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THE LEVITZKI RADICAL FOR Ω-GROUPS

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Abstract. The concept locally nilpotent ideal of an Ω-group is defined. The class of locally nilpotent Ω-groups is a Kurosh-Amitsur radical class. Furthermore, the Levitzki radical of an Ω-group is the intersection of all Ω-prime ideals P such that G/P is Levitzki semi-simple.

1. Notations and definitions. The notation and definitions of Higgins [**4**] and Buys and Gerber [**2**] will be used. For the sake of convenience we define the basic concepts. By $\mathbf{a} = (a_1, a_2, \dots, a_n) \in G$ we mean that $a_i \in G$, $i = 1, 2, \dots, n$. Higgins [**4**] called words which involve only the operations $\omega \in \Omega$, monomials. We shall call such words Ω-words. If $f(\mathbf{x}) = f(x_1, x_2, \dots, x_n)$ is an Ω-word in the indeterminates $x_1, x_2, \dots, x_n$ then $f(a) = f(a, a, \dots, a)$. Let Ω be a fixed set of operations.

1.1 *Definition.* $\omega \in \Omega$ will be called a trivial operation in the variety K of Ω-groups if $\mathbf{x}\omega = 0$ is satisfied in K. That is for all $G \in K$ and $\mathbf{a} \in G$, $\mathbf{a}\omega = 0$ holds. $\omega \in \Omega$ is a non-trivial operation if it is not trivial. An Ω-word which involves only non-trivial operations will be called a non-trivial Ω-word.

In Buys and Gerber [**2**], we defined the concept of an Ω-prime ideal for an Ω-group. That definition should actually be:

1.2 *Definition.* An ideal P of the Ω-group G is called an Ω-prime ideal if for all non-trivial $\omega \in \Omega$ and ideals $A_1, A_2, \dots, A_n$ of G such that $A_1 A_2 \dots A_n \omega^G \subseteq P$ it follows that $A_i \subseteq P$ for some $i = 1, 2, \dots, n$. All the results of Buys and Gerber [**2**] carries over with this slight alteration.

2. Locally nilpotent Ω-groups. Bhandari and Sexana [**1**] called an ideal I of a near-ring N locally nilpotent if any finite subset of I is nilpotent. They have shown that their definition coincides with the well-known definition of Levitzki defined for associative rings.

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2.1 *Definition.* A subset S of the Ω-group G is nilpotent if there exists a non-trivial Ω-word $f(\mathbf{x})$ such that $f(S) = \{f(\mathbf{s}) | \mathbf{s} \in S\}$ is zero.

2.2 *Definition.* Let A be a subset of the Ω-group G. A is called locally nilpotent if any finite subset of A is nilpotent.

2.3 COROLLARY. *If $A \subseteq B \subseteq G$ and B is locally nilpotent then A is locally nilpotent. If $A \subseteq G$ is nilpotent then A is locally nilpotent.*

2.4 LEMMA. *Let I be an ideal of the Ω-group G. G is locally nilpotent if and only if I and G/I are locally nilpotent.*

Proof. From 2.3 it follows that I is locally nilpotent. Let $\{g_1 + I, g_2 + I, \dots, g_n + I\}$ be any finite subset of G/I. Since G is locally nilpotent there exists a non-trivial Ω-word $f(\mathbf{x})$ such that $f(\mathbf{a}) = 0$ for all $\mathbf{a} \in \{g_1, g_2, \dots, g_n\}$. It follows that

$$\begin{aligned} f(\mathbf{a} + I) &= f(\mathbf{a}) + I \quad \text{(Higgins [\mathbf{4}, Theorem 3A])} \\ &= I \qquad \text{for all } \mathbf{a} + I \in \{g_1 + I, g_2 + I, \dots, g_n + I\}. \end{aligned}$$

Thus G/I is locally nilpotent.

For the converse let $\{g_1, g_2, \dots, g_n\}$ be any finite subset of G. Since G/I is locally nilpotent, there exists a non-trivial Ω-word $f(\mathbf{x})$ such that $f(\mathbf{a}+I) = 0$ for all $\mathbf{a}+I \in \{g_1+I, g_2+I, \dots, g_n+I\}$. It follows that $f(\mathbf{a}) \in I$ for all $\mathbf{a} \in \{g_1, g_2, \dots, g_n\}$. Let $A = \{f(\mathbf{a}) | \mathbf{a} \in \{g_1, g_2, \dots, g_n\}$. A is finite. Since $A \subseteq I$ there exists a non-trivial Ω-word $f_1(\mathbf{y})$ such that $f_1(\mathbf{b}) = 0$ for all $\mathbf{b} \in A$ In particular, $f_1(f(\mathbf{a})) = 0$ for all $\mathbf{a} \in \{g_1, g_2, \dots, g_n\}$. Since $f_1(f(\mathbf{x}))$ is, a non-trivial Ω-word, the lemma follows.

2.5 LEMMA. *Let I and J be locally nilpotent ideals of the Ω-group G. $I + J$ is a locally nilpotent ideal of G.*

Proof. The lemma follows from Higgins [**4**, Theorem 3**C**] and 2.4.

2.6 COROLLARY. *A finite sum of locally nilpotent ideals of G is a locally nilpotent ideal of G.*

2.7 LEMMA. *If I_α, $\alpha \in A$, are locally nilpotent ideals of G then $\sum I_\alpha$ is a locally nilpotent ideal of G.*

Proof. Since any finite subset of $\sum I_\alpha$ is contained in a finite sum of locally nilpotent ideals, the result follows from 2.6.

2.8 THEOREM. *The class $\mathcal{G} = \{G | G$ is a locally nilpotent Ω-group$\}$ is an absolutely hereditary radical class.*

Proof. Properties $R3$, $R5$ and $R7$ of Rjabuhin [**5**] respectively follow from 2.4, 2.7 and 2.4. From Rjabuhin [**5**], Theorem 1.2 it follows that G is a radical class.

From 2.3 it follows that G is an absolutely hereditary class (Rjabuhin [**5**, Definition p. 151]).

2.9 THEOREM. *Let $L(G)$ be the Levitzki radical of G that is $L(G)$ is the sum of all locally nilpotent ideals of G. $L(G) = \cap\{P_\alpha | P_\alpha$ is an Ω-prime ideal of G such that $L(G/P_\alpha) = 0\}$.*

Proof. Every locally nilpotent ideal in G and thus also $L(G)$ is contained in P_α for each Ω-prime ideal P_α with $L(G/P_\alpha) = 0$. It follows that $L(G) \subseteq \cap\{P_\alpha | P_\alpha$ is an Ω-prime ideal of G such that $L(G/P_\alpha) = 0\} = P$ (say).

Assume there exists an $a \in P$ such that $a \notin L(G)$. Since $a \notin L(G)$ every ideal I of G such that $a \in I$ is not locally nilpotent. This holds for a^G. Thus there exists an $A = \{a_1, a_2, \ldots, a_n\} \subseteq a^G$ such that A is not nilpotent. Furthermore, $\{f(\mathbf{a}) | \mathbf{a} \in A\}$ is not nilpotent for any nontrivial Ω-word $f(\mathbf{x})$. Otherwise there would exists a non-trivial Ω-word $f_1(\mathbf{y})$ such that $f_1(\{f(\mathbf{a}) | \mathbf{a} \in A\}) = 0$ and thus $f_1(f(\mathbf{a})) = 0$ for all $\mathbf{a} \in A$ contradicting the fact that A is not nilpotent. Let $\mathcal{J} = \{I | I$ is an ideal of G such that $L(G) \subseteq I$ and $\{f(\mathbf{a}) | \mathbf{a} \in A\} \not\subseteq I$ for any non-trivial Ω-word $f(\mathbf{x})\}$. $\mathcal{J} \neq \emptyset$ since $L(G) \in \mathcal{J}$. Applying Zorn's lemma $\mathcal{J}$ has a maximal element Q (say). Thus $L(G) \subseteq Q$ and $\{f(\mathbf{a}) | \mathbf{a} \in A\} \not\subseteq Q$ for any non-trivial Ω-word $f(\mathbf{x})$. We show that Q is an Ω-prime ideal with $L(G/Q) = 0$. We need only show that G/Q is an Ω-prime Ω-group (Buys and Gerber [**2**, Corollary 2.10]). Let $\omega \in \Omega$ be non-trivial and $I_1/Q, I_2/Q, \ldots, I_n/Q$ ideals of G/Q such that $(I_1/QI_2/Q \ldots I_n/Q)\omega = 0$. From Higgins [**4**] it follows that $I_1 I_2 dots I_n \omega \subseteq Q$. If $I_j/Q \neq 0$ for each $j = 1, 2, \ldots, n$ then $I_j \supset Q$. Since Q is maximal there exist non-trivial Ω-words $f_1(\mathbf{x}^{(1)}, f_2(\mathbf{x}^{(2)}, \ldots, f_n(\mathbf{x}^{(n)})$ such that $\{f(\mathbf{a}) | \mathbf{a} \in A) \subseteq I_j$ $j = 1, 2, \ldots, n$. Therefore

$$(\{f_1(\mathbf{a}) | \mathbf{a} \in A\} \ldots \{f_n(\mathbf{a}) | \mathbf{a} \in A\})\omega \subseteq I_1 I_2 \ldots I_n \omega \subseteq Q.$$

In particular we have $(f_i(\mathbf{a}) f_2(\mathbf{a}) \ldots f_n(\mathbf{a}))\omega \in Q$ for each $\mathbf{a} \in A$. Thus there exists a non-trivial Ω-word $g(\mathbf{x})$ defined by $g(\mathbf{x}) = (f_1(\mathbf{x}) f_2(\mathbf{x}) \ldots f_n(\mathbf{x}))\omega$ such that $\{g(\mathbf{a}) | \mathbf{a} \in A\} \subseteq Q$. This is a contradiction. It follows that $I_j/Q = 0$ for some j and thus that G/Q is an Ω-prime Ω-group.

Suppose that $W/Q \neq 0$ is a locally nilpotent ideal of G/Q. Then $W \supset Q$ and there exists a non-trivial Ω-word $f(\mathbf{x})$ such that $\{f(\mathbf{a}) \in A\} \subseteq W$ since Q is maximal. The family of cosets $\{f(\mathbf{a}) + Q | \mathbf{a} \in A\}$ is a finite set in W/Q. Sinco W/Q is locally nilpotent, $\{f(\mathbf{a}) + Q | \mathbf{a} \in A\}$ is nilpotent. Thus there exists a non-trivial Ω-word $f_1(\mathbf{x})$ such that $f_1(\mathbf{b}) = 0$ for every $\mathbf{b} \in \{f(\mathbf{a}) + Q | a \in A\}$. It follows that $\{f_1(f(\mathbf{a})) | bold a \in A\} \subseteq Q$ which is a contradiction. Therefore $L(G/Q) = 0$. We have proved that Q is one of the ideals P_α such that $L(G/P_\alpha) = 0$ and, therefore $P \subseteq Q$. But $A \subseteq P$ and $\{f(\mathbf{a}) | \mathbf{a} \in A\} \subseteq P$ for every Ω-word and in particular for non-trivial Ω-words. Since $P \subseteq Q$ it also holds for Q and this is a contradiction. Therefore $P \subseteq L(G)$.

As a result of the definition of Rjabuhin [**5**, p. 156], we have

2.10 THEOREM. *Every Levitzki semi-simple Ω-group is isomorphic to a sub-direct sum of Ω prime Levitzki semi-simple Ω-groups.*

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