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ON REDUCED PRODUCTS OF FORCING SYSTEMS

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Abstract. We introduce two definitions of reduced products of forcing systems and using the appropriate ultraproduct we show that for any theory T of a first order finitary language L there is a forcing system whose forcing companion intersected with $\mathrm{SENT}(L)$ gives T.

0. Through the paper the language L is a first order language with at least one constant symbol. In new definitions we introduce, it is irrelevant whether the language L is finitary or not. However, in our considerations we will restrict ourselves to dealing with a finitary language.

By $\mathrm{AT}(L)$ and $\mathrm{SENT}(L)$ we denote the sets of atomic and of all sentences of L, respectively. The definition of forcing system we use, is that presented in [**3**], hence the basic logical symbols are $\neg$, $\wedge$ and $\exists$. We assume a familiarity with the basic properties of forcing relations and ultraproducts. The notation is mainly due to [**1**] and [**3**]. Forcing conditions will be denoted, in general, by small Latin letters $p, q, \dots$ that is by $p_D, q_D, \dots$. Thus, for instance "$p \leq q$ and ..." stands for "if $p, q \in C$, $p \leq q$ and ...". A filter D over (the index set) I is always a proper filter i.e. $D \neq P(I)$. $\prod_I A_i$ will be the Cartesian product of the sets A_i, $i \in I$, and $\prod_D A_i$ the reduced product (modulo filter D).

1. *Definition* 1.1. *The standard reduced product $F = \prod_D F_i$ of forcing systems F_i, $i \in I\,(\neq \emptyset)$ $(F_i = \langle C_i, \Vdash, L\rangle)$ for a given filter D over I is the triple $F = \langle C, \Vdash, L\rangle$, where $C = \prod_D C_i$ and $\Vdash \subseteq C \times \mathrm{SENT}(L)$ is defined for $\varphi \in \mathrm{AT}(L)$ and $p_D \in C$ by $p_D \Vdash \varphi$ iff $\{i \in I \mid p(i) \Vdash_i \varphi\} \in D$ and like any forcing relation for other.*

The definition is obviously correct.

LEMMA 1.2. *The standard reduced product of forcing system is itself a forcing system.*

Proof. We will just consider the case: $p_D \Vdash t_1 = t_2$ and $p_D \Vdash \varphi(t_1)$, where t_1, t_2 are closed terms and $\varphi(\nu)$ is an atomic formula with at most one free variable.

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We are to show that for some $q_D \geq p_D$ $q_D \Vdash \varphi(t_2)$. Let for $j \in X = \{i \in I |\, p(i) \Vdash_i t_1 = t_2 \wedge \varphi(t_1)\}$ $q_j \in C_j$ be such that $p(j) \leq q_j \Vdash_j \varphi(t_2)$. Then for $q \in \prod_I C_i$ defined by $q(j) = q_j$ if $j \in X$, otherwise $q(j) = 0_j$ (the least element of C_j), q_D satisfies the above requirements.

Further, for any $\varphi \in \mathrm{SENT}\,(L)$ and any $p_D, q_D \in C$ $X_{p_D,\varphi}$ will be the set $\{i \in I |\, p(i) \Vdash_i \varphi\}$ and X_{p_D,q_D} the set $\{i \in I |\, p(i) \leq q(i)\}$.

In general, for the standard ultraproduct (D is now an ultrafilter) does not hold "Łoś theorem", i.e. it does not have to be fulfilled $p_D \Vdash \varphi$ iff $X_{p_D,\varphi} \in D$. The step: φ is of the complexity of the form $\exists\nu\psi(\nu)$ is the one which does not pass in the induction on the complexity of formula φ. For instance, it is easy to find an example in which, for the chosen sentence $\varphi \equiv \exists\nu\psi(\nu)$, $X_{0_D,\varphi} = I$ while $0_D \nVdash \varphi$. Certainly $p_D \Vdash \varphi$ implies $X_{p_D,\varphi} \in D$. However we have

THEOREM 1.3. *If the set T of closed terms of L is of cardinality $\lambda (\geq \omega_0)$ and D is λ^+-complete ultrafilter then $p_D \Vdash \varphi$ iff $X_{p_D,\varphi} \in D$.*

Proof. By induction on the complexity of the formula φ. We will consider the well ordered set T of order type λ, $T = \{t_\alpha |\, \alpha < \lambda\}$. By what was said only the case $\varphi \equiv \exists\nu\psi(\nu)$ is interesting. Let us suppose $X_{p_D,\varphi} \in D$ but $p_D \nVdash \exists\nu\psi(\nu)$. Then $p_D \nVdash \psi(t_\alpha)$ for any t_α, whence, by induction hypothesis $X_\alpha = X_{p_D,\psi}(t_\alpha) \notin D$. Thus $Y = \bigcap_{\alpha<\lambda} X_\alpha^C \in D$ and, in particular, $X_{p_D,var\phi} \cap Y \in D$. But for $i \in X_{p_D,\varphi} \cap Y$ $p(i) \Vdash_i \exists\nu\psi(\nu)$ while, on the other hand, $p(i) \nVdash_i \psi(t_\alpha)$ for all $\alpha < \lambda$, a contradiction.

COROLLARY 1.4. *If D is α λ^+-complete ultrafilter and T^C, T^{C_i} $(i \in I)$ are forcing companions of, respectively, $F = \prod_D F_i$ and F_i then $T^C = \{\varphi \in \mathrm{SENT}\,(L) |\, \{i \in I |\, \varphi \in T^{C_i}\} \in D\}$.*

We will write $T^C = \prod_D T^{C_i}$.

COROLLARY 1.5. *If D is principal ultrafilter, that is, $D = \{X \in P(I) |\, j \in X\}$ for some $j \in I$, then F is equivalent to F_j in the sense that $\prod_D C_i \simeq C_j$ and $p_D \Vdash \varphi$ iff $p(j) \Vdash_i \varphi$.*

Hence, as we can immediately suppose, only nonprincipal utraproducts are of interest. The condition that D is λ^+-complete $(\lambda \geq \omega_0)$ is rather strong. In general considerations it "eliminates" ultraproducts over countable index set and our metatheory must be extended by the axiom of the existence of a measurable cardinal. Thus one would like to have at disposal an ultraproduct which will make this condition unnecessary. The problem is, as we have seen, that in case of the standard ultraproduct we can lack in closed terms (while at the same time we have too many of term). Therefore the natural idea is to extend "the third component" in the ultraproduct. S we introduce

Definition 1.6. *The language extended reduced product of forcing systems F_i, $i \in I$* (for a given filter D over I) is triple $F' = \langle C, \Vdash', L' \rangle$, where (again)

$C = \prod_D C_i, L'$ is the language having the same function and relation symbols as L, but with the set of constants $\prod_D T$ (recall that T is the set of closed terms of L) and $\Vdash' \subseteq C \times \mathrm{SENT}(L')$ is defined for $\varphi \in \mathrm{AT}(L')$ and $p_D \in C$ by $p_D \Vdash' \varphi$ iff $\{i \in I | p(i) \Vdash_i' \varphi_i\} \in D$, where φ_i is a formula of the language L obtained by replacing in φ each constant d_D of L' by $d(i)$, $(d \in \prod_I T)$. Of course, for the sentences of greater complexity $\Vdash'$ is defined like a forcing relation.

Again it easy to verify that the definition is correct, i.e. that if

$$p_D = q_D,\ d_D^1 = e_D^1, \ldots, d_D^k = e_D^k,\ d_D^j,\ e_D^j \in \prod_D T \ \ (j = 1, \ldots, k), \text{ and}$$

$p_D \Vdash' \varphi(d_D^1, \ldots, d_D^k)$ then $p_D \Vdash' \varphi(e_D^1, \ldots, e_D^k)$ as well as $q_D \Vdash' \varphi(d_D^1, \ldots, d_D^k)$.

LEMMA 1.7. *The following extended ultraproduct of forcing systems is itself a forcing system.*

Proof. As in Lemma 1.2 we again consider only the case when $p_D \Vdash' t_1' = t_2'$ and $p_D \Vdash' \varphi(T_1')$ (t_1', t_2' are closed terms of L' and $\varphi(\nu)$ is an element of $\mathrm{AT}(L')$ with at most one free variable). For $j \in X = \{i \in I | p(i) \Vdash_i t_1^i = t_2^i \wedge \varphi_i(t_1^i)\}$, where t_k^i $(k = 1, 2)$ is a term of L obtained by replacing each constant d_D in t_k' by $d(i)$, let $q_j \in C_j$ be such that $p(j) \leq q_j \Vdash_j \varphi(t_2^j)$ and let $q(j) = q_j$ for $j \in C$, otherwise $q_j = 0_j$. Then $q_D \geq p_D$ and $q_D \Vdash' \varphi(t_2')$.

By the way, let us notice that for any closed term t' of L' there exists a constant d_D of L' such that $t^i = d(i)$ for any $i \in I$; hence $p_D \Vdash' \neg\neg(t' = d_D)$ for any p_D.

THEOREM 1.8. *"Loś theorem" holds for F', i.e. $p_D \Vdash' \varphi$ iff $\{i \in I | p(i) \Vdash_i \varphi_i\} \in D$*

Proof. By induction on the complexity of the formula φ.

Of course, only the case $\varphi \equiv \exists\nu\psi(\nu)$ will be treated. Let us suppose $p_D \Vdash' \exists\nu\psi(\nu)$. Then for some closed term t' $p_D \Vdash' \psi(t')$ and by induction hypothesis $\{i \in I | p(i) \Vdash_i \psi_i(t^i)\} \in D$. Hence also $\{i \in I | p(i) \Vdash_i \exists\nu\psi(\nu)\} \in D$.

Let $X = \{i \in I | p(i) \Vdash \exists\nu\psi_i(\nu)\} \in D$. For $i \in X$ let t_i be a closed term (of L) such that $p(i) \Vdash_i \psi(t_i)$ and let $t \in \prod_i T$ be defined by: $t(i) = t_i$ if $i \in X$, otherwise arbitrary. By induction hypothesis $p_D \Vdash' \psi(t_D)$, whence $p_D \Vdash' \exists\nu\psi(\nu)$.

2. Let T be a theory of a language L, A an uncountable set of new constants $(L \cap A = \emptyset, |A| > \omega_0)$ and for $n \in \omega$ $F_n = \langle C_n, \Vdash_n, L(A)\rangle$ n-finite forcing [**2**] (C_n in the set of finite $\sum_n, \prod_n$ sentences (by which we mean the sentences equivalent to sentences in prenex normal form with at most n blocks of quantifiers) of $L(A)$ consistent with T, ordered by inclusion and for $p \in C_n$ and $\varphi \in \mathrm{AT}(L(A))$ $p \Vdash_n \varphi$ iff $\varphi \in p$. In that case the following holds.

LEMMA 2.1. *If D is an ultrafilter over ω and F the standard ultraproduct of F_n, $n \in \omega$ $(F = \langle C, \Vdash, L(A)\rangle)$ then for any $p_D \in C = \prod_D C_n$ and any $\varphi \in$* SENT $(L(A)$ *$p_D \Vdash \neg\neg\varphi$ iff $X_{p_D, \neg\neg\varphi} \in D$.*

Proof. By induction on the complexity of φ. Let us suppose $\varphi \in \mathrm{AT}\,(L(A))$ and $p_D \Vdash \neg\neg\varphi$ but $X_{p_D,\neg\neg\varphi} \notin D$. Furthermore, for $j \in X^c_{p_D,\neg\neg\varphi}$ let q_j be such that $p(j) \leq q_j \Vdash_j \neg\varphi$ and let $q \in \prod_\omega C_i$ be defined by $q(j) = q_j$ if $j \in X^c_{p_D,\neg\neg\varphi}$; otherwise $q(j) = \emptyset$. Now if $r_D \geq q_D$ is such that $r_D \Vdash \varphi$ i.e. $X_{r_D.\varphi} \in D$ then for $j \in X_{q_D,\neg\varphi} \cap X_{r_D,\varphi} \cap X_{q_D,r_D}$ $(\in D)$ we have $q(j) \leq r(j)$, $q(j) \Vdash_j \neg\varphi$ and $r(j) \Vdash_j \varphi$, a contradiction.

If, on the other hand $X_{p_D,\neg\neg\varphi} \in D$ but not $p_D \Vdash \neg\neg\varphi$, then for some $q_D \geq p_D$ we have $q_D \Vdash \neg\varphi$; hence, as one can easily check, $X_{q_D,\neg\varphi} \in D$ and so for $j \in X_{p_D,\neg\neg\varphi} \cap X_{q_D,\neg\varphi} \cap X_{p_D,q_D} \in D$ it holds $p(j) \leq q(j)$, $p(j) \Vdash_j \neg\neg\varphi$ but $q(j) \Vdash_j \neg\varphi$, a contradiction again.

If $\varphi \equiv \psi \wedge \theta$ then by the forcing relation and filter properties it follows: $q_D \Vdash \neg\neg(\psi\wedge\theta)$ iff $p_D \Vdash \neg\neg\psi \wedge \neg\neg\theta$ iff $X_{p_D,\neg\neg\psi}, X_{p_D,\neg\neg\theta} \in D$ iff $X_{p_D,\neg\neg\psi\wedge\neg\neg\theta} \in D$ iff $X_{p_{D'},\neg\neg(\psi\wedge\theta)} \in D$

If $\varphi \equiv \neg\psi$ and $p_D \Vdash \varphi$ then for no $q_D \geq p_D$ we have $q_D \Vdash \neg\neg\psi$. Hence, by induction hypothesis, for each $q_D \geq q_D$ it holds $X_{q_D,\neg\neg\psi} \notin D$ and, obviously, the assumption $X_{p_D,\neg\psi} \notin D$ would be contradictory to the hypothesis.

It is equally easy to prove that $X_{p_D,\neg\psi} \in D$ implies $p_D \Vdash \neg\psi$.

Let now $\varphi \equiv \exists\nu\psi(\nu)$ and $p_D \Vdash \neg\neg\exists\nu\psi(\nu)$ by $X_{p_D,\neg\neg\exists\nu\psi(\nu)} \notin D$. For $j \in X^c_{p_D,\neg\neg\exists\nu\psi(\nu)}$ let $q_j \in C_j$ be such that $p(j) \leq q_j \Vdash_j \neg\exists\nu\psi(\nu)$ and let $q \in \prod_\omega C_i$ be given by $q(j) = d_j$ for $j \in X^c_{p_D,\neg\neg\exists\nu\psi(\nu)}$; otherwise $q(j) = \emptyset$. If $r_D \geq q_D$ forces $\exists\nu\psi(\nu)$ then for some closed term t it holds $r_D \Vdash \psi(t)$, whence $X_{r_D,\neg\neg\psi(t)} \in D$ and for $j \in X_{p_D,\neg\neg\exists\nu\psi(\nu)} \cap X_{r_D,\neg\psi(t)} \cap X_{q_D,r_D}$ we obtain the contradiction: $q(j) \leq r(j)$, $q(j) \Vdash_j \neg\exists\nu\psi(\nu)$ while $r(j) \Vdash \neg\neg\psi(t)$.

Finally let $X_{p_D,\neg\neg\exists\nu\psi(\nu)} \subset D$ and $q_D \geq p_D$. For $j \in X_{p_D,\neg\neg\exists\nu\psi(\nu)} \cap X_{p_D,q_D}$ let r_j and t_j be a condition of A appearing neither in r_j nor in t_j. Then $s_j = r_j \cup \{t_j = a\}$ is a condition (of C_j) and $s_j \Vdash_j \neg\neg\psi(a)$. Thus if $s \in \prod_\omega C_i$ is defined by: $s(j) = s_j$ if $j \in X_{p_D,\neg\neg\exists\nu\psi(\nu)} \cap X_{p_D,q_D}$, otherwise $s(j) = \emptyset$, then $s_D \geq q_D$ and $s_D \Vdash \neg\neg\psi(a)$. We conclude $p_D \Vdash \neg\neg\exists\nu\psi(\nu)$.

COROLLARY 2.2. *Let all assumptions be as in Lemma* 2.1 *with, in addition, D nonprincipal. Then* $T = T^C \cap \mathrm{SENT}\,(L)$, *where* T^C *is the forcing companion of* F.

Proof. Let φ be a $\prod_n$-sentence of the language L.

If $\varphi \in T$ then for any $k \geq n$ and any $p \in C_k$ we have: $q = p \cup \{\varphi\} \in C_k$ [**2**]. Thus, for $k \geq n$, $\emptyset \Vdash_k \neg\neg\varphi$ and by Lemma 2.1, being D non principal, $\emptyset_D \Vdash \neg\neg\varphi$ i.e. $\varphi \in T^C$.

On the other hand, $\varphi \notin$ gives $\{\neg\varphi\} \Vdash_k \neg\varphi$ for $k \geq n$ [**2**], whence $\emptyset \Vdash \neg\neg\varphi$, that is $\varphi \notin T^C$.

Remark. We could obtain the same result directly, using the language extended ultraproduct of forcing systems $F_n, n \in \omega$, (the analogy of Lemma 2.1 is then already given) and indetifying the language $L(A)$ with the corresponding sub-laguage of $(L(A))'$ (that is, identifying the constants a of $L(A)$ with a_D).

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