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SUMS OF PRODUCTS OF CERTAIN ARTHMETICAL FUNCTIONS

Aleksandar Ivić

Abstract. Sharp asymptotic formulae for certain sums of the type $\sum_{n\le x} f(n)g(n)$ are derived, where f is a suitable multiplicative and g a suitable additive function. The proofs are based on an analytic method which consists of considering the Dirichlet series generated by $f(n)z^{g(n)}$, z complex.

1. Introduction

To estimate sums of the type $\sum_{n\le x} f(n)g(n)$, where f is a multiplicative and g an additive function, one may use that following general analytic approach. If z is a complex number, then $h_z(n) = f(n)z^{g(n)}$ is a multiplicative function of n. Thus if a good approximation to

$$F(x,z) = \sum_{n\le x} f(n)z^{g(n)} \tag{1.1}$$

may be found by using the methods from the theory of multiplicative functions, then various arithmetical sums involving f and g may be obtained from (1.1). If $g(n)$ takes only nonnegative integer values, then $F(x,z)$ is a polynomial in z which we expect to be well approximated by functions regular for $|z| \le 1$, say. Then $F(x,z)$ may be integrated, differentiated, etc. as a function of z. In particular,

$$\sum_{n\le x} f(n)g(n) = \left.\frac{\partial F(x,z)}{\partial z}\right|_{z=1}, \tag{1.2}$$

and it remains to evaluate the function on the right-hand side of (1.2). This procedure may be carried out further to yield, for $m \ge 1$ a fixed integer,

$$\sum_{n\le x} f(n)g^m(n) = \frac{\partial}{\partial z}\left(\underbrace{z\ldots\left(z\frac{\partial F(x,z)}{\partial z}\right)}_{\text{m times}}\ldots\right)\Bigg|_{z=1}. \tag{1.3}$$

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More generally, one can estimate the sum

$$(1.4)\qquad \sum_{n\le x} f(n)g_1^m(n)\dots g_r^{m_r}(n) \qquad (m_1\ge 1,\dots m_r\ge 1 \text{ integers}),$$

where $g_1,\dots,g_r$ are suitable nonnegative, integer-valued additive functions. This follows by considering the sum

$$(1.5)\qquad F(x;z_1,\dots,z_r)=\sum_{n\le x} f(n)z_1^{g_1(n)})\dots z_r^{g_r(n)}$$

and applying the technique of (1.3) to each of the complex variables $z_1,\dots,z_r$.

If instead of the sum in (1.4) one wishes to evaluate the sum

$$(1.6)\qquad {\sum_{n\le x}}^{*} \frac{f(n)}{g_1^{m_1}(n0\dots g_r^{m_r}(n)} \quad (m_1,\dots,m_r\ge 1 \text{ integers}),$$

where $\sum^*$ denotes summation over those n for which the denorninator in (1.6) is positive, this may be also achieved via (1.5). First we estimate the portion of the sum in (1.5) for which $g_1(n)=0,\dots,g_r(n)=0$. Then dividing the remaining part of the sum in (1.5) by $z_1,\dots,z_r$ and integrating over each variable from $\varepsilon(x)$ (>0) to 1 (where $\varepsilon(x)$ is suitably chosen and satisfies $\lim_{x\to 0+}\varepsilon(x)=0$), one arrives at (1.6) with $m_1=\dots=m_r=1$. By repeating this procedure one may evaluate also the general sum (1.6).

It has been already stated that if $g(n)$ is a nonnegative, integer-valued additive function, then the sum in (1.1) is actually a polynomial in z. Hence equating the coefficients of z^q of both sides of (1.1) we are able to evaluate asymptotically the sum $\sum_{m\le x, g(n)=q} f(n)$ for a given integer $q\ge 0$, which may be termed as a "local" problem. This topic will be pursued in §**4**, while in §**2** and §**3** we shall derive sharp asymptotic formulae for various sums of the type (1.4) and (1.6).

2. Sums with the divisor function

The technique described in §1 as, at least in principle, fairly well known (see [3], [1] and [7, Ch. 14], for more details and references). The pioneering work in this field was done by **A**. Selberg [8], who was the first to estimate sums of $d_z(n)$ (see (2.4)), $z^{\omega(n)}$ and $z^{\Omega(n)}$. Using these ideas it is possible to obtain general results concerning the sums described in §**1**, but both the formulation and proofs of such theorems would be technically complicated and not very instructive, and none seem to have appeared in the literature before. Therefore it seems preferable to derive sharp asymptotic formulae for some common arithmetical functions, and to indicate how various other formulae may be derived in many other cases. For $g(n)$ we shall primarily take $\omega(n)$ or $\Omega(n)$, the number of distinct prime factors and the number of all prime factors of n, respectively. It will be clear from the sequel that the results may be generalized to other nonnegative, integer-valued additive functions such that $g(p)$ is a constant for all primes p, and $g(n)$ is in some sense of moderate growth. In this section we shall specify $f(n) = d_k(n)$, the number of ways n may be written as a product of k fixed positive integers ($d_1(n) = 1$ for all n). In general, one defines for an arbitrary complex z the multiplicative function, $d_z(n)$, commonly called the generalized divisor function, by the relation

$$\sum_{n=1}^{\infty} d_z(n)n^{-s} = \zeta^z(s), \quad (\mathrm{Re}\, s > 1), \tag{2.1}$$

so that $d_z(p^a) = \binom{a+z-1}{a}$, where a branch of $\zeta^z(s)$ in (2.1) is given by

$$\zeta^z(s) = \exp(z \log \zeta(s)) = \exp\Big(- z \sum_p \sum_{j=1}^{\infty} j^{-1} p^{-js}\Big) \quad (\mathrm{Re}\, s > 1).$$

Here and in the sequel $\zeta(s)$ is the Riemann zeta-function, and p denotes prime numbers. In [2] De Koninck and Mercier investigated certain sums of the form (1.2) by the method of §bf 1, and they stated a general theorem which implies that

$$\sum_{n \le x} d(n)\omega(n) = 2x \log x \log\log x + Ax \log x + O(x) \tag{2.2}$$

holds, where $d(n) = d_2(n)$ is the number of divisors function, and A is an explicit constant (see also [1, Ch. 9]). An asymptotic formula for the sum in (2.2) may be also easily obtained by an elementary argument. Writing $\omega(n) = \sum_{p|\, n} 1$ and noting that $d(p^2n) - 2d(pn) = -d(n)$ for all primes p and $n > 1$, we obtain

$$\sum_{n \le x} d(n)\omega(n) = 2 \sum_{pn \le x} d(n) - \sum_{p^2 n \le x} d(n).$$

Hence using classical estimates for sums of $d(n)$ and $1/p$ it follows after some simplification that

$$\sum_{n \le x} d(n)\omega(n) = 2x \log x \log\log x + Ax \log x + Bx \log\log x + Cx + O\Big(\frac{x}{\log x}\Big) \tag{2.3}$$

with explicit $A, B \ne 0$ and C. This was pointed out recently in a letter of R. Sitaramachandrarao to De Konick [9], who kindly informed me of this and indicated that by an elaboration of the above elementary argument (2.3) could be further sharpened. Thus (2.2) (and also Th. 8 of [2]) is not correct as it stands, but should have $O(x \log\log x)$ instead of only $O(x)$ as the error term. In what follows I shall use the analytic approach described in §bf 1 to derive a formula which gives as a special case a considerable sharpening of (2.3), and point out how the error committed in [2] in deriving (2.2) may be easily removed. All the theorems which follow are the sharpest ones hitherto.

For our proofs we shall need the asymptotic formula

$$\begin{aligned} D_z(x) = \sum_{n \le x} d_z(n) = c_1(z)x \log^{z-1} x + \cdots + c_N(z)x \log^{z-N} x + \\ + O(x(\log x)^{\mathrm{Re}\, z - N - 1}), \end{aligned} \tag{2.4}$$

where $N \ge 1$ is an arbitrary but fixed integer, for $j = 1, \dots, N$ we have $c_j(z) = B_j(z)/\Gamma(z-j+1)$ and each $B_j(z)$ is regular for $|z| \le A$ ($A > 0$ is arbitrary but fixed) so that $c_j(0) = 0$ for $j \ge 1$ and $c_j(1) = 0$ for $j \ge 2$. This is a result proved by R.D. Dixon [5], a proof of which may be also found in [1, Ch. 1] or [7, Ch. 14]. Dixon sharpened the result of A. Selberg [8], who proved (2.4) with the error term $O(x(\log x)^{\mathrm{Re}\, z-2})$. Now consider, for $\mathrm{Re}\, s > 1$ and $|z| \le A$ ($A > 1$ fixed, the Dirichlet series

$$F_k(s,z) = \sum_{n=1}^{\infty} d_k(n) z^{\omega(n)} n^{-s} \tag{2.5}$$
$$= \prod_p \left(1 + zkp^{-s} + \binom{k+1}{2} zp^{-2s} + \binom{k+2}{3} zp^{-3s} + \dots\right) = \zeta^{kz}(s) G_k(s,z),$$

where

$$G_k(s,z) = \sum_{n=1}^{\infty} g_k(n,z) n^{-s} = \prod_p (1-p^{-s})^k \left(1 + kzp^{-s} + \binom{k+1}{2} zp^{-2s} + \dots\right).$$

The Dirichlet series for $G_k(s,z)$ converges absolutely and uniformly for $\mathrm{Re}\, s > 1/2 + \varepsilon$ ($\varepsilon > 0$ fixed) and $|z| \le A$, where it represents a regular function of z. This follows e.g. by a lemma of H. Delange [3] (see also [1, Ch. 5] for a proof), which formulated for our purpose states the following: Suppose that up $u_p(s,z)$ and $\nu_p(s,z)$ are two sequences of complex functions defined on $s \in A$, $z \in B$, and suppose that for every prime p there exist constants U_p and V_p such that

$$|u_p|(s,z) \le U_p, \;\; |u_p(s,z) - \nu_p(s,z)| \le V_p, \;\; \sum_p U_p^2 < \infty, \;\; \sum_p V_p < \infty.$$

Then the infinite product $\prod_p (1 + u_p(s,z)) \exp(-\nu_p(s,z))$ is uniformly convergent and bounded for $s \in A$, $z \in B$.

From (2.5) it follows that $d_k(n) z^{\omega(n)}$ is the convolution of $d_{kz}(n)$ and $g_k(n,z)$. Hence using (2.4) we obtain

$$\begin{aligned}
\sum_{n \le x} d_k(n) z^{\omega(n)} &= \sum_{mn \le x} g_k(n,z) d_{kz}(m) = \sum_{n \le x} g_k(n,z) D_{kz}\left(\frac{x}{n}\right) \\
&= \sum_{n \le x/2} g_k(n,z) D_{kz}\left(\frac{x}{n}\right) + O\Big(\sum_{x/2 < n \le x} |g_k(n,z)|\Big) \\
&= x \sum_{n \le x/2} g_k(n,z) n^{-1} \sum_{j=1}^{N} c_j(kz) \left(\log \frac{x}{n}\right)^{kz-j} \\
&+ O\left(\sum_{n \le x/2} |g_k(n,z)| n^{-1} \left(\log \frac{x}{n}\right)^{\mathrm{Re}\, kz - N - 1}\right) + O(x^{1/2+\varepsilon}).
\end{aligned}$$

But for each j and $M \geq 1$ arbitrary, but fixed, we have

$$\sum_{n\leq x/2} g_k(n,z)n^{-1}\left(\log\frac{x}{n}\right)^{kz-j} = \log^{kz-j} x \sum_{n\leq x/2}\left(1-\frac{\log n}{\log x}\right)^{\mathrm{Re}\, kz-N-j} g_k(n,z)n^{-1}$$

$$= \log^{kz-j}\sum_{r=0}^{M}(-1)^r\binom{kz-j}{r}\sum_{n\leq x/2}\left(\frac{\log n}{\log x}\right)^r g_k(n,z)n^{-1} + O((\log x)^{\mathrm{Re}\, kz-M-j-1})$$

$$= \sum_{r=0}^{M} d_{r,j}(z)(logx)^{kz-j-r} + O((\log x)^{\mathrm{Re}\, kz-M-l-1}).$$

where each $d_{r,j}(z)$ is a regular function for $|z| \leq A$. Here we used the fact that, for C a constant, we have by partial summation

$$\sum_{n\leq x/2} g_k(n)n^{-1}\log^C n = \sum_{n=1}^{\infty} g_k(n,z)n^{-1}\log^C n + O(x^{-1/2+\varepsilon}),$$

since $G_k(s,z)$ is absolutely convergent for $\mathrm{Re}\, s \geq 1/2+\varepsilon$ and $|z| \leq A$. Thus from the preceding discussion we obtain

$$\sum_{n\leq x} d_k(n)z^{\omega(n)} = x\sum_{j=1}^{N} e_{k,j}(z)\log^{kz-j} x + R_{k,N}(x,z), \tag{2.6}$$

where each $e_{k,j}(z)$ and $R_{k,N}(x,z)$ are regular functions of z for $|z| \leq A$, and

$$R_{k,N}(x,z) \ll x(\log x)^{\mathrm{Re}\, kz-N-1}.$$

Moreover, we have $e_{k,j}(0) = 0$ for all $j \geq 1$ and $e_{k,j}(1) = 0$ for $j \geq k+1$, since $c_j(kz) = B_j(kz)/\Gamma(kz-j+1)$ (the c_j's are defined by (2.4)), the gamma-function has poles at nonpositive integers, and each $e_{k,j}(z)$ is seen to be a linear combination of the $c_j(kz)$'s. The formula (2.6) is a generalization of the formula obtained for $k = 1$ by Delange [3], who obtained his result by complex integration. The same could have been done for (2.6) to, but it seemed simpler to use convolution and the sharp existing formula (2.4) for $D_z(x)$. The idea of our approach is to link directly this topic to the theory of the Riemann zeta-function.

Now we choose $A = 3/2$, $r = 1/\log\log x$ and set $s = z + re^{i\theta}$, $0 \leq \theta < 2\pi$. By Cauchy's formula for derivatives of analytic functions we have, for $m \geq 1$,

$$\frac{\partial^m R_{k,N}(x,z)}{\partial z^m} = \frac{m!}{2\pi i}\int_{|s-z|=r} R_{k,N}(x,s)(s-z)^{-m-1}ds$$

$$\ll x(\log x)^{\mathrm{Re}\, kz-N-1}r^{-m}\int_0^{2\pi} d\theta \ll x(\log x)^{\mathrm{Re}\, k=-N-1}(\log\log x)^m.$$

Hence for $|z| \leq 1$ it follows that uniformly

$$\partial^m R_{k,N}(x,z)/\partial z^m \ll x(\log x)^{k-N-1}(\log\log x)^m \quad (m = 1,2,\ldots), \tag{2.7}$$

and this bound cannot be improved. It is this step in the proof of (2.2) which was carried out incorrectly in [2], where for $k = 2$, $m = N = 1$ the $\log\log x$-factor in (2.7) was missing. Now we differentiate (2.6) with respect to z. This gives, for $|z| \le 1$,

$$\sum_{n\le x} d_k(n)\omega(n)z^{\omega(n)-1} = x\sum_{j=1}^{N}(e'_{k,j}(z) + ke_{k,j}(z)\log\log x)\log^{kz-j} x + \\ + \frac{\partial R_{k,N}(x,z)}{\partial z}. \tag{2.8}$$

Replacing N by $N+1$, using (2.7) with $m = 1$ and recalling that $e_{k,j}(1) = 0$ for $j \ge k+1$, we obtain upon setting $z = 1$ in (2.8) the following

THEOREM 1. *Let $k \ge 2$ be fixed and N be an arbitrary, but fixed integer for which $N > k$. Then there exist computable constants $a_{k,j}, b_{k,j}, c_{k,j}$ $(a_k, j \ne 0)$ such that*

$$\sum_{n\le x} d_k(n)\omega(n) = x\sum_{j=1}^{N}(a_{k,j}\log\log x + b_{k,j})\log^{k-j} x \\ + x\sum_{j=k+1}^{N} c_{k,j}\log^{k-j} x + O(xlog^{k-N-1}x). \tag{2.9}$$

In the special case $k = 2$ this asymptotic expansion yields a considerable sharpening of (2.3). Proceeding from (2.6) as in (1.3), and using (2.7) with an arbitrary m, we obtain also

THEOREM 2. *Let $m, N \ge 1$ and $k \ge 2$ be fixed integers. Then there exist polynomials $P_{k,m,j}(t)$ $(j = 1, \dots, N)$ of degree m in t with computable coefficients such that*

$$\sum_{n\le x} d_k(n)\omega^m(n) = x\sum_{j=1}^{N} P_{k,m,j}(\log\log x)\log^{k-j} x \\ + O(x(\log x)^{k-N-1}(\log\log x)^m). \tag{2.10}$$

3. Further applications of the method

Theorems 1 and 2 remain valid if $\omega(n)$ is replaced by $\Omega(n)$, as hinted in §**2**, and it seems difficult to obtain formulas such as (2.10) by elementary methods. Some caution, however, must be displayed in dealing with $\Omega(n)$, since this is a "larger" function than $\omega(n)$ and the analogue of (2.6) is valid only for $|z| \le 2 - \varepsilon$, which follows if one considers the analogue of (2.5) for $\Omega(n)$. The ideas connected with (1.3) and (1.6) may be combined to deduce from the asymptotic expansion of $\sum_{n\le x} d_k(n)z^{\omega(n)}w^{\Omega(n)}$, by successive differentiation and integration (the latter is explained in full detail in [1, Chs. 2 and 5]), an asymptotic formula for the summatory function of $d_k(n)\omega^m(n)\Omega^r(n)$, when m and r are fixed integers (not necessarily positive!). The details of the analysis are omitted and the result, which is a generalization of both Theorem 1 and Theorem 2, is the following

THOREM 3. *Let $k \ge 2$, m, r be fixed integers (not necessarily positive), and let $N \ge 1$ be an arbitrary, fixed integer. Then*

$$\sum_{2\le n\le x} d_k(n)\omega^m(n)\Omega^r(n) = x\sum_{j=1}^{N} c_j L_j(x)\log^{k-j} x + O(x(\log x)^{k-N-1}(\log\log x)^{m+r}, \tag{3.1}$$

where for any fixed integer $M \ge 1$ there exist computable constants $b_{1,j},\dots,b_{M,j}$ such that for $j = 1,\dots,N$

$$L_j(x) = (\log\log x)^{m+r}\left\{1 + \frac{b_{1,j}}{\log\log x} + \frac{b_{M,j}}{(\log\log x)^M} + O\Big(\frac{1}{(\log\log x)^{M+1}}\Big)\right\}$$

if either $m \le -1$ or $r \le -1$, otherwise $L_j(x)$ is a polynomial in $\log\log x$ of degree $m + r$.

One may obtain the analogues of Theorems 1 – 3 if $d_k(n)$ in replaced by some other common multiplicative functions such as

$$f(n) = \mu^2(n),\ \ \frac{1}{4}r(n) = \frac{1}{4}\sum_{n=a^2+b^2} 1,$$

$a(n)$ (the number of nonisomorphic abelian groups with n elements) which corresponds to $k = 1$ on the right-hand side of (3.1), $f(n) = d(n^2)(k = 3)$, $f(n) = d^2(n)$ $(k = 4)$ etc. The analysis is again very similar to the previous case and therefore the details are omitted.

Another possibility is to take for $f(n)$ a multiplicative function for which $f(p)$ is not exactly a constant, but $f(p) = C + o(1)$ as $p \to \infty$ for some $C > 0$. Examples such as

$$f(n) = \frac{\sigma(n)}{n} = \sum_{d\mid n}\frac{1}{d},\quad f(n) = \frac{\varphi(n)}{n} = \prod_{p\mid n}\left(1 - \frac{1}{p}\right)$$

and their unitary analogues $\sigma^*(n)/n$, $\varphi^*(n)/n$ immediately come to mind (recall that $\sigma^*(p^a) = p^a + 1$ and $\varphi^*(p^a) = p^a + 1$). For simplicity we shall consider now $f(n) = \varphi(n)/n$, but obviously the argument is fairly general. For $\operatorname{Re} s > 1$ and $|z| \le A$ $(A > 1$ fixed) we have the Dirichlet series representation

$$\sum_{n=1} f(n)z^{\omega(n)}n^{-s} = \prod_p\left(1 + \Big(1 - \frac{1}{p}\Big)zp^{-s} + \Big(1 - \frac{1}{p}\Big)zp^{-2s} + \dots\right)$$

$$= \zeta^z(s)\prod_p(1 - p^{-s})^z\left(1 + \frac{(p-1)z}{p^{s+1} - p}\right) = \zeta^z(s)\sum_{n=1}^{\infty} g(n,z)n^{-s} = \zeta^z(s)G(s,z),$$

say. Following the discussion that leads to (2.6), it is seen that the Dirichlet series for $G(s,z)$ is absolutely convergent for $\mathrm{Re}\, s \geq 1/2+\varepsilon$ and $|z| \leq A$. Hence by the method of §**2** we obtain (this corresponds to Theorem 1 with $k=1$)

$$\sum_{n\leq x} f(n)\omega(n) = x(A\log\log x + B) + x\sum_{j=1}^{N} c_j \log^{-j} x + O(x\log^{-N-1} x) \tag{3.2}$$

with some computable constants $A \neq 0$, B, $c_1,\dots,c_N$, and $N \leq 1$ arbitrary, but fixed. By partial summation we have

$$\sum_{n\leq x} \varphi(n)\omega(n) = x\sum_{n\leq x} f(n)\omega(n) - \int_1^x \Big(\sum_{n\leq t} f(n)\omega(n))dt,$$

hence using (3.2) and simplifying we deduce

THEOREM 4. *If $N \geq 1$ is an arbitrary, but fixed integer, then there exist computable constants $C > O, D, d_1,\dots,d_N$ such that*

$$\sum_{n\leq x} \varphi(n)\omega(n) = Cx^2\log\log x + Dx^2 + x^2\sum_{j=1}^{N} d_j \log^{-j} x + O(x^2\log^{-N-1} x). \tag{3.3}$$

Obviously (3.3) remains valid (with perhaps different constants) if $\omega(n)$ is replaced by $\Omega(n)$, and if φ is replaced by $\sigma, \varphi^*, \sigma^*$, or more generally, by a suitable "polynomial-like" multiplicative function, i.e. by a function for which $f(p^a)$ is a monic polynomial of degree a in p. Multiplicative functions f which are "larger" than φ or σ may be also considered, but they should be first appropriately normalized. Consider, for example, the function $\varphi^2(n)\sigma(n)$. Then setting $f(n) = \varphi^2(n)\sigma(n)n^{-3}$ we have $f(p) = 1 + O(1/p)$, and a suitable analogue of (3.2) holds. Partial summation yields then the analogue of (3.3) for $\varphi^2(n)\sigma(n)$ with x^2 replaced by x^4 on the right-hand side of (3.3). Also by the foregoing methods more general sums than (3.3), such as

$$\sum_{2\leq n\leq x} \varphi(n)\omega^m(n)\Omega^r(n), \quad (m, r \text{ fixed integers})$$

may be estimated, analogously as in Theorem 3.

A special arithmetic sum was investigated by the author in [6], where it was shown that

$$\sum_{1\leq n\leq x} \left(\frac{\omega(n) - \log\log n}{a(n)}\right)^2 = Ax\log\log x + O(x) \tag{3.4}$$

holds for some constant $A > 0$ ($a(n)$ is the number of nonisomorphic abelian groups with n elements). The motivation for this result is the classical formula of P. Turán [7, Ch. 14) that (3.4) holds without $a(n)$, in which case a sharp asymptotic formula may be obtained by the method of Delange [3, p. 136]. One obtains (3.4) in [6] by squaring $\omega(n) - \log\log n$ and estimating each ensuing sum separately (since $\log\log n - \log\log x + O(1)$ for $x^{1/2} < n \le x$, one may replace $\log\log n$ by $\log\log x$ in (3.4)). Using the technique of Theorem 1 one may considerably sharpen (3.4) and show that

(3.5)
$$\sum_{2\le n\le x}\left(\frac{\omega(n)-\log\log n}{a(n)}\right)^2 = Ax\log\log x + Bx + x\sum_{j=1}^{N} P_j(\log\log x)\log^{-j} x + O\left(\frac{x(\log\log x)^2}{(\log x)^{N+1}}\right)$$

for some computable constants $A \ne 0, B$, and quadratic functions $P_j (j = 1, \ldots, N)$. Similar results hold if $a(n)$ is replaced by a multiplicative function $f(n)$ such that $f(p) = 1 + O(p^{-n})$ for some $\eta > 0$ as $p \to \infty$ and $f(n)$ is in a suitable sense of moderate growth. An appropriate analogue of (3.5) may be derived if $a(n)$ is replaced by $d_k(n), \omega(n)$ by $\Omega(n)$ etc.

4. Local problems

As stated in §**1**, by a local problem we shall mean estimations of the sums over $n \le x$ for which $g(n) - q$. The method of **2** allows us to estimate sums of the form $\sum_{n\le x, g(n)=q} f(n)$ when $f(n) = d_k(n)$ (or any of the other multiplicative functions mentioned in §**3**, such as $\mu^2(n)$, $a(n)$, etc.), and $g(n)$ is a suitable nonnegative, integer-valued additive function. In particular, one may derive sharp asymptotic formulae for the above sum when $g(n) = \omega(n)$, $\Omega(n)$ or $\Omega(n) - \omega(n)$. In the sequel we shall suppose that $q \ge 0$ is a fixed integer, since the case when $q = q(x)$ is a function of x is much more difficult.

We start from (2.6), noting that the left-hand side is a polynomial in z whose coefficient of z^q is exactly $\sum_{n\le x,\omega(n)=q} d_k(n)$. Further we shall use the series expansion

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$$\log^{kz-j} x = \log^{-j} x \sum_{r=0}^{\infty} k^r z^r \frac{(\log\log x)^r}{r!},$$

and we recall that $e_{k,j}(0) = 0$ for $j \ge 1$. Thus finding the coefficient of z^q on the right-hand side of (2.6) we obtain

THEOREM 5. *Let k, $q \ge 1$ be fixed integers, and $N \ge 1$ be an arbitrary but fixed integer. Then there exist polynomials $P_1(t), \ldots, P_N(t)$ of degree $q-1$ in t whose coefficients are computable constants depending on k and g such that*

(4.1)
$$\sum_{n\le x,\omega(n)=q} d_k(n) = x\sum_{j=1}^{N} P_j(\log\log x)\log^{-j} x + O(x(\log x)^{-N-1}(\log\log x)^{q-l}).$$

When $k = 1$ (i.e., $d_1(n) = 1$ for all n) this is a classical result, proved first by Delange [3] (see also [1, Ch. Sj), in this degree of sharpness. It is possible to obtain similar formulae if $d_k(n)$ is replaced by other multiplicative functions $f(n)$. We formulate only one example:

$$(4.2)\qquad \sum_{n\le x,\omega(n)=q}\sigma(n) = x^2\sum_{j=1}^{N}P_j(\log\log x)\log^{-j}x + O(x^2(\log x)^{-N-1}(\log\log x)^{q-1}),$$

where q, N and P_j have a similar meaning as in (4.1). Both (4.1) and (4.2) holds if $\omega(n)$ is replaced by $\Omega(n)$.

We continue our discussion by considering the local problem connected with $\Omega(n)-\omega(n)$. For $\mathrm{Re}\, s > 1$ and $|z| \le A$ ($A > 1$ fixed) we have

$$(4.3)\qquad \begin{aligned}&\sum_{n=1}^{\infty} d_k(n) z^{\Omega(n)-\omega(n)} n^{-s} =\\ &= \left(1 + kp^{-s} + \binom{k+1}{2} zp^{-2s} + \binom{k+2}{3} z^2p^{-3s} + \ldots\right) =\\ &= \zeta^k(s)G(s,z),\end{aligned}$$

where

$$\begin{aligned}G(s,z) &= \sum_{n=1}^{\infty} g(n,z)n^{-s} =\\ &= \prod_p (1-p^{-s})^k \left(1 + kp^{-s} + \binom{k+1}{2} zp^{-2s} + \binom{k+2}{3} z^2p^{-3s} + \ldots\right)\end{aligned}$$

is a Dirichlet series which converges absolutely and uniformly for $\mathrm{Re}\, s \ge 1/2 + \varepsilon$, $|z| \le A$, where it represents a regular function of z. Hence by (4.3) we have

$$\sum_{n\le x} d_k(n) z^{\Omega(n)-\omega(n)} = \sum_{mn\le x} g(n,z)d_k(m) = \sum_{n\le x} g(n,z)D_k(x/n).$$

The estimation of D_k is known in the literature (see [7, Ch. 13], for a detailed discussion) as general Diriclet divisor problem, and one has

$$(4.5)\quad D_k(x) = \sum_{n\le x} d_k(n) = \operatorname*{Res}^{n\le x} \zeta^k(s)x^s s^{-1} + O(x^{\alpha_k+\varepsilon} = xP_{k-1}(\log x) + O(x^{\alpha_k+\varepsilon})$$

for a suitable polynomial $P_{k-1}(t)$ of degree $k-1$ in t and a constant a_k which satisfies $(k-1)/2k \le \alpha_k \le (k-1)/(k+1)$ for all $k \ge 2$. Inserting (4.5) in (4.4) we obtain, after some simplification

$$(4.6)\qquad \sum_{n\le x} d_k(n) z^{\Omega(n)-\omega(n)} = x\sum_{j=0}^{k-1} h_j(z)\log^j x + R(x,z),$$

where, for $|z| \le A$, $R(x,z)$ and each $h_j(z)$ are regular functions of z and uniformly

$$R(x,z) \ll x^{1/2+\varepsilon} + x^{\alpha_k+\varepsilon}. \tag{4.7}$$

This formula serves as the analytic basis for the derivation of results involving $d_k(n)$ and $\Omega(n) - \omega(n)$. We have

THEOREM 6. *Let $m, q \ge 0$ and $k \ge 1$ be fixed integers. Then there exist computable constants $A_{1,m}, \ldots, A_{k-1,m}$ and $B_{1,q}, \ldots, B_{k-l,q}$, such that*

$$\sum_{n\le x} d_k(n)(\Omega(n) - \omega(n))^m = x\sum_{j=0}^{k-1} A_{j,m}\log^j x + O(x^{1/2+\varepsilon} + x^{\alpha_k+\varepsilon}) \tag{4.8}$$

and

$$\sum_{n\le x, \Omega(n)-\omega(n)=q} d_k(n) = x\sum_{j=0}^{k-1} B_{j,k}\log^j x + O(x^{1/2+\varepsilon} + x^{\alpha_k+\varepsilon}), \tag{4.9}$$

where α_k is defined by (4.5).

One obtains (4.8) by differentiating (4.6) in the manner of (1 .3), while (4.9) follows by equating the coefficients of z^q on both sides of (4.6). On may replace $d_k(n)$ in Theorem 6 by various other multiplicative functions on the same lines as in Theorem 5. Both (4.8) and (4.9) for $k = 1$ are well-known in the literature. In fact, the estimation of the sum in (4.9) when $k = 1$ is known as "Rényi's problem". The sharpest known formula for this sum is due to Delange [4] (see also [1, Ch. 5], for a proof). In this case Delang 'es formula is sharper than (4.9) with $k = 1$. Since $\alpha_2 \le 35/108$ and $\alpha_3 \le 43/96$ is known to hold in (4.5) (all the other known bounds for α_k are not smaller than 1/2; see [7, ch. 13]), one can presumably replace the error term in (4.9) for $k = 2$ and $k = 3$ by a more precise expression, and if the Lindelöf hypothesis that $\zeta(1/2 + it) \ll |t|^\varepsilon$ is true, then for all $k \ge 2$. This would follow by suitably adapting the elaborate method of Delange [4] used for $k = 1$. We shall not go into this matter here.

In concluding, let it be mentioned that the foregoing methods may be used to investigate "double' and "multiple" local problems. As an example, we state the asymptotic expansion

$$\sum_{n\le x, \Omega(n)-\omega(n)=q} d_k(n) = x\sum_{j=1}^{N} P_j(\log\log x)\log^{-j} x + O(x(\log x)^{-N-1}(\log\log x)^{q-1}), \tag{4.10}$$

where $q, k \ge 1$, $r \ge 0$ are fixed integers, $N \ge 1$ is an arbitrary but fixed integer, and each $P_j(t)$ is a polynomial in t of degree $q - 1$ whose coefficients depend on k, q, r. The formula (4.10) follows on comparing the coefficients of $z^q w^r$ on both sides of the relation

$$\sum_{n\le x} d_k(n) z^{\omega(n)} w^{\Omega(n)-\omega(n)} = x\sum_{j=1}^{N} f_j(z,w)\log^{kz-j} x + O(x(\log x)^{\mathrm{Re}\, kz-N-1}),$$

valid for some $f_j(z,w)$ $(f_j(0,w)=0,\ j=1,\dots,N)$ which are regular functions of both z and w for $|z|\le 3/2$, $|w|\le 3/2$ and $N\ge 1$ is arbitrary, but fixed.

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Katedra Matematike
RGF-a Univerziteta u Beogradu
Đušina 7, 11000 Beograd
Jugoslavija

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