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FINSLER SPACE WITH RUND'S h-CURVATURE TENSOR $\mathbf{K}^i_{ljk}$ OF A SPECIAL FORM

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Abstract. We propose a special form of Rund's h-curvature tensor K^i_{ljk} and deal with some special Finsler spaces characterized by such a special form of K^i_{ljk}.

1. Introduction. Let us consider an n-dimensional Finsler space $F_n (n \geq 3)$, equipped with a metric tensor $g_{ij}(x, y)$, $(y = x)$ and a metric function $L(x, y)$. The h-covariant, v-covariant and o-covariant derivatives are denoted by $|j, |j$ and $||j$ respectively.

A Finsler space is called h-isotropic [**2**] if Cartan's h-curvature tensor R_{hijk} is written in the form

$$R_{hijk} = R(g_{hj}g_{ik} - g_{hk}g_{ij}), \tag{1.1}$$

where R is non-zero scalar curvature. We have

$$R_{iok} = g_{ir}R^r_{ok} = RL^2 h_{ik}, \tag{1.2,}$$

where $h_{ik} = g_{ik} - l_i l_k$ is the angular metric tensor, R^i_{jk} is the (v) h-torsion tensor and the suffix '0' means contraction with y^i.

A C-reducible Finsler space [**3**] is defined by

$$C_{ijk} = (C_i h_{jk} + C_j h_{ki} + C_k h_{ij})/(n + 1) \tag{1.3}$$

where $C_{ijk} \overset{\text{def}}{=} \dot{\partial}_k g_{ij}/2 = g_{ij||k}/2$,

$$C_i = C_{ijk}g^{jk}, \ \ C_i y^i = 0, \tag{1.4}$$

A P-reducible Finsler space [**6**] is defined by

$$P_{ijk} = G_i h_{jk} + G_j h_{ki} + G_k h_{ij} \tag{1.5}$$

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where

$$P_{ijk} = C_{ijk\,|\,0} \text{ and } G_i = C_{i\,|\,0}/(n+1). \tag{1.6}$$

A Finsler space is called Landsberg if $C_{ijk\,|\,o} = 0$.

It is well known [**1**,**5**] that

(1.7) a) $R^i_{jk\|l} - K^i_{l_{jk}}\{P^i_{j_r}P^r_{k_l} + P^i_{k\,l\,|\,j}\} = 0$

b) $K^i_{l_{jk}} = R^i_{l_{jk}} - C^i_{lr}R^r_{jk}$, c) $R^i_{ojk} = R^i_{jk}$,

d) $K^i_{l_{jk}} + K^i_{j_{kl}} + K^i_{k_{lj}} = 0$,

e) $H^i_{ljk} = R^i_{jk\|l}$, f) $H^i_{ljk} + H^i_{j\,kl} + H^i_{k\,lj} = 0$,

g) $K^i_{ljk} = -K^i_{l\,kj}$, $K_{lijk} = g_{ir}K^r_{l\,jk}$

h) $H^i_{ljk} = K^i_{ljk} + U_{(jk)}\{C^i_{jl|o|k} + C^i_{kr|o}C^r_{jl|o}\}$,

where $U_{(jk)}\{A_jB_k\} = A_jB_k - A_kB_j$. The angular metric tensors h_{ij} and h^i_j ($= g^{ik}h_{kj}$) have the following covariant differentiations

a) $h_{ij\,|\,k} = 0$, $h_{ij\,|\,k} = 0$

(1.8) b) $h_{ij\|k} = 2C_{ijk} - L^{-2}(y_ih_{jk} + y_jh_{ik})$

c) $h^i_{j\|k} = -L^{-2}(y_jh^i_k + y^ih_{jk})$.

We shall use the following lemma proved by Matsumoto [**4**].

LEMMA. *If the equation $v_{hi}h_{jk} + v_{ij}h_{hk} + v_{jh}h_{ik} = 0$ holds in F_n, then we have* (1) $v_{ij} = 0$, $(n \geq 4)$ *and* (2) $v_{ij} = v(m_in_j - m_jn_i)$, *with reference to Moor frame* (l^i, m^i, n^i) *where* v *is a scalar.*

2. Special form of Rund's h-curvature tensor K^i_{ljk}. Let F_n be a Finsler space with Rund's h-curvature tensor K^i_{ljk} of the form

$$K^i_{ljk} = U_{(jk)}\{A_{jk}h^i_l + B_{jl}h^i_k + D^i_kh_{ji}\} \tag{2.1}$$

where A_{jk}, B_{jl} and D^i_k are Finsler tensor fields.

THEOREM 2.1. *A P-reducible Finsler space of non zero scalar curvature has the special form* (2.1) *of Rund's h-curvature tensor K^i_{ljk}.*

Proof. From (1.7 a)

$$K^i_{ljk} = R^i_{jk\|l} + U_{(jk)}\{P^i_{jr}P^r_{kl} + P^i_{kl\,|\,j}\} \tag{2.2}$$

since $R^i_{jk} = (R^i_{ok\|j} - R^i_{oj\|k})/3$.

From (1.2) and (1.8 c) we get

$$R^i_{jk} = a_jh^i_k - a_kh^i_j \tag{2.3}$$

where $a_j = (R_{\|j}L^2 + 3Ry_j)/3$.

From (1.5) and (2.3) we have

$$P^i_{kl\,|\,j} = G_{k\,|\,j}h^i_l + G_{l\,|\,j}h^i_k + G^i_{|\,j}h_{kl} \tag{2.4}$$

(2.5)
$$R^i_{jk\|l} = L^{-2}(a_k y_j - a_j y_k)h^i_l + a_{j\|l}h^i_k - a_{k\|l}h^i_j + L^{-2}a_k y^i h_{jl} - L^{-2}a_j y^i h_{kl}$$

respectively. Using (1.5), (2.4) and (2.5) in (2.1), we get

(2.6)
$$\begin{aligned} K^i_{ljk} = &\{(L^{-2}a_k y_j + G_{k\,|\,j}) - (L^{-2}a_j y_k + G_{j\,|\,k})\}h^i_l + (a_{j\|l} + G_{l\,|\,j} - G_j G_l)h^i_k \\ &- (a_{k\|l} + G_{l\,|\,k} - G_k G_l)h^i_j + (L^{-2}a_k y^i - M^i_k)h_{jl} - (L^{-2}a_i y^i - M^i_j)h_{kl}, \end{aligned}$$

where $M^i_k = G_k G^i + G_r G^r h^i_k + G^i_{|\,k}$.

The equation (2.6) reduces to the form (2.1) with

$$A_{jk} = L^{-2}a_k y_j + G_{k\,|\,j},\;\; B_{jl} = a_{j\|l} + G_{l\,|\,j} - G_j G_l,\;\; D^i_k = L^{-2}a_k y^i - M^i_k.$$

COROLLARY 1. *A Landsberg space of non-zero scalar curvature has the special form* (2.1) *of Rund's h-curvature tensor.*

Proof. For a Landsberg space $P_{ijk} = 0$; hence from (1.7 a) and (2.5) we have (2.1) with

$$A_{jk} = L^{-2}a_k y_j, \quad B_{jl} = a_{j\|l}, \quad D^i_k = L^{-2}a_k y^i.$$

THEOREM 2.2. *An h-isotropic and C-reducible Finsler space G_n has the special form* (2.1) *of Rund's h-curvature tensor K^i_{ljk}.*

Proof. If the space is h-isotropic and C-reducible then by (1.1) and (1.3) the equation (1.7 b) takes the form

(2.7)
$$\begin{aligned} K^i_{ljk} =&(C_j y_k - C_k y_j)h^i_l/(n+1) + R(h_{lj} + l_l l_j - C_l y_j/(n+1))h^i_k \\ &-R(h_{lk} + l_l l_k - C_l y_k/(n+1))h^i_j + R(l^i l_k + C^i y_k/(n+1))h_{lj} \\ &-R(l^i l_j + C^i y_j/(n+1))h_{lk}, \end{aligned}$$

which reduces to the form (2.1) with

$$A_{jk} = R(C_j y_k)/(n+1),\; B_{jl} = R(h_{ij} + l_i l_j - C_l y_j/(n+1)),\; D^i_k = R(l^i l_k + C^i y_k/(n+1)).$$

THEOREM 2.3. *If the Rund's h-curvature tensor has the special form* (2.1), *then F_n is a space of scalar curvature $B_{oo}L^{-2}$.*

Proof. From (1.7b) and (2.1) we have

$$R^i_{ljk} - C^i_{ir}R^r_{jk} = U_{(jk)}\{A_{jk}h^i_l + B_{jl}h^i_k + D^i_k h_{jl}\}. \tag{2.8}$$

Transvecting (2.8) by y^l we get

$$R^i_{jk} = B_{jo}h^i_k - B_{ko}h^i_j. \tag{2.9}$$

Tarnsvecting (2.9) by y^j we get

$$R^i_{ok} = B_{oo} h^i_k. \tag{2.10}$$

Comparing (2.10) with (1.2) we get $R = B_{oo} L^{-2}$.

THEOREM 2.4. *A C-reducible Finsler space satisfying* (2.1) *has Cartan's h-curvature tensor* R^i_{ljk} *of the form*

$$R^i_{ljk} = U_{(jk)}\{Q_{jk} h^i_l + N_{jl} h^i_k + E^i_k h_{jl}\}. \tag{2.11}$$

Proof. Using (1.3), (2.1) and (2.9) in (1.7 b) we get

(2.12)

$$\begin{aligned} R^i_{ljk} &= U_{(jk)}\{(A_{(jk)} + C_j B_{ko}/(n+1))h^i_l + (B_{jl} + C_l B_{jo}/n+1))h^i_k \\ &\quad + (D^i_k - C^i B_{ko}/(n+1))h_{jl}\} \end{aligned}$$

which reduces to the form (2.11) with

$$Q_{jk} = A_{jk} + C_j B_{ko}/(n+1),\ N_{jl} = B_{jl} + C_l B_{jo}/(n+1),\ E^i_k = D^i_k - C^i B_{ko}/(n+1)$$

THEOREM 2.5. *If Rund's h-curvature tensor* K^i_{ljk} *is of the form* (2.1), *then we have*

(2.13)

$$\begin{aligned} &(a)\ A_{jk} - A_{kj} = B_{jk} - B_{kj}, \quad b)\ B_{jo} = (R_{\|j} L^2 + 3R y_j)/3, \\ &(c)\ (B_{jo\|k} - B_{ko\|j}) g^{jk} = 0. \end{aligned}$$

Proof. Using the Bianchi identity (1.7d) in (2.1) we get

$$(A_{jk} - A_{kj} + B_{kj} - B_{jk})h^i_l + (A_{kl} - A_{lk} + B_{lk} - B_{kl})h^i_j + (A_{lj} - A_{jl} + B_{jl} - B_{lj})h^i_l = 0$$

which due to lemma yields (2.13a). Again from (2.3) and (2.10) we get (2.13b).

Now differentiating (2.9) 0-covarinatly and using (1.7 e) and (1.8 c) we get

$$H^i_{ljk} = R^i_{jk\|l} = L^{-2}(B_{ko} y_j - B_{jo} y_k) h^i_j + U_{(jk)}\{B_{jo\|l} h^i_k + L^{-2} B_{ko} y^i h_{jl}\},$$

which due to (1.7 f) yields $U_{jk} h^i_l + U_{kl} h^i_j + U_{lj} h^i_k = 0$, where $U_{jk} = (B_{ko} y_j - B_{jo} y_k) L^{-2} + B_{ko\|j} - B_{jo\|k}$. Applying the lemma and transvecting by g^{jk} we get y_k) (2.13 c).

THEOREM 2.6. *A Finsler space with Rund's h-curvature of the form* (2.1) *is a Riemannian space of constant curvature* $B_{oo} L^{-2}$.

Proof. Using Theorem 2.3 and Corollary 1 and [**5**, Th. 30.6] we obtain the theorem.

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