \right) \mathbf{e}^\dagger < 1. \end{aligned}$$

So

$$\begin{cases} |\mathrm{tr}^2(f_1) - 4| + |\mathrm{tr}(f_1 g_1 f_1^{-1} g_1^{-1}) - 2| < 1, \\ |\mathrm{tr}^2(f_2) - 4| + |\mathrm{tr}(f_2 g_2 f_2^{-1} g_2^{-1}) - 2| < 1. \end{cases} \quad (21)$$

It is easy to check that $|c_{0,1}| < 1$.

The relation (19) yields

$$\begin{aligned} a_{n+1,1} &= 1 - a_{n,1}c_{n,1}, & a_{n+1,2} &= 1 \\ b_{n+1,1} &= a_{n,1}^2, & b_{n+1,2} &= a_{n,2}^2 \\ c_{n+1,1} &= -(-c_{0,1})^{2^n}, & c_{n+1,2} &= 0 \\ d_{n+1,1} &= 1 + a_{n,1}c_{n,1}, & d_{n+1,2} &= 1, \end{aligned}$$

where

$$a_{n,2} = \begin{cases} 1 & \text{if } n \geq 1 \\ a_{0,2} & \text{if } n = 0. \end{cases}$$

Since $|c_{0,1}| < 1$, we have $|c_{n,1}| \rightarrow 0$, and $|a_{n,1}| \leq n + |a_{0,1}|$. So $a_{n,1}c_{n,1} \rightarrow 0$, $|a_{n,1}| \rightarrow 1$, and $|d_{n,1}| \rightarrow 1$.

If $\langle f, g \rangle$ is discrete, $\mathcal{B}_{n+1} = \mathcal{A}$, for sufficiently large n . According to Theorem 3.20 we have $\langle f, g \rangle$ is elementary.

Similarly, if $c_{0,1} = 0$, $c_{0,2} \neq 0$ or $c_{0,1} \neq 0$, $c_{0,2} \neq 0$, we have $\langle f, g \rangle$ is elementary.

Case 2. f has two fixed point in $\mathbb{BC}$.

According to Lemma 3.9 and Lemma 3.15, we may assume that

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{e} + \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix} \mathbf{e}^\dagger \quad (u \neq 0, 1.) \\ \mathcal{B} &= \begin{pmatrix} a_{0,1} & b_{0,1} \\ c_{0,1} & d_{0,1} \end{pmatrix} \mathbf{e} + \begin{pmatrix} a_{0,2} & b_{0,2} \\ c_{0,2} & d_{0,2} \end{pmatrix} \mathbf{e}^\dagger, \end{aligned}$$

where $c_{0,1} \neq 0$ or $b_{0,2}c_{0,2} \neq 0$ (else $\langle f, g \rangle$ is elementary). Without loss of generality, suppose that $c_{0,1} \neq 0$ and $b_{0,2}c_{0,2} \neq 0$.

By (20), we have

$$|c_{0,1}| < 1 \quad \text{and} \quad \nu = (1 + |b_{0,2}c_{0,2}|) \left| u - \frac{1}{u} \right|^2 < 1.$$

The relation (19) yields

$$\begin{aligned} a_{n+1,1} &= 1 - a_{n,1}c_{n,1} & a_{n+1,2} &= a_{n,2}d_{n,2}u - \frac{b_{n,2}c_{n,2}}{u} \\ b_{n+1,1} &= a_{n,1}^2 & b_{n+1,2} &= a_{n,2}b_{n,2}\left(\frac{1}{u} - u\right) \\ c_{n+1,1} &= -(-c_{0,1})^{2^n} & c_{n+1,2} &= c_{n,2}d_{n,2}\left(u - \frac{1}{u}\right) \\ d_{n+1,1} &= 1 + a_{n,1}c_{n,1} & d_{n+1,2} &= \frac{a_{n,2}d_{n,2}}{u} - b_{n,2}c_{n,2}u. \end{aligned}$$

So

$$\begin{aligned} |b_{n+1,2}c_{n+1,2}| &\leq \left| b_{n,2}c_{n,2}(1 + b_{n,2}c_{n,2})\left(u - \frac{1}{u}\right)^2 \right| \\ &\leq \nu^n |b_{0,2}c_{0,2}|. \end{aligned}$$

Thus

$$b_{n,2}c_{n,2} \rightarrow 0 \text{ and } a_{n,2}d_{n,2} \rightarrow 1.$$

Also, we obtain

$$a_{n+1,2} \rightarrow u, \quad d_{n+1,2} \rightarrow \frac{1}{u}.$$

Now

$$\left| \frac{b_{n+1,2}}{b_{n,2}} \right| = \left| a_{n,2} \left(\frac{1}{u} - u \right) \right| \leq \nu^{\frac{1}{2}} |u|.$$

Thus

$$\frac{b_{n,2}}{u^n} \rightarrow 0.$$

Similarly, $c_{n,2}u^n \rightarrow 0$. So

$$\begin{aligned} &\mathcal{A}^{-n} \mathcal{B}_{2n} \mathcal{A}^n \\ &= \begin{pmatrix} a_{2n,1} - nc_{2n,1} & n(a_{2n,1} - nc_{2n,1}) + b_{2n,1} - nd_{2n,1} \\ c_{2n,1} & d_{2n,1} \end{pmatrix} \mathbf{e} + \begin{pmatrix} a_{2n,2} & \frac{b_{2n,2}}{u^{2n}} \\ c_{2n,2}u^{2n} & d_{2n,2} \end{pmatrix} \mathbf{e}^\dagger \\ &= \begin{pmatrix} a_{2n,1} - nc_{2n,1} & n(a_{2n,1} - d_{2n,1}) - n^2c_{2n,1} + b_{2n,1} \\ c_{2n,1} & d_{2n,1} \end{pmatrix} \mathbf{e} + \begin{pmatrix} a_{2n,2} & \frac{b_{2n,2}}{u^{2n}} \\ c_{2n,2}u^{2n} & d_{2n,2} \end{pmatrix} \mathbf{e}^\dagger. \end{aligned}$$

$$a_{2n,1} - nc_{2n,1} = a_{2n,1} + n(-c_{0,1})^{2^{2n-1}}$$

$$n(a_{2n,1} - d_{2n,1}) - n^2c_{2n,1} + b_{2n,1} = -2na_{2n-1,1}c_{2n-1,1} - n^2(-c_{0,1})^{2^{2n-1}} + b_{2n,1}.$$

Since $|c_{0,1}| < 1$, we have

$$|-2na_{2n-1,1}c_{2n-1,1}| \leq 2n(2n-1 + |a_{0,1}|)|c_{0,1}|^{2^{2n-1}}.$$

Thus

$$a_{2n,1} - nc_{2n,1} \rightarrow 1 \text{ and } n(a_{2n,1} - d_{2n,1}) - n^2c_{2n,1} + b_{2n,1} \rightarrow 1.$$

As $\langle f, g \rangle$ is discrete, we must have

$$\mathcal{A}^{-n} \mathcal{B}_{2n} \mathcal{A}^n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{e} + \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix} \mathbf{e}^\dagger \quad (u \neq 0, 1.)$$

for all large n . So $\mathcal{B}_{2n} = \mathcal{A}$ for infinitely many n . According to Theorem 3.20, we have $\langle f, g \rangle$ is elementary.

Similarly, if $c_{0,1} = 0, b_{0,2}c_{0,2} \neq 0$ or $c_{0,1} \neq 0, b_{0,2}c_{0,2} = 0$, we have $\langle f, g \rangle$ is elementary.

Case 3. f has four fixed points. Let

$$\mathcal{A} = \begin{pmatrix} v & 0 \\ 0 & \frac{1}{v} \end{pmatrix} \mathbf{e} + \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix} \mathbf{e}^\dagger \quad (u, v \neq 0, 1.)$$

$$\mathcal{B} = \begin{pmatrix} a_{0,1} & b_{0,1} \\ c_{0,1} & d_{0,1} \end{pmatrix} \mathbf{e} + \begin{pmatrix} a_{0,2} & b_{0,2} \\ c_{0,2} & d_{0,2} \end{pmatrix} \mathbf{e}^\dagger,$$

where $b_{0,1}c_{0,1} \neq 0$ or $b_{0,2}c_{0,2} \neq 0$ (else $\langle f, g \rangle$ is elementary).

The relation (19) yields

$$\begin{aligned} a_{n+1,1} &= a_{n,1}d_{n,1}v - \frac{b_{n,1}c_{n,1}}{u} & a_{n+1,2} &= a_{n,2}d_{n,2}u - \frac{b_{n,2}c_{n,2}}{u} \\ b_{n+1,1} &= a_{n,1}b_{n,1}\left(\frac{1}{v} - v\right) & b_{n+1,2} &= a_{n,2}b_{n,2}\left(\frac{1}{u} - u\right) \\ c_{n+1,1} &= c_{n,1}d_{n,1}\left(v - \frac{1}{v}\right) & c_{n+1,2} &= c_{n,2}d_{n,2}\left(u - \frac{1}{u}\right) \\ d_{n+1,1} &= \frac{a_{n,1}d_{n,1}}{v} - b_{n,1}c_{n,1}v & d_{n+1,2} &= \frac{a_{n,2}d_{n,2}}{u} - b_{n,2}c_{n,2}u. \end{aligned}$$

Similar discussions as case 2, if $\langle f, g \rangle$ is discrete, we have

$$\mathcal{B}_{2n} = \mathcal{A}$$

for all sufficiently large n . This shows that $\langle f, g \rangle$ is elementary. $\square$

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References

- [1] AHLFORS, L. V., On the fixed points of Möbius transformation in R^n , *Ann. Fenn. Math.* **10** (1985), 15-27. MR0802464, Zbl 0586.30045, doi: 10.5186/aasfm.1985.1005. 819
- [2] ALAQAD, H; GONG, J; MARTIN, G. J., Chebyshev polynomials and inequalities for Kleinian groups. *Commun. Contemp. Math.* **25** (2021), no. 2, 22 pages. MR4546881, Zbl 1508.30004, doi: 10.1142/S0219199721501029. 819
- [3] AGARWAL, R; GOSWAMI, P; AGARWAL, R. P; BALEANU, D., Solution of maxwell's wave equations in bicomplex space. *Romanian J. Phys.* **62** (2017), 1-11. 819
- [4] AGARWAL, R; SHARMA, U. P; AGARWAL, R. P., Bicomplex Mittag-Leffler function and properties. *J. Nonlinear Sci. Appl.* **15** (2022), 48-60. MR4282934, Zbl 2103.10324, doi: 10.22436/jnsa.015.01.04. 819, 820

- [5] ALEXANDER, B; JUAN, R; LUNA-ELIZARRARÁS, M. E; MICHAEL, S., Cantor-type sets in hyperbolic numbers. *Fractals* **24** (2016), no. 4, 12 pages. MR3583983, Zbl 1357.28008, doi: 10.1142/S0218348X16500511. 819
- [6] ALPAY, D; LUNA-ELIZARRARÁS, M. E; SHAPIRO, M; STRUPPA, D. C., Basics of functional analysis with bicomplex scalars, and bicomplex schur analysis. Springer (2014). MR3309382, Zbl 1319.46001, doi: 10.1007/978-3-319-05110-9. 819, 820, 821
- [7] BASMAJIAN, A; MINER, R., Discrete subgroups of complex hyperbolic motions. *Invent. Math.* **131** (1998), 85-136. MR1489895, Zbl 0901.32018, doi: 10.1007/s002220050198. 818, 819
- [8] BEARDON, A. F., The geometry of discrete groups, Springer, New York (1983). MR0698777, Zbl 0528.30001, doi: 10.1007/978-1-4612-1146-4. 818, 825, 826, 827
- [9] BANERJEE, A; DATTA, K; HOQUE, A., Fourier transform and its inverse for functions of bicomplex variables. arXiv:1404.4236v1. doi: 10.48550/arXiv.1404.4236. 819
- [10] BANERJEE, A; DATTA, K; HOQUE, A., Inverse Laplace transform for bicomplex variables. arXiv:1403.3313v1. doi: 10.48550/arXiv.1403.3313. 819
- [11] BANERJEE, A., Bicomplex modules with indefinite inner product. *Adv. Appl. Clifford Algebras* **29** (2019), 25 pages. MR3959551, Zbl 1418.30043, doi: 10.1007/s00006-019-0975-y. 819
- [12] BANERJEE, A., Bicomplex harmonic and isotonic oscillators: The excited states. *Adv. Appl. Clifford Algebras* **27** (2017), 2321–2332. MR3688826, Zbl 1380.30034, doi: 10.1007/s00006-017-0772-4. 819
- [13] BORY, R; PÉREZ, R; CESAR, O; SHAPIRO, M., Cauchy type integral in bicomplex setting and its properties. *Complex Anal. Oper. Theory* **8** (2019), 3807–3815. MR4011254, Zbl 1445.30022, doi: 10.1007/s11785-019-00918-x. 819
- [14] CHEN, M., Discreteness and convergence of Möbius groups. *Geom. Dedicata.* **104** (2022), 61-69. MR2043954, Zbl 1053.30038, doi: 10.1023/B:GEOM.0000023036.55318.40. 819
- [15] CHEN, L; DAI, B., The fixed points and cross-ratios of hyperbolic Möbius transformations in bicomplex space. *Adv. Appl. Clifford Algebr.* **32** (2022), 23 pages. MR4475298, Zbl 1502.30138, doi: 10.1007/s00006-022-01231-1. 819, 822, 823, 824, 830
- [16] CAO, C., Some trace inequalities for discrete groups of Möbius transformation. *Proc. Amer. Math. Soc.* **123** (1995), 3807-3815. MR1301013, Zbl 0843.30037, doi: 10.2307/2161910. 819
- [17] CAO, W; PARKER, J. R., Jørgensen's inequality and collars in n-dimensional quaternionic hyperbolic space. *Q. J. Math.* **62** (2011), 523-543. MR2825470, Zbl 1244.20045, doi: 10.1093/qmath/haq003. 819
- [18] CHINMAY, G., Bicomplex Möbius transformation. *Bull. Calcutta Math. Soc.* **110** (2018), 141-150. MR3821936, Zbl 1410.30026, doi: 10.48550/arXiv.1706.07699. 822, 824, 825
- [19] DAI, B; XIE S., On algebraic limits and topological properties of sets of Möbius groups $\bar{R}^n$. *Complex Var. Theory Appl.* **47** (2002), 149-154. MR1892515, Zbl 1038.30019, doi: 10.1080/02781070290010896.
- [20] DAI, B; FANG, A; NAI, B., Discreteness criteria for subgroups in complex hyperbolic space. *Proc. Japan Acad. Ser. A Math. Sci.* **77** (2001), 168-172. MR1873738, Zbl 1009.32016, doi: 10.3792/pjaa.77.168.
- [21] EMANUELLO, J. A; NOLDER, C. A., Projective compactification in $R^{1,1}$ and its Möbius geometry. *Complex Var. Elliptic Equ.* **62** (2017), 1266-1286. MR3311942, Zbl 1309.30042, doi: 10.1007/s11785-014-0363-5. 819, 822, 823
- [22] EMANUELLO, J. A., Analysis of functions of split-complex, multicomplex, and split-quaternionic variables and their associated conformal geometries. Thesis (Ph.D.)-The Florida State University(2015). MR3389108. 818, 822, 823, 824, 825
- [23] FRIEDLAND, S; HERSONSKY, S., Jørgensen inequality for discrete groups in normed algebras. *Duke Math J.* **69** (1993), 593-614. MR1208812, Zbl 0799.30033, doi: 10.1215/S0012-7094-93-06924-4. 819

- [24] GEHRING, F. W; MARTIN, G. J., Inequalities for Möbius transformations and discrete groups. *J. Reine Angew Math.* **418** (1991), 31-76. MR1111201, Zbl 0722.30034, doi: 10.1515/crll.1991.418.31. 818
- [25] GOLBERG, A; LUNA-ELIZARRARÁS, M. E., hyperbolic conformality in multidimensional hyperbolic spaces. *Math. Meth. Appl. Sci.* **47** (2020), 7862–7878. MR4760454, Zbl 1551.30089, doi: 10.1002/mma.7109. 819, 822, 825, 829, 830
- [26] GONGOPADHYAY, K., On generalized Jørgensen inequality in infinite dimension. *New York J. Math.* **42** (2018), 865-869. MR3874953, Zbl 1405.30045.
- [27] GILMAN, J., A geometric approach to the hyperbolic Jørgensen inequality. *Bull. Amer. Math. Soc.* **16** (1987), 91-92. MR0866021, Zbl 0722.30033, doi: 10.1090/S0273-0979-1987-15467-X.
- [28] JØRGENSEN, T., On discrete groups of Möbius transformations. *Amer. J. Math.* **98** (1976), 739-749. MR0427627, Zbl 0336.30007, doi: 10.2307/2373814. 818
- [29] JIANG, Y; KAMIYA, S; PARKER, J. R., Jørgensen's inequality for complex hyperbolic space. *Geom. Dedicata.* **97** (2003), 55-80. MR2003690, Zbl 1037.20053, doi: 10.1023/A:1023627324013. 819
- [30] KRISHNENDU, G., Conjugacy classes in Möbius groups. *Geom. Dedicata.* **151** (2011), 245-258. MR2780749, Zbl 1253.51013, doi: 10.1007/s10711-010-9531-6. 819
- [31] LI, L., A generalization of Jørgensen's inequality to infinite dimension. *New York J. Math.* **17** (2011), 41-49. MR2781907, Zbl 1220.30056. 819
- [32] LAVOIE, R. G; MARCHILDON, L; ROCHON, D., Finite-dimensional bicomplex Hilbert spaces. *Adv. Appl. Clifford Algebra* **3** (2011), 561-581. MR2825022, Zbl 1237.46009, doi: 10.1007/s00006-014-0503-z. 823
- [33] LUNA-ELIZARRARÁS, M. E; PÉREZ-REGALADO, C. O; MICHAEL, S., On linear functionals and hahn-Banach theorems for hyperbolic and bicomplex modules. *Adv. Appl. Clifford Algebra* **24** (2014), 1105-1129. MR3277692, Zbl 1308.30059, doi: 10.1007/s00006-010-0274-0. 819, 820
- [34] LUNA-ELIZARRARÁS, M. E; PÉREZ-REGALADO, C. O; MICHAEL, S., Complex Laplacian and derivatives of bicomplex functions. *Complex Anal. Oper. Theory* **7** (2013), 1675-1711. MR3103314, Zbl 1492.30098, doi: 10.1007/s11785-022-01197-9. 819, 820
- [35] LUNA-ELIZARRARÁS, M. E; PÉREZ-REGALADO, C. O; MICHAEL, S., Integration of functions of a hyperbolic variable. *Complex Anal. Oper. Theory* **6** (2022), no. 3, 21 pages. MR4396704, Zbl 1278.30052, doi: 10.1007/s11785-013-0284-8. 821
- [36] LUNA-ELIZARRARÁS, M. E; SHAPIRO, M; STRUPPA, D. C; VAJAC, A., Bicomplex holomorphic functions: The Algebra, geometry and analysis of bicomplex numbers, frontiers in Mathematics. Birkhäuser: Springer, (2015). MR3410909, Zbl 1345.30002, doi: 10.1007/978-3-319-24868-4. 819, 821
- [37] LUNA-ELIZARRARÁS, M. E; PEREZ-REGALADO, C. O; SHAPIRO, M., Singularities of bicomplex holomorphic functions. *Math. Meth. Appl. Sci.* **47** (2024), 7933–7948. MR4760459, Zbl 1551.32037, doi: 10.1002/mma.7522. 819, 821
- [38] LUNA-ELIZARRARÁS, M. E; PÉREZ-REGALADO, C. O; SHAPIRO, M., On the Laurent series for bicomplex holomorphic functions. *Complex Var. Elliptic Equ.* **62** (2017), 1266-1286. MR3662505, Zbl 1375.30077, doi: 10.1080/17476933.2016.1250404. 819, 821, 822
- [39] MARTIN, G. J., On discrete Möbius groups in all dimensions: a generalization of Jørgensen's inequality. *Acta Math.* **163** (1989), 253-289. MR1032075, Zbl 0698.20037, doi: 10.1007/BF02392737. 819
- [40] MARKHAM, S; PARKER, J. R., Jørgensen's inequality for metric spaces with application to the octonions. *Adv. Geom.* **7** (2007), 19–38. MR2290637, Zbl 1121.30020, doi: 10.1515/ADVGEOM.2007.002. 819
- [41] POGORUI, A. A; DAGNINO, R. M., On the set of zeros of bicomplex polynomials. *Complex Var. Elliptic Equ.* **7** (2006), 725-730. MR2242492, Zbl 1104.30005, doi: 10.1080/17476930600757139. 825

- [42] TAYLOR, P. B., Jørgensen's inequality for discrete convergence groups. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **25** (2000), 131-150. MR1737431, Zbl 0945.30035. 819
- [43] TAN, D., On Two-Generator Discrete Groups of Möbius Transformations. *Proc. Amer. Math. Soc.* **106** (1989), 763-770. MR0969527, Zbl 0793.15019, doi: 10.2307/2047433. 819
- [44] WATERMAN, P. L., Möbius Transformations in Several Dimensions. *Adv. Math.* **101** (1993), 87-113. MR1239454, Zbl 0673.20023, doi: 10.1006/aima.1993.1043. 819
- [45] WANG, X; LI, L; CAO, W., Discreteness criteria for Möbius groups acting on $\overline{\mathbb{R}^n}$, *Israel J. Math.* **150** (2005), 357-368. MR2255815, Zbl 1330.30042, doi: 10.1007/BF02762387. 819
- [46] YÜCE, S., Jørgensen's inequality and purely loxodromic two-generator free Kleinian groups. *Turkish J. Math.* **43** (2019), 833-861. MR3935676, Zbl 1414.30046, doi: 10.3906/mat-1808-101. 819

(Zekun Li) SCHOOL OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU 510275, P. R. CHINA

lizk25@mail2.sysu.edu.cn

(Binlin Dai) SCHOOL OF MATHEMATICS, SHANGHAI UNIVERSITY OF FINANCE AND ECONOMICS, SHANGHAI 200433, P. R. CHINA

bldai@mail.shufe.edu.cn

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