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A FIXED POINT THEOREM IN A REFLEXIVE BANACH SPACE

Zvonimir Mavar

In [**1**] the following theorem is proved:

THEOREM A *Let B a reflexive Banach space, K a nonempty bounded closed and convex subset of B and $T : K \to K$ a mapping satisfying the following conditions:*

$$\|Tx - Ty\| \leq \max\{\|x - Tx\|, \|y - Ty\|, (\|x - ty\| + \|y - Tx\|)/3, \\ (\|x - y\| + \|x - Tx\| + \|yTy\|)/3\}, \ \ x, y \in K$$

and

$$\sup_{z \in D} \|z - Tz\| \leq \delta(D)/2,$$

where D is any nonempty closed convex subset of K which is mapped into itself by T and $\delta(D) = \sup_{x,y \in D} \|x - y\|$ the diameter of D. Then T has a unique fixed point in K.

In the present note we shall prove a theorem which is certain generalization of Theorem A, and its proof is simpler than that of Theorem A. Namely, we have the following:

THEOREM 1. *Let B a reflexive Banach space, K a nonempty bounded closed and convex subset of B and $T : K \to K$ a mapping satisfying the following conditions:*

$$\begin{aligned} \|Tx - Ty\| \leq \max\{\|x - Tx\|, \|y - Ty\|, a\|x - Ty\| + b\|y - Tx\|, \\ (\|x - y\| + \|x - Tx\| + \|y - Ty\|)/3 \\ x, y \in K, \ \ a \geq 0, b \geq 0, a + b < 1 \end{aligned} \tag{1}$$

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$$\sup_{z\in D}\|z-Tz\| \le r\delta(D),\ \ 0\le r=r(D)<1, \tag{2}$$

where D and $\delta(D)$ have the same meaning as in Theorem A. Then T has a unique fixed point in K.

Proof. Let $\mathcal{F}$ denote the family of all nonempty bounded closed convex subsets of K, which are mapped by T into itself. $\mathcal{F}$ is nonempty since $K\in\mathcal{F}$. If $\|F_\alpha\}$ is any nonincreasing sequence in $\mathcal{F}$, then by the well known result [**2**] of Smulian, $F=\bigcap_\alpha F_\alpha$ is in $\{$. Now, by Zorn's lemma it follows that $\mathcal{F}$ has a minimal element. If C is such a minimal element of $\mathcal{F}$, we shall prove that C contains only one point, i.e. that T has a fixed point in K. Supposing that C contains more than one element we obtain

$$\sup_{x,y\in C}\|x-y\| = \delta(C) > 0 \tag{3}$$

Since $T(C)\subseteq C$, for any $x,y\in C$ we have, by (1), (2) and (3)

$$\|Tx-Ty\| \le \max\{r\delta(C),(a+b)\delta(C),(\delta(C)+2r\delta(C))/3\}$$

Putting $\overline{r}=\max\{a+b,(1+2r)/3\}<1$ we have

$$\|Tx-Ty\| \le \{r\delta(C),(\overline{r}<1)\ \text{ for each }\ x,y\in C. \tag{4}$$

If by $\operatorname{co}D$ we denote the convex hull of D, and by $\overline{\operatorname{co}}D$ the closed convex hull of D we have

$$\overline{\operatorname{co}}T(C)\subseteq\overline{\operatorname{co}}C=\overline{C}=C \tag{5}$$

because $C\in\mathcal{F}$, $T(C)\subseteq C$ and C is closed and convex. Therefore

$$T(\overline{\operatorname{co}}T(C))\subseteq T(C)\subseteq\operatorname{co}T(C)\subseteq\overline{\operatorname{co}}T(C). \tag{6}$$

Since C is a minimal element of $\mathcal{F}$, by (5) and (6) we have $\overline{\operatorname{co}}T(C)=C$. Let $\overline{x},\overline{y}\in\operatorname{co}T(C)$. Then we can write

$$\overline{x}=\sum_{i=1}^{n}a_iTx_i,\ a_i\ge 0\,(i=1,\dots,n),\quad \sum_{i=1}^{n}a_i=1,\ x_i\in C$$
$$\overline{y}=\sum_{j=1}^{m}b_jTy_j,\ b_j\ge 0\,(j=1,\dots,m),\quad \sum_{j=1}^{n}b_j=1,\ y_j\in C$$

Now, by (4), and $\sum_{i,j}a_ib_j=1$

$$\|\overline{x}-\overline{y}\| = \left\|\sum_{i=1}^{n}a_iTx_i-\sum_{j=1}^{m}b_iTy_j\right\| = \left\|\sum_{i,j}a_ib_jTx_i-\sum_{i,j}a_ib_jTy_j\right\| =$$
$$= \left\|\sum_{i,j}a_ib_j(Tx_i-Ty_j)\right\| \le \sum_{i,j}a_ib_j\overline{r}\delta(C)=\overline{r}\delta(C)$$

Hence $\|\overline{x}-\overline{y}\| \le \overline{r}\delta(C)$, for every $\overline{x},\overline{y} \in \operatorname{co} T(C)$ and therefore

$$\sup_{\overline{x},\overline{y}\in \operatorname{co} T(C)} \|\overline{x}-\overline{y}\| \le \overline{r}\delta(C) \tag{7}$$

Now, $\delta(C) = \sup_{x,y\in C} \|x-y\|$ and $\overline{\operatorname{co}} T(C) = C$ implies

$$\delta(C) = \sup_{\overline{x},\overline{y}\in \operatorname{co} T(C)} \|x-y\|, \text{ and by (7) we obtain } \delta \le \overline{r}\delta.$$

where $\overline{r} < 1$ and $\delta > 0$. This contradiction proves that C contains only one point, i.e. that T has a fixed point in K. Now we shall complete the proof demonstrating that the fixed point of T is unique. Let x_0 and y_0 be two fixed points of T.

Then by (1) we have

$$\begin{aligned}\|x_0 - y_0\| = \|Tx_0 - Ty_0\| \le \max\{&\|x_0 - Tx_0\|, \|y_0 - Ty_0\|,\\ &a\|x_0 - Ty_0\| + b\|y_0 - Tx_0\|,\\ &(\|x_0 - y_0\| + \|x_0 - Tx_0\| + \|y_0 - Ty_0\|)/3\}\\ = \max\{&0, 0, (a+b)\|x_0 - y_0\|, \|x_0 - y_0\|/3\}\end{aligned}$$

From this inequality immediately follows $x_0 = y_0$.

REFERENCES

[1] LJ. B. Ćirić, *On fixed point theorems in Banach space*, Publ. inst. Math. (Beograd) (N. S.) **19 (33)** (1975), 43–50.

[2] V. Smulian, *On the principle of inclusion in the space of type (B)*, Math. Sb. **5** (1939)

Mašinski fakultet
79000 Mostar
Jugoslavija

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