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ALMOST TRIVIAL GROUPOIDS

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Berglund and Mislove defined semigroups with almost trivial multiplications and proved that such a semigroup belongs to one of five well known semigroup classes (see [1]). In a similar way, without associativity condition, we defined almost trivial groupoids and proved Th 1 analogous to Th 1 of [1].

Using this result, we solved generalized associativity equation on almost trivial groupoids. As an example we obtained 1344 solutions of generalized associativity equation on two-element groupoids.

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1. If groupoid $(S, \cdot)$ (or simply $\cdot$) is given, then functions

$$\lambda_x, \rho_y \colon S \to S \ \ (x, y \in S),$$

defined by:

$$\lambda_x y = \rho_y x = xy$$

are respectively left and right translations of $(S, \cdot)$.

In [1], semigroup with almost trivial multiplications is defined as a semigroup whose translations are either surjections or constant.

Almost trivial groupoids (ATG) we define as groupoids with translations which are either permutations or constant.

LEMMA 1.1. *Any isotope of* ATG *is also an* ATG.

PROOF: Let $(S, \cdot)$ be an ATG and $x * y = \varphi^{-1}(\alpha x \cdot \beta y)$ where $\alpha, \beta, \varphi \colon T \to S$ are bijections.

Let $\lambda'(\rho')$ be left (right) translations of $(T, *)$.

Then for all $a \in T$:

$$\lambda_a{}'x = a * x = \varphi^{-1}(\alpha a \cdot \beta x) = \varphi^{-1}\lambda_{\alpha_a}\beta x$$
$$\rho_a{}'x = x * a = \varphi^{-1}(\alpha x \cdot \beta a) = \varphi^{-1}\rho_{\beta_a}\alpha x$$

so $\lambda_a{}'$ and $\rho_a{}'$ are also either permutations or constant functions on T and consequently $(T, *)$ is also an ATG.

DEFINITION. $(S, \cdot)$ is a quasigroup with quasizero (p, q, r) iff:

– $px = r$

– $xq = r$

– for all $a, b \in S (a \neq p)$, equation $ax = b$ has the unique solution

– for all $a, b \in S (a \neq q)$, equation $xa = b$ has the unique solution.

$(S, \cdot)$ is a left (right) groupoid iff:

$$xy = \varphi x \qquad (xy = \varphi y)$$

where φ is a permutation of S.

LEMMA 1.2. *Left (right) groupoid* $xy = \varphi x (xy = \varphi y)$ *is a semigroup iff* $\varphi = \varepsilon$.

PROOF: If $xy = \varphi x$ is associative, then

$$\varphi\varphi x = \varphi(xy) = xy \cdot z = x \cdot yz = \varphi x$$

and $\varphi = \varepsilon$. The converse is trivial.

It is clear that a quasigroup with quasizero is not a quasigroup. If $p = q = r$ (i.e. r is zero), S is a quasigroup with zero. Both these notions are generalizations of well known notion of group with zero (see [2]).

THEOREM 1. *Any* ATG *is one of:*

(S_0) *zero-semigroup*

(S_L) *left groupoid*

(S_R) *right groupoid*

(Q) *quasigroup*

(Q_p) *quasigroup with quasizero.*

PROOF: It is easy to see that all groupoids we mention, are ATG.

To prove the converse, we define:

$$L_0 = \{a \in S \mid \lambda_a \text{ is constant}\}$$
$$L_1 = \{a \in S \mid \lambda_a \text{ is a permutation}\}$$
$$R_0 = \{a \in S \mid \rho_a \text{ is constant}\}$$
$$R_1 = \{a \in S \mid \rho_a \text{ is a permutation}\}$$

(a) Let $L_0 = \emptyset$.

If both $R_0 \neq \emptyset$ and $R_1 \neq \emptyset$ then there are $x \in R_0$ and $y \in R_1$ such that $Sx = b$ for some $b \in S$ and $Sy = S$, so there is an $a \in S$ such that $ay = b$. From $ax = b$ and $ay = b$ it follows that λ_a is constant contrary to hypothesis.

It must be either $R_0 = \emptyset$ or $R_1 = \emptyset$.

(a′) Let $R_0 = \emptyset$.

Then any translation is a permutation, so $\cdot$ is a quasigroup.

(a″) Let $R_1 = \emptyset$.

Then any left translation is a permutation while any right translation is constant and there is a permutation φ such that $xy = \varphi y$.

(b) Let $L_1 = \emptyset$.

If both $R_0 \neq \emptyset$ and $R_1 \neq \emptyset$ then there are $x \in R_0$ and $y \in R_1$ such that for some $a, b \in S (a \neq b)$:

$$ay = ax = bx = by$$

and ρ cannot be permutation, contrary to hypothesis.

It must be either $R_0 = \emptyset$ or $R_1 = \emptyset$.

(b′) Let $R_0 = \emptyset$.

Then any right translations is a permutation, while any left translation is constant and there is a permutation φ such that $xy = \varphi x$.

(b″) Let $R_1 = \emptyset$.

From $x, y, u, v \in S$ it follows that $x \in L_0$, $v \in R_0$ and $xy = xv = uv$ so $\cdot$ is a zero-semigroup.

(c) Let $L_0 \neq \emptyset$ and $L_1 \neq \emptyset$.

Then also $R_0 \neq \emptyset$ and $R_1 \neq \emptyset$. It is easy to see that L_0 and R_0 have only one element. Let $a \in L_0$, $b \in R_0$ and $c = ab$.

Except λ_a, all λ_x are permutations. Also, except ρ_b, all ρ_x are permutations and $\cdot$ is a quasigroup with quasizero (a, b, c).

Definition. Principal ATG's are:

(S_0) zero-semigroup

(A_L) left zero semigroup

(A_R) right zero semigroup

(L) loop

(L_0) loop with zero.

Principal ATG's from a family $\{(S, *_i) \mid i \in I\}$ are compatible if:

– all ATG's (from the family) with zero, have a common zero d

– all ATG's (from the family) with unit, have a common unit e

– if d and c are as above, then $d \neq e$.

LEMMA 1.3. *Any* ATG *is (principal) isotope of some principal* ATG.

PROOF: In the first three cases the proof is trivial, while in the fourth it follows from the well known theorem of Albert ([3]).

Let $\cdot$ be a quasigroup with quasizero (p,q,r), α a transposition of p and r and β a transposition of q and r.

Let also $x * y = \alpha x \cdot \beta y$. Then:

$$r * x = \alpha r \cdot \beta x = p \cdot \beta x = r$$
$$x * r = \alpha x \cdot \beta r = \alpha x \cdot q = r.$$

Also, for $x \neq r$ $\lambda_x{}' y = x * y = \alpha x \cdot \beta y = \lambda_{\alpha x} \beta y$ so $\lambda_x{}'$ is a permutation such that $\lambda_x{}' r = r$.

Analogously $\rho_x{}' \restriction S \backslash \{r\}$ is a permutation and $* \restriction (S\backslash\{r\})^2$ is a quasigroup. Since any quasigroup is (principally) isotopic to a loop, $(S,*)$ is (principally) isotopic to a loop with zero.

COROLLARY 1.4. *Any* ATG *is (principal) isotope of one of:*

(S_0) *zero-semigroup*

(A_L) *left zero semigroup*

(A_R) *right zero semigroup*

(L) *loop*

(L_0) *loop with zero*

COROLLARY 1.5. *Any* ATG *which is a semigroup is one of:*

(S_0) *zero-semigroup*

(A_L) *left zero semigroup*

(A_R) *right zero semigroup*

(G) *group*

(G_0) *group with zero*

PROOF:

(a) Zero-semigroup is a semigroup.

(b) According to L 1.1.

(c) According to L 1.1.

(d) Associative quasigroup is a group.

(e) Let $\cdot$ be a quasigroup with quasizero (p,q,r). For any $x,y,z \in S$ we have:

$$rz = py \cdot z = p \cdot yz = r = pz \quad \text{so} \quad p = r$$
$$xr = x \cdot yq = xy \cdot q = r = xq \quad \text{so} \quad q = r.$$

Consequently $\cdot$ is an associative quasigroup with zero i.e. group with zero.

Corollary 1.5 is smoewhat weaker than Th 1 from [1], where the same conclusion follows from weaker hypothesis about semigroups with almost trivial multiplications.

COROLLARY 1.6. *Any* ATG *with unit is one of:*

(L) *loop*

(L_0) *loop with zero.*

COLLORARY 1.7. *Any* ATG *with zero is one of:*

(S_0) *zero-semigroup*

(Q_0) *quasigroup with zero.*

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2. In Th 2 the general solution is given of the generalized associativity equation on ATG:

$$A(x, B(y, z)) = C(D(x, y), z). \tag{1}$$

We are using the following conventions:

– A, B, C, D are always given by the formulas (2) and their respective isotopes $\cdot$, $\circ$, $*$, Δ.

– $S_\circ$, S_L, A_L, S_R, A_R, Q, L, G, Q_q, $Q_\circ$, $L_\circ$, $G_\circ$ are types i.e. various kinds of ATG's as follows:

$S_\circ$ – zero-semigroup

$S_L(S_R)$ – left (right) groupoid

$A_L(A_R)$ – left (right) zero semigroup

$Q(Q_q, Q_\circ)$ – quasigroup (with quasizero, with zero)

$L(L_\circ)$ – loop (with zero)

$G(G_\circ)$ – group (with zero)

– a, b, c, d denote types

– $\langle a, b, c, d\rangle$ means that $\cdot$ is of type a, $\circ$ of type b, $*$ of type c and Δ of type d. If for example $b = d$, then $\circ$ and Δ are the same operation.

$\langle a, b, c, b'\rangle$ means that operations $\circ$ and Δ are of the same type but not necessarily identical.

So $\langle S_\circ L S_\circ L'\rangle$ means that zero-semigroups $\cdot$ and $*$ are identical while $\circ$ and Δ are loops which are not necessarily identical.

THEOREM 2. *The general solution of generalized associativity equation* (1) *on* ATG's, *is given by:*

$$\begin{aligned} A(x,y) &= A_1x \cdot A_2y \\ B(x,y) &= A_2^{-1}(A_2B_1x \circ A_2B_2y) \\ C(x,y) &= C_1x * C_2y \\ D(x,y) &= C_1^{-1}(C_1D_1x \,\Delta\, C_1D_2y) \end{aligned} \tag{2}$$

where A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2 *are arbitrary permutations on* S *such that:*

$$(3) \qquad A_1 = C_1 D_1 \qquad A_2 B_1 = C_1 D_2 \qquad A_2 B_2 = C_2$$

and where $\cdot$, $\circ$, $*$, $\triangle$ *are arbitrary principal* ATG's *such that either one of the following conditions are fulfilled:*

$\langle S_\circ S_\circ S_\circ S_\circ\rangle$	$\langle S_\circ S_\circ S_\circ A_L\rangle$	$\langle S_\circ S_\circ S_\circ A_R\rangle$	$\langle S_\circ S_\circ S_\circ L\rangle$	$\langle S_\circ S_\circ S_\circ L_\circ\rangle$
$\langle S_\circ A_L S_\circ S_\circ\rangle$	$\langle S_\circ A_L S_\circ A_L\rangle$	$\langle S_\circ A_L S_\circ A_R\rangle$	$\langle S_\circ A_L S_\circ L\rangle$	$\langle S_\circ A_L S_\circ L_\circ\rangle$
$\langle S_\circ S_\circ A_L S_\circ\rangle$	$\langle S_\circ S_\circ L_\circ S_\circ\rangle$	$\langle S_\circ A_L A_L S_\circ\rangle$	$\langle S_\circ A_L L_\circ S_\circ\rangle$	$\langle S_\circ A_R S_\circ S_\circ\rangle$
$\langle S_\circ A_R S_\circ A_L\rangle$	$\langle S_\circ A_R S_\circ A_R\rangle$	$\langle S_\circ A_R S_\circ L\rangle$	$\langle S_\circ A_R S_\circ L_\circ\rangle$	$\langle S_\circ A_R A_L S_\circ\rangle$
$\langle S_\circ A_R L_\circ S_\circ\rangle$	$\langle S_\circ L S_\circ S_\circ\rangle$	$\langle S_\circ L S_\circ A_L\rangle$	$\langle S_\circ L S_\circ A_R\rangle$	$\langle S_\circ L S_\circ L'\rangle$
$\langle S_\circ L S_\circ L_\circ\rangle$	$\langle S_\circ L A_L S_\circ\rangle$	$\langle S_\circ L L_\circ S_\circ\rangle$	$\langle S_\circ L_\circ S_\circ S_\circ\rangle$	$\langle S_\circ L_\circ S_\circ A_L\rangle$
$\langle S_\circ L_\circ S_\circ A_R\rangle$	$\langle S_\circ L_\circ S_\circ L\rangle$	$\langle S_\circ L_\circ A_L S_\circ\rangle$	$\langle A_L S_\circ A_L A_L\rangle$	$\langle A_L A_L A_L A_L\rangle$
$\langle A_L A_R A_L A_L\rangle$	$\langle A_L L A_L A_L\rangle$	$\langle A_L L_\circ A_L A_L\rangle$	$\langle A_R S_\circ S_\circ S_\circ\rangle$	$\langle A_R S_\circ S_\circ A_L\rangle$
$\langle A_R S_\circ S_\circ A_R\rangle$	$\langle A_R S_\circ S_\circ L\rangle$	$\langle A_R S_\circ S_\circ L_\circ\rangle$	$\langle A_R S_\circ A_L S_\circ\rangle$	$\langle A_R S_\circ L_\circ S_\circ\rangle$
$\langle A_R A_L A_L A_R\rangle$	$\langle A_R A_R A_R S_\circ\rangle$	$\langle A_R A_R A_R A_L\rangle$	$\langle A_R A_R A_R A_R\rangle$	$\langle A_R A_R A_R L\rangle$
$\langle A_R A_R A_R L_\circ\rangle$	$\langle A_R L L A_R\rangle$	$\langle A_R L_\circ L_\circ A_R\rangle$	$\langle L A_L A_L L\rangle$	$\langle L A_R L A_L\rangle$
$\langle L_\circ S_\circ S_\circ S_\circ\rangle$	$\langle L_\circ S_\circ S_\circ A_L\rangle$	$\langle L_\circ S_\circ S_\circ A_R\rangle$	$\langle L_\circ S_\circ S_\circ L\rangle$	$\langle L_\circ S_\circ A_L S_\circ\rangle$
$\langle L_\circ S_\circ L_\circ' S_\circ\rangle$	$\langle L_\circ A_L A_L L_\circ\rangle$	$\langle L_\circ A_R L_\circ A_L\rangle$	$\langle GGGG\rangle$	$\langle G_\circ G_\circ G_\circ G_\circ\rangle$

where operations $\cdot$, $\circ$, $*$, $\triangle$ *are compatible, or one of the following:*

$\langle S_\circ L_\circ S_\circ L_\circ'\rangle$, $\cdot$ *and* $\circ$ *have a common zero,* $\circ$ *and* $\triangle$ *have a common unit*

$\langle S_\circ L_\circ L_\circ' S_\circ\rangle$, $\cdot$ *and* $*$ *have a common zero,* $\circ$ *and* $*$ *have a common unit*

$\langle A_R A_R L S_\circ\rangle$, *the unit of the loop* $*$ *is a zero of* $\triangle$

$\langle A_R A_R L_\circ S_\circ\rangle$, *the unit of the loop with zero* $*$ *is a zero of* $\triangle$

$\langle L S_\circ A_L A_L\rangle$, *the unit of loop* $\cdot$ *is a zero of* $\circ$

$\langle L_\circ S_\circ S_\circ L_\circ'\rangle$, $\cdot$ *and* $*$ *have a common zero,* $\cdot$ *and* $\triangle$ *have a common unit*

$\langle L_\circ S_\circ A_L A_L\rangle$, *the unit of the loop with zero* $\cdot$ *is a zero of* $\circ$.

Sketch of the proof: We can easily check that quadruple (A, B, C, D) given by (2), satisfying (3) and one of conditions $\langle abcd\rangle$, is a solution of (1).

Conversely, let T be a ternary operation defined by:

$$T(x, y, z) = A(x, B(y, z))$$

(a) T does not depnd on x, y, z.

For A, B we have the following possibilities:

– $A(x, y) = 0$, B is arbitrary

– $A(x,y)=\alpha y$, $B(x,y)=b$

– A is a guasigroup with quasizero $(a,b,0)$, $B(x,y)=b$ and analogous possibilities for C, D.

Only for three of 49 solutions, we prove that they are of the required form.

(a′) $A(x,y)=0$, B is a guasigroup

$C(x,y)=0$, D is a guasigroup.

Let $p,q,r\in S$ and A_1 arbitrary permutation on S such that $A_1\ p\neq 0$. We define:

$$B_1x=B(x,r),\quad B_2x=B(q,x),\quad D_1x=D(x,q),\quad D_2x=D(p,x),$$
$$C_1=A_1D_1^{-1},\quad A_2=C_1D_2B_1^{-1},\quad C_2=A_2B_2,\quad e=A_1p$$

and

$$x\cdot y=A(A_1^{-1}x,A_2^{-1}y)$$
$$x\circ y=A_2B((A_2B_1)^{-1}x,\ (A_2B_2)^{-1}y)$$
$$x\bigtriangleup y=C_1D((C_1D_1)^{-1}x,\ (C_1D_2)^{-1}y).$$

Then it is easy to prove (3), $\langle S_\circ L S_\circ L'\rangle$ and compatibility of $\cdot$, $\circ$, $\bigtriangleup$.

(a″) $A(x,y)=0$, B is a quasigroup with quasizero (b_1,b_2,b)

$D(x,y)=d$, C is a quasigroup with quasizero $(d,c,0)$.

Let $q,r,r'\in S$ such that $q\neq b_1$, $r\neq b_2$, $r\neq c$, $r'\neq d$ and D_1 arbitrary permutation on S. We define:

$$B_1x=B(x,r),\ \ B_2x=B(q,x),\ \ C_1x=C(x,r),\ \ C_2x=C(r',x),$$
$$e=C(r',r),\ \ A_1=C_1D_1,\ \ A_2=C_2B_2^{-1},\ \ D_2=C_1^{-1}A_2B_1$$

and

$$x\cdot y=A(A_1^{-1}x,A_2^{-1}y)$$
$$x\circ y=A_2B((A_2B_1)^{-1}x,\ (A_2B_2)^{-1}y)$$
$$x*y=C(C_1^{-1}x,\ C_2^{-1}y).$$

Then it is easy to prove (3) and $\langle S_\circ L_\circ L_\circ{}' S_\circ\rangle$ where $\cdot$ and $*$ have zero 0, $*$ and $\circ$ have unit e and $\circ$ has zero A_2b. Operations $\cdot$, $\circ$, $*$ are compatible iff $A_2b=0$ i.e. $b_2=c$.

(a‴) A is a quasigroup with quasizero $(a,b,0)$, $B(x,y)=b$

C is a quasigroup with quasizero $(d,c,0)$, $D(x,y)=d$.

Let $p, p', r \in S$, $p \neq a$, $p' \neq b$, $r \neq c$ and let B_1 be an arbitrary permutation on S. Since C is a quasigroup with quasizero and $r \neq c$, there is unique solution of equation $C(x, r) = A(p, p')$. Let us denote it by r' and define:

$$A_1 x = A(x, p'),\ \ A_2 x = A(p, x),\ \ e = A(p, p'),\ \ C_1 x = C(x, r),$$
$$C_2 x = C(r', x),\ \ B_2 = A_2^{-1} C_2,\ \ D_1 = C_1^{-1} A_1,\ \ D_2 = C_1^{-1} A_2 B_1$$

and

$$x \cdot y = A(A_1^{-1} x, A_2^{-1} y)$$
$$x \circ y = A_2 B((A_2 B_1)^{-1} x,\ \ (A_2 B_2)^{-1} y)$$
$$x * y = C(C_1^{-1} x,\ \ C_2^{-1} y).$$

It is easy to prove (3), $\langle L_\circ S_\circ L_\circ{}' S_\circ \rangle$ and combatibility of $\cdot$, $\circ$, $*$.

(b) T depends on x only (and (a) does not hold).

We have the following possibilities for A and B:

– $A(x, y) = \alpha x$, B is arbitrary

– A is a quasigroup, $B(x, y) = b$

– A is a quasigroup with quasizero $(a_1, a_2, 0)$, $B(x, y) = b$, $b \neq a_2$.

For C and D there is only one posibility:

– $C(x, y) = \gamma x$, $D(x, y) = \delta x$.

Only for one of 7 solutions, we prove that it is of the requred form.

(b′) A is a quasigroup, $B(x, y) = b$,

$$C(x, y) = \gamma x,\ \ D(x, y) = \delta x,\ \ A(x, b) = \gamma \delta x.$$

Let $p \in S$ and B_1, B_2 be arbitrary permutations on S. We define:

$$C_1 = \gamma,\ \ D_1 = \delta,\ \ A_1 = C_1 D_1,\ \ A_2 = A(p, x),\ \ C_2 = A_2 B_2,$$
$$D_2 = C_1^{-1} A_2 B_1$$

and

$$x \cdot y = A(A_1^{-1} x, A_2^{-1} y)$$
$$x \circ y = A_2 B((A_2 B_1)^{-1} x,\ \ (A_2 B_2)^{-1} y)$$
$$x * y = C(C_1^{-1} x, C_2^{-1} y).$$

Consequenty (3), $\langle L S_\circ A_L A_L \rangle$ and $A_2 b$ is both unit of $\cdot$ and zero of zero semigroup $\circ$.

(c) T depends on y only (and (a) does not hold).

Only one case (of type $\langle A_R A_L A_L A_R \rangle$) is possible.

(d) T depends on z only (and (a) does not hold).

This case is "dual" to the case (b). There are 7 solutions.

(e) T depends on x, y only ((a), (b), (c) does not hold).

There are two possibilities for A and B:

– A is a quasigroup, $B(x,y) = \beta x$

– A is a quasigroup with quasizero $(a,b,0)$, $B(x,y) = \beta x$.

There are two possibilities for C and D:

– $C(x,y) = \gamma x$, D is a quasigroup

– $C(x,y) = \gamma x$, D is a quasigroup with quasizero $(c,d,0')$.

It follows from (1) that $A(x,\beta y) = \gamma D(x,y)$ i.e. A and D are isotopic and consequently both are either quasigroups or quasigroups with quasizero.

There are two solutions (of types $\langle L A_L A_L L \rangle$ and and $\langle L_\circ A_L A_L L_\circ \rangle$).

(f) T depends on x, z only ((a), (b), (d) does not hold).

As in the case (e), A and C are isotopic, so only two solutions (of types $\langle L A_R L A_L \rangle$ and $\langle L_\circ A_R L_\circ A_L \rangle$) remain.

(g) T depends on y, z only ((a), (c), (d) does not hold).

This case is "dual" to the case (e). The types of solutions are $\langle A_R L L A_R \rangle$ and $\langle A_R L_\circ L_\circ A_R \rangle$.

(h) T depends on x, y, z (and previous cases does not hold).

Any one of A, B, C, D can be either a quasigroup or a quasigroup with quasizero. Let $p \in S$ (and $p \neq a_1$ if (a_1, a_2, a) is a quasizero of A). Also, let $A_2 x = A(p,x)$ and $D_2 x = D(p,x)$. It follows from (1) that $A_2 B(y,z) = C(D_2 y, z)$ so D_2 is also a permutation and consequently B and C are isotopic.

Analogously, other operations are isotopic to B and C, so all are simultaneously either quasigroups or quasigroups with quasizero.

In the first case, using procedure from [4] (or [5]), we deduce (3) and $\langle GGGG \rangle$.

Analogously we obtain the solution of type $\langle G_\circ G_\circ G_\circ G_\circ \rangle$.

COROLLARY 2.1. *The general solution of generalized associativity equation* (1) *on* ATG*'s is given by* (2) *where* $A_1, A_2, \ldots, D_2$ *are arbitrary permutations on* S *satisfying* (3) *and* $\cdot$, $\circ$, $*$, $\triangle$ *are arbitrary principal* ATG*'s such that:*

$$x \cdot (y \circ z) = (x \triangle y) * z. \tag{4}$$

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EXAMPLE. Let $S = \{0,1\}$. Then all groupoids on S are ATG. We shall solve the generalized associativity equation on S.

We denote the operations on S in a following way:

x	y	$x0y$	$x\wedge y$	$x\mapsto y$	$x \llcorner y$	$x\mapsfrom y$	xRy	$x+y$	$x\vee y$	$x\downarrow y$	$x\leftrightarrow y$	$x\bar{R}y$	$x\leftarrow y$	$x\, \bar{\llcorner}\, y$	$x\rightarrow y$	$x\uparrow y$	$x1y$
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Repeating the procedure from Th 2 we obtain all solutions of (1). Solutions like $(0,B,0,D)$ and $(0,B,L,0)$ stand for solutions where B and D are substituted by arbitrary operation on S.

(a) $(0,B,0,D)$ $(1,B,1,D)$ $(0,B,\llcorner,0)$ $(0,B,\overline{\llcorner},1)$
$(1,B,\llcorner,1)$ $(1,B,\overline{\llcorner},0)$ $(0,B,\wedge,0)$ $(0,B,\mapsto,0)$
$(0,B,\mapsfrom,1)$ $(0,B,\downarrow,1)$ $(1,B,\vee,1)$ $(1,B,\leftarrow,1)$
$(1,B,\rightarrow,0)$ $(1,B,\uparrow,0)$ $(R,0,0,D)$, $(R,1,1,D)$
$(\bar{R},0,1,D)$ $(\bar{R},1,0,D)$ $(R,0,\llcorner,0)$ $(R,0,\overline{\llcorner},1)$
$(R,1,\llcorner,1)$ $(R,1,\overline{\llcorner},0)$ $(\bar{R},0,\llcorner,1)$ $(\bar{R},0,\overline{\llcorner},0)$
$(\bar{R},1,\llcorner,0)$ $(\bar{R},1,\overline{\llcorner},1)$ $(R,0,\wedge,0)$ $(R,0,\mapsto,0)$
$(R,0,\mapsfrom,1)$ $(R,0,\downarrow,1)$ $(R,1,\vee,1)$ $(R,1,\leftarrow,1)$
$(R,1,\rightarrow,0)$ $(R,1,\uparrow,0)$ $(\bar{R},1,\wedge,0)$ $(\bar{R},1,\mapsto,0)$
$(\bar{R},1,\mapsfrom,1)$ $(\bar{R},1,\downarrow,1)$ $(\bar{R},0,\vee,1)$ $(\bar{R},0,\leftarrow,1)$
$(\bar{R},0,\rightarrow,0)$ $(\bar{R},0,\uparrow,0)$ $(\wedge,0,0,D)$ $(\mapsto,1,0,D)$
$(\mapsfrom,0,0,D)$ $(\downarrow,1,0,D)$ $(\vee,1,1,D)$ $(\leftarrow,0,1,D)$
$(\rightarrow,1,1,D)$ $(\uparrow,0,1,D)$ $(\wedge,0,\llcorner,0)$ $(\wedge,0,\overline{\llcorner},1)$
$(\mapsto,1,\llcorner,0)$ $(\mapsto,1,\overline{\llcorner},1)$ $(\mapsfrom,0,\llcorner,0)$ $(\mapsfrom,0,\overline{\llcorner},1)$
$(\downarrow,1,\llcorner,0)$ $(\downarrow,1,\overline{\llcorner},1)$ $(\vee,1,\llcorner,1)$ $(\vee,1,\overline{\llcorner},0)$
$(\leftarrow,0,\llcorner,1)$ $(\leftarrow,0,\overline{\llcorner},0)$ $(\rightarrow,1,|,1)$ $(\rightarrow,1,\overline{\llcorner},0)$
$(\uparrow,0,\llcorner,1)$ $(\uparrow,0\overline{\llcorner},0)$ $(\wedge,0,\wedge,0)$ $(\wedge,0,\mapsto,0)$
$(\wedge,0,\mapsfrom,1)$ $(\wedge,0,\downarrow,1)$ $(\mapsto,1,\wedge,0)$ $(\mapsto,1,\mapsto,0)$
$(\mapsto,1,\mapsfrom,1)$ $(\mapsto,1,\downarrow,1)$ $(\mapsfrom,0,\wedge,0)$ $(\mapsfrom,0,\mapsto,0)$
$(\mapsfrom,0,\mapsfrom,1)$ $(\mapsfrom,0,\downarrow,1)$ $(\downarrow,1,\wedge,0)$ $(\downarrow,1,\mapsto,0)$
$(\downarrow,1,\mapsfrom,1)$ $(\downarrow,1,\downarrow,1)$ $(\vee,1,\vee,1)$ $(\vee,1,\leftarrow,1)$
$(\vee,1,\rightarrow,0)$ $(\vee,1,\uparrow,0)$ $(\leftarrow,0,\vee,1)$ $(\leftarrow,0,\leftarrow,1)$
$(\leftarrow,0,\rightarrow,0)$ $(\leftarrow,0,\uparrow,0)$ $(\rightarrow,1,\vee,1)$ $(\rightarrow,1,\leftarrow,1)$
$(\rightarrow,1,\rightarrow,0)$ $(\rightarrow,1,\uparrow,0)$ $(\uparrow,0,\vee,1)$ $(\uparrow,0,\leftarrow,1)$
$(\uparrow,0,\rightarrow,0)$ $(\uparrow,0,\uparrow,0)$

(b)	$(\mathsf{L}, B, \mathsf{L}, \mathsf{L})$	$(\mathsf{L}, B, \overline{\mathsf{L}}, \overline{\mathsf{L}})$	$(\overline{\mathsf{L}}, B, \mathsf{L}, \overline{\mathsf{L}})$	$(\overline{\mathsf{L}}, B, \overline{\mathsf{L}}, \mathsf{L})$
	$(+, 0, \mathsf{L}, \mathsf{L})$	$(+, 0, \overline{\mathsf{L}}, \overline{\mathsf{L}})$	$(+, 1, \mathsf{L}, \overline{\mathsf{L}})$	$(+, 1, \overline{\mathsf{L}}, \mathsf{L})$
	$(\leftrightarrow, 0, \mathsf{L}, \mathsf{L})$	$(\leftrightarrow, 0, \overline{\mathsf{L}}, \mathsf{L})$	$(\leftrightarrow, 1, \mathsf{L}, \mathsf{L})$	$(\leftrightarrow, 1, \overline{\mathsf{L}}, \overline{\mathsf{L}})$
	$(\wedge, 1, \mathsf{L}, \mathsf{L})$	$(\wedge, 1, \overline{\mathsf{L}}, \overline{\mathsf{L}})$	$(\mapsto, 0, \mathsf{L}, \mathsf{L})$,	$(\mapsto, 0, \overline{\mathsf{L}}, \overline{\mathsf{L}})$
	$(\mapsfrom, 1, \mathsf{L}, \overline{\mathsf{L}})$	$(\mapsfrom, 1, \overline{\mathsf{L}}, \mathsf{L})$	$(\downarrow, 0, \mathsf{L}, \overline{\mathsf{L}})$	$(\downarrow, 0, \overline{\mathsf{L}}, \mathsf{L})$
	$(\vee, 0, \mathsf{L}, \mathsf{L})$	$(\vee, 0, \overline{\mathsf{L}}, \overline{\mathsf{L}})$	$(\mapsfrom, 1, \mathsf{L}, \mathsf{L})$	$(\mapsfrom, 1, \overline{\mathsf{L}}, \overline{\mathsf{L}})$
	$(\rightarrow, 0, \mathsf{L}, \overline{\mathsf{L}})$	$(\rightarrow, 0, \overline{\mathsf{L}}, \mathsf{L})$	$(\uparrow, 1, \mathsf{L}, \overline{\mathsf{L}})$	$(\uparrow, 1, \overline{\mathsf{L}}, \mathsf{L})$
(c)	$(R, \mathsf{L}, \mathsf{L}, R)$	$(R, \mathsf{L}, \overline{\mathsf{L}}, \bar{R})$	$(R, \overline{\mathsf{L}}, \mathsf{L}, \bar{R})$	$(R, \overline{\mathsf{L}}, \overline{\mathsf{L}}, R)$
	$(\bar{R}, \mathsf{L}, \mathsf{L}, \bar{R})$	$(\bar{R}, \mathsf{L}, \overline{\mathsf{L}}, R)$	$(\bar{R}, \overline{\mathsf{L}}, \mathsf{L}, R)$	$(\bar{R}, \overline{\mathsf{L}}, \overline{\mathsf{L}}, \bar{R})$
(d)	(R, R, R, D)	$(R, \bar{R}, \bar{R}, D)$	$(\bar{R}, R, \bar{R}, D)$	$(\bar{R}, \bar{R}, R, D)$
	$(R, R, +, 0)$	$(R, R, \leftrightarrow, 1)$	$(R, \bar{R}, +, 1)$	$(R, \bar{R}, \leftrightarrow, 0)$
	$(\bar{R}, R, +, 1)$	$(\bar{R}, R, \leftrightarrow, 0)$	$(\bar{R}, \bar{R}, +, 0)$	$(\bar{R}, \bar{R}, \leftrightarrow, 1)$
	$(R, R, \wedge, 1)$	$(R, R, \leftarrow, 0)$	$(R, R, \vee, 0)$,	$(R, R, \rightarrow, 1)$
	$(R, \bar{R}, \mapsto, 1)$	$(R, \bar{R}, \downarrow, 0)$	$(R, \bar{R}, \leftarrow, 0)$	$(R, \bar{R}, \uparrow, 1)$
	$(\bar{R}, R, \mapsto, 1)$	$(\bar{R}, R, \downarrow, 0)$	$(\bar{R}, R, \leftarrow, 0)$	$(\bar{R}, R, \uparrow, 1)$
	$(\bar{R}, \bar{R}, \wedge, 1)$	$(\bar{R}, \bar{R}, \leftarrow, 0)$	$(\bar{R}, \bar{R}, \vee, 0)$	$(\bar{R}, \bar{R}, \rightarrow, 1)$
(e)	$(+, \mathsf{L}, \mathsf{L}, +)$	$(+, \mathsf{L}, \overline{\mathsf{L}}, \leftrightarrow)$	$(+, \overline{\mathsf{L}}, \mathsf{L}, \leftrightarrow)$	$(+, \overline{\mathsf{L}}, \overline{\mathsf{L}}, +)$
	$(\leftrightarrow, \mathsf{L}, \mathsf{L}, \leftrightarrow)$	$(\leftrightarrow, \mathsf{L}, \overline{\mathsf{L}}, +)$	$(\leftrightarrow, \overline{\mathsf{L}}, \mathsf{L}, +)$	$(\leftrightarrow, \overline{\mathsf{L}}, \overline{\mathsf{L}}, \leftrightarrow)$
	$(\wedge, \mathsf{L}, \mathsf{L}, \wedge)$	$(\wedge, \mathsf{L}, \overline{\mathsf{L}}, \uparrow)$	$(\wedge, \overline{\mathsf{L}}, \mathsf{L}, \mapsto)$	$(\wedge, \overline{\mathsf{L}}, \overline{\mathsf{L}}, \rightarrow)$
	$(\mapsto, \mathsf{L}, \mathsf{L}, \mapsto)$	$(\mapsto, \mathsf{L}, \overline{\mathsf{L}}, \rightarrow)$	$(\mapsto, \overline{\mathsf{L}}, \mathsf{L}, \wedge)$,	$(\mapsto, \overline{\mathsf{L}}, \overline{\mathsf{L}}, \uparrow)$
	$(\mapsfrom, \mathsf{L}, \mathsf{L}, \mapsfrom)$	$(\mapsfrom, \mathsf{L}, \overline{\mathsf{L}}, \leftarrow)$	$(\mapsfrom, \overline{\mathsf{L}}, \mathsf{L}, \downarrow)$	$(\mapsfrom, \overline{\mathsf{L}}, \overline{\mathsf{L}}, \vee)$
	$(\downarrow, \mathsf{L}, \mathsf{L}, \downarrow)$	$(\downarrow, \mathsf{L}, \overline{\mathsf{L}}, \vee)$	$(\downarrow, \overline{\mathsf{L}}, \mathsf{L}, \mapsfrom)$	$(\downarrow, \overline{\mathsf{L}}, \overline{\mathsf{L}}, \leftarrow)$
	$(\vee, \mathsf{L}, \mathsf{L}, \vee)$	$(\vee, \mathsf{L}, \overline{\mathsf{L}}, \downarrow)$	$(\vee, \overline{\mathsf{L}}, \mathsf{L}, \leftarrow)$	$(\vee, \overline{\mathsf{L}}, \overline{\mathsf{L}}, \mapsfrom)$
	$(\leftarrow, \mathsf{L}, \mathsf{L}, \leftarrow)$	$(\leftarrow, \mathsf{L}, \overline{\mathsf{L}}, \mapsfrom)$	$(\leftarrow, \overline{\mathsf{L}}, \mathsf{L}, \vee)$	$(\leftarrow, \overline{\mathsf{L}}, \overline{\mathsf{L}}, \downarrow)$
	$(\rightarrow, \mathsf{L}, \mathsf{L}, \rightarrow)$	$(\rightarrow, \mathsf{L}, \overline{\mathsf{L}}, \mapsto)$	$(\rightarrow, \overline{\mathsf{L}}, \mathsf{L}, \uparrow)$	$(\rightarrow, \overline{\mathsf{L}}, \overline{\mathsf{L}}, \wedge)$
	$(\uparrow, \mathsf{L}, \mathsf{L}, \uparrow)$	$(\uparrow, \mathsf{L}, \overline{\mathsf{L}}, \wedge)$	$(\uparrow, \overline{\mathsf{L}}, \mathsf{L}, \rightarrow)$	$(\uparrow, \overline{\mathsf{L}}, \overline{\mathsf{L}}, \mapsto)$
(f)	$(+, R, +, \mathsf{L})$	$(+, R, \leftrightarrow, \overline{\mathsf{L}})$	$(+, \bar{R}, \leftrightarrow, \mathsf{L})$	$(+, \bar{R}, +, \overline{\mathsf{L}})$
	$(\leftrightarrow, R, \leftrightarrow, \mathsf{L})$	$(\leftrightarrow, R, +, \overline{\mathsf{L}})$	$(\leftrightarrow, \bar{R}, +, \mathsf{L})$	$(\leftrightarrow, \bar{R}, \leftrightarrow, \overline{\mathsf{L}})$
	$(\wedge, R, \wedge, \mathsf{L})$	$(\wedge, R, \mapsfrom, \overline{\mathsf{L}})$	$(\wedge, \bar{R}, \mapsto, \mathsf{L})$	$(\wedge, \bar{R}, \downarrow, \overline{\mathsf{L}})$
	$(\mapsto, R, \mapsto, \mathsf{L})$	$(\mapsto, R, \downarrow, \overline{\mathsf{L}})$	$(\mapsto, \bar{R}, \wedge, \mathsf{L})$,	$(\mapsto, \bar{R}, \mapsfrom, \overline{\mathsf{L}})$
	$(\mapsfrom, R, \mapsfrom, \mathsf{L})$	$(\mapsfrom, R, \wedge, \overline{\mathsf{L}})$	$(\mapsfrom, \bar{R}, \downarrow, \mathsf{L})$	$(\mapsfrom, \bar{R}, \mapsto, \overline{\mathsf{L}})$
	$(\downarrow, R, \downarrow, \mathsf{L})$	$(\downarrow, R, \mapsto, \overline{\mathsf{L}})$	$(\downarrow, \bar{R}, \mapsfrom, \mathsf{L})$	$(\downarrow, \bar{R}, \wedge, \overline{\mathsf{L}})$
	$(\vee, R, \vee, \mathsf{L})$	$(\vee, R, \rightarrow, \overline{\mathsf{L}})$	$(\vee, \bar{R}, \leftarrow, \mathsf{L})$	$(\vee, \bar{R}, \uparrow, \overline{\mathsf{L}})$
	$(\leftarrow, R, \leftarrow, \mathsf{L})$	$(\leftarrow, R, \uparrow, \overline{\mathsf{L}})$	$(\leftarrow, \bar{R}, \vee, \mathsf{L})$	$(\leftarrow, \bar{R}, \rightarrow, \overline{\mathsf{L}})$

	$(\rightarrow, R, \rightarrow, \llcorner)$	$(\rightarrow, R, \vee, \overline{\llcorner})$	$(\rightarrow, \bar{R}, \uparrow, \llcorner)$	$(\rightarrow, \bar{R}, \leftarrow, \overline{\llcorner})$
	$(\uparrow, R, \uparrow, \llcorner)$	$(\uparrow, R, \leftarrow, \overline{\llcorner})$	$(\uparrow, \bar{R}, \rightarrow, \llcorner)$	$(\uparrow, \bar{R}, \vee, \overline{\llcorner})$
(g)	$(R, +, +, R)$	$(R, +, \nleftrightarrow, \bar{R})$	$(R, \nleftrightarrow, +, \bar{R})$	$(R, \nleftrightarrow, \nleftrightarrow, R)$
	$(\bar{R}, +, +, \bar{R})$	$(\bar{R}, +, \nleftrightarrow, R)$	$(\bar{R}, \nleftrightarrow, +, R)$	$(\bar{R}, \nleftrightarrow, \nleftrightarrow, \bar{R})$
	$(R, \wedge, \wedge, R)$	$(R, \wedge, \mapsfrom, \bar{R})$	$(R, \mapsto, \mapsto, R)$	$(R, \mapsto, \downarrow, \bar{R})$
	$(R, \mapsfrom, \mapsfrom, R)$	$(R, \mapsfrom, \wedge, \bar{R})$	$(R, \downarrow, \downarrow, R)$	$(R, \downarrow, \mapsto, \bar{R})$
	$(R, \vee, \vee, R)$	$(R, \vee, \rightarrow, \bar{R})$	$(R, \leftarrow, \leftarrow, R)$	$(R, \leftarrow, \uparrow, \bar{R})$
	$(R, \rightarrow, \rightarrow, R)$	$(R, \rightarrow, \vee, \bar{R})$	$(R, \uparrow, \uparrow, R)$	$(R, \uparrow, \leftarrow, \bar{R})$
	$(\bar{R}, \wedge, \uparrow, R)$	$(\bar{R}, \wedge, \leftarrow, \bar{R})$	$(\bar{R}, \mapsto, \rightarrow, R)$	$(\bar{R}, \mapsto, \vee, \bar{R})$
	$(\bar{R}, \mapsfrom, \leftarrow, R)$	$(\bar{R}, \mapsto, \uparrow, \bar{R})$	$(\bar{R}, \downarrow, \vee, R)$	$(\bar{R}, \downarrow, \rightarrow, \bar{R})$
	$(\bar{R}, \vee, \downarrow, R)$	$(\bar{R}, \vee, \mapsto, \bar{R})$	$(\bar{R}, \leftarrow, \mapsfrom, R)$	$(\bar{R}, \leftarrow, \wedge, \bar{R})$
	$(\bar{R}, \rightarrow, \mapsto, R)$	$(\bar{R}, \rightarrow, \downarrow, \bar{R})$	$(\bar{R}, \uparrow, \wedge, R)$	$(\bar{R}, \uparrow, \mapsfrom, \bar{R})$
(h)	$(+, +, +, +)$	$(+, +, \nleftrightarrow, \nleftrightarrow)$	$(+, \nleftrightarrow, +, \nleftrightarrow)$	$(+, \nleftrightarrow, \nleftrightarrow, +)$
	$(\nleftrightarrow, +, +, \nleftrightarrow)$	$(\nleftrightarrow, +, \nleftrightarrow, +)$	$(\nleftrightarrow, \nleftrightarrow, +, +)$	$(\nleftrightarrow, \nleftrightarrow, \nleftrightarrow, \nleftrightarrow)$
	$(\wedge, \wedge, \wedge, \wedge)$	$(\wedge, \wedge, \mapsfrom, \uparrow)$	$(\wedge, \mapsto, \mapsto, \wedge)$	$(\wedge, \mapsto, \downarrow, \uparrow)$
	$(\wedge, \mapsfrom, \wedge, \mapsto)$	$(\wedge, \mapsfrom, \mapsfrom, \rightarrow)$	$(\wedge, \downarrow, \mapsto, \mapsto)$	$(\wedge, \downarrow, \downarrow, \rightarrow)$
	$(\mapsto, \vee, \mapsto, \mapsto)$	$(\mapsto, \vee, \downarrow, \rightarrow)$	$(\mapsto, \leftarrow, \wedge, \mapsto)$	$(\mapsto, \leftarrow, \mapsfrom, \rightarrow)$
	$(\mapsto, \rightarrow, \mapsto, \wedge)$	$(\mapsto, \rightarrow, \downarrow, \uparrow)$	$(\mapsto, \uparrow, \wedge, \wedge)$	$(\mapsto, \uparrow, \mapsfrom, \uparrow)$
	$(\mapsfrom, \wedge, \wedge, \mapsfrom)$	$(\mapsfrom, \wedge, \mapsfrom, \leftarrow)$	$(\mapsfrom, \mapsto, \mapsto, \mapsfrom)$	$(\mapsfrom, \mapsto, \downarrow, \leftarrow)$
	$(\mapsfrom, \mapsfrom, \wedge, \downarrow)$	$(\mapsfrom, \mapsfrom, \mapsfrom, \vee)$	$(\mapsfrom, \downarrow, \mapsto, \downarrow)$	$(\mapsfrom, \downarrow, \downarrow, \vee)$
	$(\downarrow, \vee, \mapsto, \downarrow)$	$(\downarrow, \vee, \downarrow, \vee)$	$(\downarrow, \leftarrow, \wedge, \downarrow)$	$(\downarrow, \leftarrow, \leftarrow, \vee)$
	$(\downarrow, \rightarrow, \mapsto, \mapsfrom)$	$(\downarrow, \rightarrow, \downarrow, \leftarrow)$	$(\downarrow, \uparrow, \wedge, \mapsfrom)$	$(\downarrow, \uparrow, \mapsfrom, \leftarrow)$
	$(\vee, \vee, \vee, \vee)$	$(\vee, \vee, \rightarrow, \downarrow)$	$(\vee, \leftarrow, \leftarrow, \vee)$	$(\vee, \leftarrow, \uparrow, \downarrow)$
	$(\vee, \rightarrow, \vee, \leftarrow)$	$(\vee, \rightarrow, \rightarrow, \mapsfrom)$	$(\vee, \uparrow, \leftarrow, \leftarrow)$	$(\vee, \uparrow, \uparrow, \mapsfrom)$
	$(\leftarrow, \wedge, \leftarrow, \leftarrow)$	$(\leftarrow, \wedge, \uparrow, \mapsfrom)$	$(\leftarrow, \mapsto, \vee, \leftarrow)$	$(\leftarrow, \mapsto, \rightarrow, \mapsfrom)$
	$(\leftarrow, \mapsfrom, \mapsfrom, \vee)$	$(\leftarrow, \mapsfrom, \uparrow, \downarrow)$	$(\leftarrow, \downarrow, \downarrow, \vee)$	$(\leftarrow, \downarrow, \rightarrow, \downarrow)$
	$(\rightarrow, \vee, \vee, \rightarrow)$	$(\rightarrow, \vee, \rightarrow, \mapsto)$	$(\rightarrow, \leftarrow, \leftarrow, \rightarrow)$	$(\rightarrow, \leftarrow, \uparrow, \mapsto)$
	$(\rightarrow, \rightarrow, \mapsfrom, \uparrow)$	$(\rightarrow, \rightarrow, \rightarrow, \wedge)$	$(\rightarrow, \uparrow, \leftarrow, \uparrow)$	$(\rightarrow, \uparrow, \uparrow, \wedge)$
	$(\uparrow, \wedge, \leftarrow, \uparrow)$	$(\uparrow, \wedge, \uparrow, \wedge)$	$(\uparrow, \mapsto, \vee, \uparrow)$	$(\uparrow, \mapsto, \rightarrow, \wedge)$
	$(\uparrow, \mapsfrom, \leftarrow, \rightarrow)$	$(\uparrow, \mapsfrom, \uparrow, \mapsto)$	$(\uparrow, \downarrow, \vee, \rightarrow)$	$(\uparrow, \downarrow, \rightarrow, \mapsto)$

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