PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 37 (51), 1985, pp. 7–15

# SEMANTICS FOR SOME INTERMEDIATE LOGICS

**Milan Božić**

**Abstract**. We give semantics for intermediate logics of the form $H + \vee S$, where $\vee S$ is the schema

$$\bigvee_{(i,j)\in S} (A_i \to A_j)$$

and $S$ is a nonempty subset of $\{1, \dots, n\}^2$. It is proved that such a logic is complete with respect to the class of Kripke frames $(X, R)$ which satisfy the universal closure of the formula

$$\bigvee_{(i,j),(k,i)\in S} x_{ij} R x_{ki}$$

## 0. Introduction

We shall give Kripke semantics for some intermediate logics, i. e., for some of the logics which lie in between the intuitionistic propositional calculus $H$ and the classical propositional calculus. Logics for which we shall give semantics are obtained by adding to $H$ axiom-schemata of the form

$$(\vee\mathrm{S}) \qquad \bigvee_{(i,j)} \{A_i \to A_j \mid (i,j) \in S\}$$

where $S$ is a nonempty subset of $\{1, \dots, n\}^2$ $(n \geq 1)$. Some special cases of these schemate were considered in [**2, 3**] and [**5**].

## 1. Prelimaries

The reader which is acquainted with Kripke semantics for $H$, and related techniques of canonical models and frames, can skip this section. Proofs of the results quoted here can be found in [**1**] and [**4**]. The connectives $\forall, \exists$ *and, or,* $\Rightarrow$, *iff*, and *not* belong to the meta-logic, which is here classical. The use of the other mathematical signs should be clear from the context.

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*AMS Subject Classification* (1980): Primary 03 B 55.

1.1. *Definition.* 1.1.1. The logic $H$ is the propositional calculus in the *language* $L = \{\rightarrow, \wedge, \vee, \neg\}$ over the set of *variables* $V = \{p_i \mid i \in \omega\}$ given with the axiom-schemata

(H1) $A \rightarrow (B \rightarrow A)$

(H2) $\big(A \rightarrow (B \rightarrow C)\big) \rightarrow \big((A \rightarrow B) \rightarrow (A \rightarrow C)\big)$

(H3) $A \wedge B \rightarrow A$

(H4) $A \wedge B \rightarrow B$

(H5) $A \rightarrow (B \rightarrow A \wedge B)$

(H6) $A \rightarrow A \vee B$

(H7) $B \rightarrow A \vee B$

(H8) $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$

(H9) $\neg A \rightarrow (A \rightarrow B)$

(H10) $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$

and the ruke-schema

(MP) $$\frac{A, A \rightarrow B}{B}$$

where $A, B$ and $C$ range over the set of all formulas, denoted by For $L$.

1.1.2. The notions of proof, proof from hypotheses and theorem are defined in the standard way. If $x \subseteq \mathrm{For}\,L$ and $A \in \mathrm{For}\,L$, then $x \vdash A$ stands for $A$ *is derived from* $x$ *in* $H$. The propositional calculus $S$ is an *extension of* $H$ iff it has the same language and variables as $H$, among its theorems it has all the axioms of $H$, and $MP$ is its only rule. $\mathrm{Th}(S)$ will denote the set of all theorems of $S$. If $S$ is an extension of $H$ and $x \subseteq \mathrm{For}\,L$, then $x \vdash_S A$ stands for $x \cup \mathrm{Th}(S) \vdash A$.

1.2. *Definition.* 1.2.1 $\mathfrak{X} = (X, R)$ is an $H$ *frame* iff $R$ is a reflexive and transitive relation on the nonempty set $X$ ($\mathrm{dom}\,\mathfrak{X} \overset{\text{def}}{=} X$, $K(H) \overset{\text{def}}{=} \{\mathfrak{X} \mid \mathfrak{X}$ is an $H$ frame$\}$).

1.2.2 $\mathfrak{M} = (\mathfrak{X}, v)$ is an $H$ *model* iff $(i)$ $\mathfrak{X} \in K(H)$ *and* $(ii)$ $v : \mathrm{dom}\,\mathfrak{X} \rightarrow P(V)$ such that

(her) $$xRy \Rightarrow v(x) \subseteq v(y) \text{ for all } x, y \in \mathrm{dom}\,\mathfrak{X}.$$

($\mathrm{Fr}\,\mathfrak{M} \overset{\text{def}}{=} \mathfrak{X}$, $\mathrm{dom}\,\mathfrak{M} \overset{\text{def}}{=} \mathrm{dom}\ \mathrm{Fr}\,\mathfrak{M}$, $v$ is a *valuation*, $\mathrm{val}\,\mathfrak{M} \overset{\text{def}}{=} v$, $M(H) \overset{\text{def}}{=} \{\mathfrak{M} \mid \mathfrak{M}$ is an $H$ model$\}$).

1.2.3. Let $\mathfrak{X} \in K(H)$ and $a \in \mathrm{dom}\,\mathfrak{X}.\mathfrak{X}_a = (\mathrm{dom}\,\mathfrak{X}_a, R_a)$ is a *truncation of the frame* $\mathfrak{X}$ *at the point* $a$ *iff* (i) $\mathrm{dom}\,\mathfrak{X} \overset{\text{def}}{=} \{x \in \mathrm{dom}\,\mathfrak{X} \mid aRx\}$ *and* (ii) $xR_ay$ *iff* $xRy$ for all $x, y \in$ $\mathrm{dom}\,\mathfrak{X}_a$.

1.2.4. Let $\mathfrak{M} \in M(H)$ and $a \in \mathrm{dom}\,\mathfrak{M}$. Then $\mathfrak{M}_a = ((\mathrm{Fr}\,\mathfrak{M})_a, (\mathrm{val}\,\mathfrak{M})_a)$ *is a truncation of the model* $\mathfrak{M}$ *at the point* $a$ *iff* $(\mathrm{val}\,\mathfrak{M})_a(x) = \mathrm{val}\,\mathfrak{M}(x)$ for all $x \in \mathrm{dom}(\mathrm{Fr}\,\mathfrak{M})_a$.

1.2.5. Let $\mathfrak{M} \in M(H)$ and $x \in \operatorname{dom}\mathfrak{M}$, $A \in \operatorname{For} L$. The predicate *formula $A$ holds in the point $x$ of the model* $\mathfrak{M}((\mathfrak{M}, x) \models A$, or, if the context is clear, briefly, $x \models A)$ is defined by the recursion on the number of connectives of the formula $A$. The axioms of this recursion are:

| | | | |
|---|---|---|---|
| $(p_i)$ | $A = p_i,$ | $x \models p_i$ | *iff* $p_i \in \operatorname{val}\mathfrak{M}(x)$ |
| $(\wedge)$ | $A = B \wedge C,$ | $x \models B \wedge C$ | *iff* $x \models B$ *and* $x \models C$ |
| $(\vee)$ | $A = B \vee C,$ | $x \models B \vee C$ | *iff* $x \models B$ *or* $x \models C$ |
| $(\rightarrow)$ | $A = B \rightarrow C,$ | $x \models B \rightarrow C$ | *iff* $\forall y(xRy$ *and* $y \models B \Rightarrow y \models C)$ |
| $(\neg)$ | $A = \neg B,$ | $x \models \neg B$ | *iff* $\forall y(xRy \Rightarrow$ *not* $y \models B)$ |

(in the formulas above the quantifiers range over dom $\mathfrak{M}$)

*$B$ holds in the model* $\mathfrak{M}(\mathfrak{M} \models A)$ *iff* $\forall x(x \in \operatorname{dom}\mathfrak{M} \Rightarrow x \models A)$.

*$A$ holds in the frame* $\mathfrak{X}(\mathfrak{X} \models A)$ *iff* $\forall \mathfrak{M}(\mathfrak{M} \in M(H)$ *and* $\operatorname{Fr}\mathfrak{M} = \mathfrak{X} \Rightarrow \mathfrak{X} \models A)$.

*$A$ is $H$ valid* $(\models_H A)$ *iff* $\forall \mathfrak{X}(\mathfrak{X} \in K(H) \Rightarrow \mathfrak{X} \models A)$.

Let $K \subseteq K(H)$; *$A$ is $K$ valid* $(K \models A)$ *iff* $\forall \mathfrak{X}(\mathfrak{X} \in K \Rightarrow \mathfrak{X} \models A)$.

1.3. Theorem (Intuitionistic Heredity) *For all* $\mathfrak{M} \in M(H)$, $A \in \operatorname{For} L$

(Her) $\qquad xRy$ *and* $x \models A \Rightarrow y \models A$ *for all* $x, y \in \operatorname{dom}\mathfrak{M}$.

1.4. Theorem $A \in \operatorname{Th}(H) \Rightarrow \models_H A$.

1.5 Theorem *For every* $\mathfrak{X} \in K(H)$, $\mathfrak{M} \in M(H)$ *and* $a \in \operatorname{dom}\mathfrak{X}$ *(or* $\operatorname{dom}\mathfrak{M}$).

1.5.1. $\mathfrak{X}_a \in K(H)$, 1.5.2. $\mathfrak{M}_a \in M(H)$.

1.6 Theorem *For Every* $\mathfrak{M} \in M(H)$, $A \in \operatorname{For} L$, $a \in \operatorname{dom}\mathfrak{M}$ *and every* $x \in \operatorname{dom}\mathfrak{M}$ *such that* $aRx$,

$$(\mathfrak{M}, x) \models A \text{ iff } (\mathfrak{M}_a, x) \models A.$$

1.7. *Definition*. Let $S$ be an extension of $H$.

1.7.1. $\mathfrak{X}^c(S) = (X^c(S), R^c)$ is the *$S$ canonical $H$ frame* iff (i) $X^c(S)$ is the set of all *$S$ nice* sets of formulas over the languange $L$, i. e., the set of all $x \subseteq \operatorname{For} L$ which satisfy

1° $x \vdash_S A \Rightarrow A \in x$($x$ is *$S$ deductively closed) and*

2° $A \vee B \in x \Rightarrow A \in x$ or $B \in x$($x$ is *prime) and*

3° *not* $(\forall A \in \operatorname{For} L) x \vdash_S A$*($x$ is $S$ consistent)*

(ii) $xR^cy$ *iff* $x \subseteq y$, for all $x, y \in X^c(S)$.

1.7.2. $\mathfrak{M}^c(S) = \mathfrak{X}^c(S), v^c$ is the *$S$ canonical model iff* $v^c(x) = x \cap V$, for all $x \in X^c(S)$($v^c$ is the *canonical valuation).*

1.8 Theorem 1.8. *If $S$ is an exstension of $H$ $A$ a formula over the language $L$, and $x$ a set of formulas over the language $L$ such that not $x \vdash_S A$ then there exists an $S$ nice set of formulas $y$ such that $x \subseteq y$ and not $A \in y$.*

1.8.2. *If $S$ is an extension of $H$, then*

$$A \in \mathrm{Th}(S) \text{ iff } \forall x(x \text{ is } S \text{ nice } \Rightarrow A \in x).$$

1.9 Theorem. *If $S$ is an extension of $H$, then*

1.9.1 $\mathfrak{X}(S) \in K(H)$, 1.9.2 $\mathfrak{X}^c(S) \in M(H)$.

1.10 Theorem. *If $S$ is an extension of $H$ and $x$ is an $S$ nice set of formulas, then $(\mathfrak{M}^c(S), x \models A$ iff $A \in x$.*

1.11 Theorem $A \in \mathrm{Th}(H)$ *iff* $\models_H A$.

1.12 Theorem. *Let $S$ be an extension of $H$ and $x \subseteq \mathrm{For}\,L$. Then $x$ is $S$ consistent iff there exists an $H$ model $\mathfrak{M}$ such that $\forall A(x \vdash_S A \Rightarrow \mathfrak{M} \models A)$.*

From this theorem it follows that, in particular (for $x = \varnothing$), $S$ is $H$ consistent *iff* it has an $H$ model.

## 2. The logic $H + \vee S$ and its semantics

2.1 *Definition*. Let $S$ be a nonempty subset of $\{1, \ldots, n\}^2$ $(n \geq 1)$. The logic $H + \vee S$ is the extension of $H$ with the axiom-schema

$$(\vee S) \qquad \bigvee_{(i,j)} \{A_i \to A_j \mid (i,j) \in S\}.$$

2.2 *Definition. $Sijk$* stands for $(i,j) \in S$ *and* $(k,i) \in S$.

2.3 Theorem $H + \vee S$ *is consistent iff* $\exists i\, \exists j\, \exists k\ Sijk$.

*Proof*. $(\Rightarrow)$ We prove the contraposition. Suppose *not* $\exists i\, \exists j\, \exists k\, Sijk$ i.e.

$$(^*) \qquad \textit{not } \exists i(\exists j(i,j) \in S \textit{ and } \exists k\,(k,i) \in S).$$

Then formula $(p_0 \to p_0) \to p_0 \wedge \neg p_0$ is an instance of the schema $\vee S$ obtained by substituting $p_0 \to p_0$ for all $A_i$ such that $\exists j(i,j) \in S$ and $p_0 \wedge \neg p_0$ for all $A_i$ such that $\exists k(k,i) \in S$. This substitution is correct as, in the first place, such $A_i's$ occur as $S$ is nonempty and only such $A_i'$ occur in $\vee S$; in the second, they are different because of $(^*)$. Thus, $p_0 \wedge \neg p_0$ is a theorem of $H + \vee S$; i. e. $H + \vee S$ is inconsistent.

$(\Leftarrow)$ Suppose $\exists i\, \exists j\, \exists k\ Sijk$, i.e. for some $(i_0, j_0, k_0)$, $(i_0, j_0) \in S$ and $(k_0, i_0) \in S$. We will construct an $H$ model for $H + \vee S$, which by Theorem 1.12, yields its consistency. This model is an element $H$ model $(\{a\}, R, v)(xRy$ *iff* $x = y = a$, $v(a) = V$. As an $H$ model, this model verifies all axioms and the rule of $H$. It also verifies the schema $\vee S$ as it verifies the disjunction $(A_{i_0} \to A_{j_0}) \vee (A_{k_0} \to A_{i_0})$ because it is true in all (i.e. one) points of our model. Namely, $a \models (A_{i_0} \to$

$A_{j_0}) \vee (A_{k_0} \to A_{i_0})$ is equivalent with $(a \models A_{i_0} \Rightarrow a \models A_{j_0})$ *or* $(a \models A_{k_0} \Rightarrow a \models A_{i_0})$ which is true. This ends the proof.

2.4 *Definition.* Let $S$ be $a$ subset of $\{1, \ldots, n\}^2 (n \geq 1)$ such that $\exists i \exists j \exists k\, Sijk$. Then $K(\vee S)$ is the class of all $H$ frames which satisfy the universal closure of the first-order formula

$$(\mathrm{k}(\vee S)) \qquad \bigvee_{(i,j)} \{tRx_{ij} \mid (i,j) \in S\} \Rightarrow \bigvee_{(i,j,k)} \{x_{ij} R x_{ki} \mid Sijk\}.$$

Note that the condition imposed on $S$ implies that the disjunction on the right-hand side of $k(\vee S)$ is nonempty. The following theorem shows that $K(\vee S)$ is the largest class of $H$ frames which satisfy the schema $\vee S$.

2.5. THEOREM *Let $S$ be a subset of $\{1, \ldots, n\}^2$ $(n \leq 1)$ such that $\exists i \exists j \exists k\, Sijk$. For every $\mathfrak{X} \in K(H)$*

$$\mathfrak{X} \models \vee S \ \textit{ iff } \ \mathfrak{X} \in K(\vee S)$$

*Proof*.($\Leftarrow$) Let $\mathfrak{X} \in K(\vee S)$ and let $\mathfrak{M}$ be an $H$ model such that $Fr\mathfrak{M} = \mathfrak{X}$. As $\mathfrak{M}$ verifies all the axioms and the rule of $H$, it is enough to prove that $\mathfrak{M} \models \vee S$. Suppose the opposite. Then there exists a $t \in \operatorname{dom} \mathfrak{M}$ such that *not* $t \models \vee_{i,j}\{A_i \to A_j \mid (i,j) \in S\}$ which implies that for all $(i,j) \in S$ there exists an $x_{ij}$ such that

$$(^*) \qquad tRx_{ij} \ \textit{ and } \ x_{ij} \models A_i \ \textit{ and not } \ x_{ij} \models A_j.$$

As $\mathfrak{X} \in K(\vee S)$ we have that $\bigwedge_{(i,j)}\{tRx_{ij} \mid (i,j) \in S\}$ implies, for some $i_0, j_0, k_0$ such that $Si_0j_0k_0$, $x_{i_0j_0} R x_{k_0i_0}$. Because of (*), $x_{i_0j_0} \models A_{i_0}$, which, because of $x_{i_0j_0} R x_{k_0i_0}$ implies, by Heredity, $x_{k_0i_0} \models A_{i_0}$. Since this contradicts (*), we have $\mathfrak{M} \models \vee S$.

($\Rightarrow$) We prove the contraposition. Suppose $x \notin K(\vee S)$. Then there exist $a, b_{ij} \in \operatorname{dom} \mathfrak{X}$ (for all $(i,j) \in S$) such that

$$(^{**}) \qquad \bigwedge_{(i,j)} \{aRb_{ij} \mid (i,j) \in S\} \ \textit{ and } \ \bigwedge_{(i,j,k)} \{\textit{not } b_{ij} R b_{ki} \mid Sijk\}$$

Define $v : \operatorname{dom} \mathfrak{X} \to P(V)$ with $p_i \in v(t)$ *iff* $\exists j((i,j) \in S$ *and* $b_{ij}Rt)$. It is easy to check that, as $R$ transitive, $v$ is a valuation. In the obtained model $\mathfrak{M} = (\mathfrak{X}, v)$ we have $b_{ij} \models p_i$ (for all $(i,j) \in S$, as $R$ is reflexive) and *not* $b_{ij} \models p_j$ (for all $(i,j) \in S$, since otherwise $b_{i_0j_0} \models p_{j_0}$, for some $(i_0, j_0) \in S$, which means $p_{j_0} \in v(b_{i_0j_0})$, and this, by the definition of $v$ yields $b_{j_0k_0} R b_{i_0j_0}$ for some $k_0$ such that $(j_0, k_0) \in S$ and $Sj_0k_0i_0$, which contradicts (**). Because of that we obtain $aRb_{ij}$ *and* $b_{ij} \models p_i$ *and not* $b_{ij} \models p_j$, for all $(i,j) \in S$, which yields *not* $a \models p_i \to p_j$ for all $(i,j) \in S$, i.e. $\vee_{(i,j)}\{p_i \to p_j \mid (i,j) \in S\}$ fails, in a and consequently in $\mathfrak{M}$, which ends the proof.

Now we prove the main theorem of this section.

2.6 THEOREM. *Let $S$ be a subset of $\{1, \ldots, n\}^2$ such that $\exists i \exists j \exists k\ Sijk$. For any $A \in For L$,*

$$A \in Th(H + \wedge S) \ \textit{ iff } \ K(\vee S) \models A.$$

*Proof*. The ($\Rightarrow$) part follows from the previous theorem. ($\Leftarrow$) To prove this we prove that $\mathfrak{X}^c(H + \vee S) \in K(\vee S)$. Then we will have

$$\begin{aligned} K(\vee S) \models A &\Rightarrow \mathfrak{X}^c(H + \vee S) \models A \\ &\Rightarrow \mathfrak{M}^c(H + \vee S) \models A \\ &\Rightarrow \forall x(x \in X^c(H + \vee S) \Rightarrow x \models A \\ &\Rightarrow \forall x(x \in X^c(H + \vee S) \Rightarrow A \in x) \quad \text{(by 1.10)} \\ &\Rightarrow A \in Th(H + \vee S) \quad \text{(by 1.8.2)} \end{aligned}$$

So, we have to prove that $k(\vee S)$ holds in $\mathfrak{X}^c(H + \vee S)$, i. e. that

$$(^*) \qquad \bigwedge_{(i,j)} \{t \subseteq x_{ij} \mid (i,j) \in S\} \Rightarrow \bigvee_{(i,j,k)} \{x_{ij} \subseteq x_{ki} \mid Sijk\}$$

holds for all $H + \vee S$ nice sets $t, x_{ij}((i,j) \in S)$. Note that $R$ becomes $\subseteq$ in the canonical model. Suppose that (*) fails to hold for some $H + \vee S$ nice sets $t, x_{ij}$, i. e. that the antecedent of (*) holds and that the consequent of (*) doesn't hold. Then we have:

$$(1) \qquad \bigwedge_{(i,j,k)} \{not\ x_{ij} \subseteq x_{ki} \mid Sijk\}$$

Because of (1), for any triple $(i,j,k)$ such that $Sijk$ (at least one such triple exists because of the assumption of the theorem), there exists $a$ formula $A_{ijk}$ such that

$$(2) \qquad A_{ijk} \in x_{ij} \ \ and\ not\ \ A_{ijk} \in x_{ki}.$$

For all $(\iota, k)$ such that $\exists j\ Sijk$ define

$$A_{ik} \overset{\text{def}}{=} \vee_j \{A_{ijk} \mid Sijk\}.$$

Obviously, $\vdash_H A_{ijk} \to A_{ik}$ for all $(i,j,k)$ such that $Sijk$; so, as $A_{ijk} \in x_{ij}$, we have

$$(3) \qquad A_{ik} \in x_{ij} \quad \text{for all } (i,j,k) \text{ such that } Sijk.$$

Also,

$$(4) \qquad not\ A_{ik} \in x_{ki} \quad \text{for all } (i,k) \text{ such that } \exists j\ Sijk$$

because, in the opposite case, $A_{ik} \in x_{ki}$ implies, $x_{ki}$ being prime (it is an $H + \vee S$ nice set), $\exists j(A_{ijk} \in x_{ki}\ and\ Sijk)$, which contradicts (2).

For all i such that $\exists j \exists j\ Sijk$ define

$$A_i \overset{\text{def}}{=} \bigwedge_k \{A_{ik} \mid \exists j\ Sijk\}$$

Obiviously, $\vdash A_i \to A_{ik}$ for all $(i,k)$ such that $\exists j\ Sijk$; so, because of (3),

$$(5) \qquad A_i \in x_{ij} \quad \text{for all } (i,j) \text{ such that } \exists k\ Sijk.$$

Also, because of (4) and the remark above,

$$(6) \qquad \textit{not } A_i \in x_{ki} \quad \text{for all } (i,k) \text{ such that } \exists j \; Sijk.$$

Formulas $A_i$ are defined for all i such that $\exists j \exists k \; Sijk$, i. e. such that $\exists j (i,j) \in S$ *and* $\exists k (k,i) \in S$.

Define $\top \overset{\text{def}}{=} p_0 \to p_0$ and $\bot \overset{\text{def}}{=} p_0 \wedge \neg p_0$ and, for all $1 \leq i \leq n$,

$$A'_i \overset{\text{def}}{=} \begin{cases} A_i \text{ if } \exists j(i,j \in S \;\; \textit{and} \;\; \exists k(k,i) \in S \in S \\ \top \text{ if } \exists j(i,j) \in S \;\; \textit{and not} \;\; \exists k(k,i) \in S \\ \bot \text{ if } \textit{not} \;\; \exists j(i,j) \in S \;\; \textit{and} \;\; \exists k(k,i) \in S \\ p_0 \text{ if } \textit{not} \;\; \exists j(i,j) \in S \;\;\; \textit{and not} \;\; \exists k(k,i) \in S. \end{cases}$$

The last case in the definition above is unimportant as such an $A'_i$ doesn't occur in the schema $\vee S$. The formula $\bigvee_{(i,j)}\{A'_i \to A'_j \mid (i,j) \in S\}$ is a theorem of $H + \vee S$ and, consequently, belongs to all $H + \vee S$ nice sets. In particular it belongs to $t$ and, as $t$ is prime, for some $(i_0, j_0) \in S$, $A'_{i_0} \to A'_{j_0} \in t$. We have supposed that the antecedent of $(*)$ holds, so, $t \subseteq x_{fj}$ for all $(i,j) \in S$. In particular, $t \subseteq x_{i_0 j_0}$. Thus,

$$(7) \qquad A'_{i_0} \to A'_{j_0} \in x_{i_0 j_0}.$$

Because of $(i_0, j_0) \in S$, two cases may occur: $\exists k(k, i_0) \in S$ or *not* $\exists k(k, i_0) \in S$. In he first case ($(i_0, j_0) \in S$ *and* $\exists k(k, i_0) \in S$) we have that $A'_{i_0} = A_{i_0}$ and $\exists k S i_0 j_0 k$. Because of (5) this implies $A'_{i_0} \in x_{i_0 j_0}$. In the second case ($(i_0, j_0) \in S$ *and not* $\exists k(k, i_0) \in S$) we have that $A'_{i_0} = \top$. Because $x_{i_0 j_0}$ contains all theorems of $H$, we also obtain that $A'_{i_0} \in x_{i_0 j_0}$. So, in both cases $A'_{i_0} \in x_{i_0 j_0}$ and, because of (7) we have

$$(8) \qquad A'_{i_0} \in x_{i_0 j_0}.$$

Again two cases may occur: $\exists k(j_0, k) \in S$ or *not* $\exists k(j_0, k) \in S$. In the first case ($\exists k(j_0, k) \in S$ *and* $(i_0, j_0) \in S$) we have $A'_{i_0} = A_{i_0}$ and $\exists k S j_0 k i_0$ which brings into contradiction (6) and (8). In the second case (*not* $\exists k(j_0, k) \in S$ *and* $(i_0, j_0) \in S$ we have $A'_{i_0} = \bot$, which, because of (8), contradicts the consistency of $x_{i_0 j_0}$. Thus, the assumption that $(*)$ fails leads to contradiction. This ends the proof.

The previous theorem states that the logic $H + \vee S$ is complete with respect to the largest class of $H$ frames in which $\vee S$ holds. Now we prove that this class can be made smaller and simpler.

2.7. *Definition.* Let $S$ be a subset of $\{1, \ldots, n\}^2$ such that $\exists i \exists j \exists k \; Sijk$. $K(C(\vee S))$ is the class of all $H$ frames which satisfy the universal closure of the first-order formula

$$(C(k(\vee S))) \qquad \bigvee_{(i,j,k)} \{x_{ij} R X_{ki} \mid Sijk\}$$

Note that $C(k(\vee S))$ is the consequent of $k(\vee S)$, so, as $C(k(\vee S)) \Rightarrow k(\vee S)$, we have that $K(C(\vee S) \subseteq K(\vee S)$. It is easy to prove that this inclusion is proper.

2.8. THEOREM *Let* $S$ *be a subset of* $\{1,\dots,n\}^2$ *such that* $\exists i \exists j \exists k\ Sijk$, *and let* $\mathfrak{X} \in K(\vee S)$ *and* $a \in dom\,\mathfrak{X}$. *Then* $\mathfrak{X}_a \in K(C(\vee S))$.

*Proof*. Because $\mathrm{dom}\mathfrak{X}_a = \{x \in \mathrm{dom}\mathfrak{X} \mid aRx\}$ and $\mathrm{dom}\mathfrak{X}_a \subseteq \mathrm{dom}\mathfrak{X}$ we have that the universal closure of $k(\vee S)$ holds in $\mathfrak{X}_a$. By substituting $a$ for $t$ in $k(\vee S)$ we obtain, because of the definition of $\mathfrak{X}_a$, that the antecedent of $k(\vee S)$ holds for all $x_{ij} \in \mathrm{dom}\,\mathfrak{X}_a$ $(i,j)) \in S)$. Thus, in $\mathfrak{X}_a$ holds the consequent of $k(\vee S)$ too. This consequent is $C(k(\vee S))$, so, $\mathfrak{X}_a \in K(C(\vee S))$.

2.9. THEOREM. *Let* $S$ *be a subset of* $\{1,\dots,n\}^2$ *such that* $\exists i \exists j \exists k\ Sijk$. *For any* $A \in F$ *or* $L$,

$$A \in Th(H + \vee S) \quad \textit{iff} \quad K(C(\vee S)) \models A.$$

*Proof*. We prove that $K(C(\vee S)) \models A$ iff $K(\vee S) \models A$, which, because of Theorem 2.6, implies our theorem. The $(\Rightarrow)$ part follows because of $K(C(\vee S)) \subseteq K(\vee S)$. $(\Rightarrow)$ Suppose *not* $K(\vee S) \models A$. Then for some $H$ model $\mathfrak{M}$ such that $Fr\mathfrak{M} \in K(\vee S)$ and for some $a \in \mathrm{dom}\,\mathfrak{M}$ we have $not(\mathfrak{M}, a) \models A$. Because of Theorem 1.6 we obtain $not(\mathfrak{M}_a, a) \models A$ and, consequently *not* $Fr(\mathfrak{M}_a) \models A$. But $Fr(\mathfrak{M}_a) = (Fr(\mathfrak{M}))_a$ which by Theorem 2.8, belongs to $K(C(\vee S))$. Thus, *not* $K(C(\vee S)) \models A$, which ends the proof.

Let us mention, at the end of this section, that the condition $\exists i \exists j \exists k\ Sijk$ imposed on the index set $S$ can be eliminated from all statements above if we define the empty disjunction as $\perp$ and the empty conjuctions as $\top$. In this way the formulations (though not the proofs!) of all theorems and definitions can be made simpler. However, as this condition is, by Theorem 2.3, equivalent with the consistency of the logic $H + \vee S$, we have decided to formulate our results for consistent logics only.

## 3. Some remarks

We have mentioned at the beginning of this paper that some intermediate logics of the form $H + \vee S$ have been considered in **[3]** and **[5]**. In **[5]** López-Escobar has suggested that the sequence of logics $H + c_n$, obtained with

$$(c_n) \qquad (A_1 \to A_2) \vee (A_2 \to A_3) \vee \cdots \vee (A_{n-1} \to A_n) \vee (A_n \to A_1) \ \ (n \geq 2)$$

iz strictly decreasing. Boričić has proved in **[3]** that this is not the case. In this paper he also proves that two sequences of intermediate logics $H + a_n$ and $H + b_n$, obtained with

$$(a_n) \qquad \bigvee_{1 \leq i < j \leq n} (A_i \to A_j), \quad n \geq 2$$

$$(b_n) \qquad \bigvee_{1 \leq i < j \leq n} (A_i \to A_j) \vee (A_n \to A_1) \quad n \geq 2$$

and strictly decreasing and that $\bigcap_{n\geq 2} Th(H+a_n) = \bigcap_{n\geq 2} Th(H+b_n) = Th(H)$. Our paper has originated in the search for the semantics for these two sequences of logics. Now, using the results from section 2, it is easy to write down the appropriate characteristic formulas for the semantics of the logics. Let us mention that in López-Escobar's case $S = \{(1,2),\dots,(n-1,n),\ (n,1)\}$ the appropriate formula is

$$x_{i2}Rx_{n1} \vee x_{23}Rx_{12} \vee \cdots \vee x_{n-1,n}Rx_{n-2,n-1} \vee x_{n1}Rx_{n-1,n}$$

which is equaivalent with

$$(k_n) \qquad x_1Rx_2 \vee x_2Rx_3 \vee \cdots \vee x_{n-1}Rx_n \vee x_nRx_1.$$

It is easy to prove that for $n \in 2N$, $k_n$ is equivalent with $k_2$ and less easy, but still elementary (the transitivity of $R$ and invariance of $k_n$ with respect to the cyclic permutation of the variables must be used), to prove that for $n \in 2N+1$, $k_n$ is equivalent with $k_3$, which is another proof of the falsity of López-Escobar's suggestion. It is possible that our semantics for the intermediate logics of the form $H + \vee S$ can be used for the investigation of some classes of logics with specific $S$. However, no results worth mentioning are known to us.

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Institut za matematiku
Prirodno-matematički fakultet
11000 Beograd
Jugoslavija

(Received 23 04 1984)