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SOME PROPERTIES OF THE QUASIASYMPTOTIC OF SCHWARTZ DISTRIBUTIONS PART I: QUASIASYMPTOTIC AT $\pm\infty$

Stevan Pilipović

Abstract. Using the results for the quasiasymptotic at $+\infty$ of tempered distributions from $\mathcal{S}'_+$ and the results from [**5**] we give several properties of the quasiasymptotic at $+\infty$ of Schwartz distributions.

1. Introduction

The quasiasymptotic at ∞ of tempered distributions which have their supports in $[0,\infty)$ (the space of such distributions is denoted by $\mathcal{S}'_+$) have been studied by Soviet mathematicians Drožžinov, Vladimirov and Zavialov in many pepers (see [**8**] and references therein).

We extend this notion to the space of Schwartz distributions (on the real line):

Definition 1. It is said that an $f \in \mathcal{D}'$ has the quasiasymptotic at $\pm\infty$ with recpect to some positive measurable function $c(k),\ k \in (a,\infty),\ a > 0$ if for some $g \in \mathcal{D}',\ g \neq 0$,

$$\lim_{k\to\infty} \langle f(kx)/c(k),\ \Phi(x)\rangle = \langle g(x),\ \Phi(x)\rangle,\quad \Phi \in \mathcal{D} \tag{1}$$

In this case we write $f \sim^q g$ at $\pm\infty$ with respect to $c(k)$.

Let us recall (see [**4**]) that a positive measurable function $L(x),\ x \in (a,\infty)$, resp. $x \in (0,a),\ a > 0$, is called slowly varying at ∞, resp. 0^+, if for any $\lambda > 0$

$$\lim_{x\to\infty} L(\lambda x)/L(x) = 1,\ \text{ resp. } \lim_{x\to 0^+} L(\lambda x)/L(x) = 1.$$

For the properties of slowly varying functions we refer to [**4**].

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Because of [**4,** 1.4] we can and we shall always assume that L is a continuous function.

The following theorem is proved in [**5**].

THEOREM 1. *Let $f \in \mathcal{D}'$ have the quasiasympotic at $\pm\infty$ with respect to some positive continuous function $c(k)$, $k > a$. Then: (i) $f \in \mathcal{S}'$; (ii) There are $\nu \in R$ and a slowly varying function $L(k)$, $k > a$, such that $c(k) = k^\nu L(k)$, $k > a$; moreover, (g) is a homogeneous distribution with the order of homogenity ν; (iii) if $\nu \in R\backslash(-N)$, then (1) holds in the sense of convergence in $\mathcal{S}'$ (i. e. for $\varphi \in \mathcal{S}$).*

Using the quoted results from [**5**] and the results of the mentioned Soviet mathematicians we shall give in this paper several properties of the quasiasymptotic behaviour at $\pm\infty$ of Schwartz distributions.

2. Some properties

Several trivial properties of the quasiasymptotic at $\pm\infty$ are given in the following theorem.

THEOREM 1. *Let $f \in \mathcal{D}'$ and $f \sim^q g$ at $\pm\infty$ with respect to $k^\nu L(k)$. Then: (i) $f^{(\alpha)}$ at $\pm\infty$ with respect to $k^{\nu-\alpha}L(k)$, $\alpha \in N$ (we assume $g^{(\alpha)} \neq 0$); (ii) if $\nu \notin -N$ and $m \in N$ then $x^m f \sim^q x^m g$ at $\pm\infty$ with respect to $k^{\nu+m}L(k)$; (iii) if $\nu = -n$, $n \in N$, and $m \in N$ such that $m < n$, then (ii) holds as well; (iv) If $\Phi \in \mathcal{E}$ and c_1 is a measurable positive function on some interval (a, ∞), $a > 0$, such that*

$$\Phi(kx)/c_1(k) \to \Phi_0(x) \ \text{ in } \mathcal{E},\ k \to,\ x \in R,$$

then $f\Phi \sim^q g\Phi_0$ at $\pm\infty$ with respect to $c_1(k)k^\nu L(k)$.

Let us only remark that (iv) follows from [**7**, T. I, p. 72, Théorème X].

THEOREM 2. *Let $f \in \mathcal{E}'$ and $f \sim^q g$ at $\pm\infty$ with respect to $k^\nu L(k)$. Then $L(k) = 1$, $k > a$ for some $a > 0$, and $\nu \in -N$. Moreover, the limit in (1) can be exstended on $\mathcal{S}$.*

Proof. It is well-known that f can be written in the from $f = \sum_{k=o}^m f_k^{(k)}$, where f_k, $k = 0, \dots, m$, are continuous functions with compact supports. If F is a continuous function with the compact support, then one can easily prove that for some C

$$\lim_{k\to\infty} \left\langle \frac{F(kx)}{k^{-1}},\ \Phi(x) \right\rangle = \langle C\delta(x),\ \Phi(x)\rangle, \quad \Phi \in \mathcal{S}.$$

This implies the assertion.

Let us recall ([**7**, p. 88]) that the scale of distribution f_{nu+1}, $\nu \in R$; is defined in the following way.

$$f_{\nu+1}(x) = H(x)x^\nu\Gamma(\nu+1) \ \text{ for } \ \nu > -1$$

$$x \in R,$$

$$f_{\nu+1}(x) = f_{\nu+n+1}^{(n)}(x) \ \text{ for } \ \nu \leq -1$$

where $n \in N$ and $n + \nu > -1$. H is the Heaviside function.

As it us usual, we identify locally integrable function with the corresponding distributions.

THEOREM 3. *Let F be a locally integrable function and $\nu \in R$, $\nu > -1$, such that*

$$\lim_{\substack{x\to+\infty\\ x\to-\infty}} \frac{F(x)}{|\, x\,|^{\nu}\, L(|\, x\,|)} = C_{\pm} \text{ where } (C_+, C_-) \neq (0,0).$$

Then $F \sim^q g$ at $\pm\infty$ with respect to $k^\nu L(k)$ where

$$g(x) = \overline{C}_+ f_{\nu+1}(x) + \overline{C}_- f_{\nu+1}(-x),\ x \in R \text{ and } (\overline{C}_+, \overline{C}_- \neq (0,0).$$

Proof. Let us put $F_+(x) = H(x)F(x)$ and $F_-(x) = H(-x)F(x)$, $x \in R$. It is well-known ([1]) that for any $\Phi \in \mathcal{S}$

$$\Big\langle \frac{F_+(kx)}{k^\nu L(k)},\ \Phi(x)\Big\rangle \to \langle g_+(x),\quad \Phi(x)\rangle,\quad k \to \infty,$$

$$\Big\langle \frac{F_-(kx)}{k^\nu L(k)},\ \Phi(x)\Big\rangle \to \langle g_-(x),\quad \Phi(x)\rangle\quad k \to \infty$$

where

$$g_\pm(x) = \overline{C}_\pm f_{\nu+1}(\pm x),\ x \in R,\ \text{ with } (\overline{C}_+, \overline{C}_-) \neq (0,0).$$

This implies the assertion.

THEOREM 4. *Let $f \in \mathcal{D}'$ and $f \sim^q$ at $\pm\infty$ with respect to $k^\nu L(k)$ where $\nu \in R\backslash(-N)$. There are $m \in N_0$ and a locally integrable function F such that*

$$f = F^{(m)} \text{ and } \lim_{\substack{x\to+\infty\\ x\to-\infty}} \frac{F(x)}{|\, x\,|^{\nu+m}\, L(|\, x\,|)} = C_\pm$$

where $(C_+, C^-) \neq (0,0)$.

Proof. Since $f \in \mathcal{S}'$ (Theorem I), let $f = f_+ + f_-$, where $f_+ \in \mathcal{S}'_+$ and $f_- \in \mathcal{S}'_-$ (supp $f_\subset(-\infty, 0]$. Theorem I implies that for every $\Phi \in \mathcal{S}$

$$\Big\langle \frac{f(kx)}{k^\nu L(k)},\ \Phi(x)\Big\rangle \to \langle g(x),\quad \Phi(x)\rangle\quad k \to \infty.$$

Now [**2**, Lemmas 2.2 and 2.3] implies (see Part II, the proof of Theorem 9) that

$$\Big\langle \frac{f_\pm(kx)}{k^\nu L(k)},\ \Phi(x)\Big\rangle \to \langle \widetilde{C}_\pm f_{\nu+1}(\pm x),\quad \Phi(x)\rangle\quad k \to \infty.$$

The structural theorem [**1**, Theorem I] implies that there are locally integrable functions F_1 and F_2 with supp $F_1 \subset [0,\infty)$, supp $F_2 \subset (-\infty, 0]$, and $m \in N_0$ such that

$$f_+(x) = F_1^{(m)}(x),\ \ f_-(x) = F_2^{(m)}(x),\quad x \in R,$$

and

$$\lim_{x\to\infty}\frac{F_1(x)}{x^{\nu+m}L(x)}=C_+,\quad \lim_{x\to-\infty}\frac{F_2(x)}{|\,x\,|^{\nu+m}\,L(|\,x\,|)}=C_-.$$

This completes the proof.

THEOREM 5. *Let $f\in\mathcal{S}'$ and $\Phi_0\in\mathcal{D}$ such that $\int\Phi_0(t)dt=1$. Let $f'\sim^q g$ at $\pm\infty$ with respect to $k^\nu L(k)$, $\nu\in R$, and*

$$\Big\langle\frac{f(kx)}{k^{\nu+1}L(k)},\quad \Phi_0(x)\Big\rangle\to\langle g_0(x),\quad \Phi_0(x)\rangle$$

for some $g_0\in\mathcal{S}'$ for which there holds: $g_0'=g$. Then $f\sim^q g_0$ at $\pm\infty$ with respect to $k^{\nu+1}L(k)$, (g ang g_o are determined by Theorem 1.)

Proof. It is well-known that for any $\Phi\in\mathcal{D}$ there exist $\psi\in\mathcal{D}$ such that

$$\Phi(x)=\Phi_0(x)\int\Phi(t)dt+\psi(x),\quad x\in R,$$

and ψ is of the form $\psi=\psi_1'$ for some $\psi_1\in\mathcal{D}$. We have

$$\begin{aligned}&\Big\langle\frac{f'(kx)}{k^{\nu+1}L(k)},\quad \Phi(x)\Big\rangle=\Big\langle\frac{f(kx)}{k^{\nu+1}L(k)},\quad \Phi_0(x)\Big\rangle\int\Phi(t)dt\\ &-\Big\langle\frac{f'(kx)}{k^{\nu}L(k)},\quad \psi(x)\Big\rangle\to\langle g_0(x),\quad \Phi_0(x)\rangle\int\Phi(t)dt\\ &-\langle g(x),\quad \psi_1(x)\rangle=\langle g_0(x),\quad \Phi(x)\rangle\ \text{ as }\ k\to\infty.\end{aligned}$$

This proves the assertion.

3. The Fourier transformation and the quasiasymptotic

The Fourier and inverse Fourier transformation in $\mathcal{S}$ and $\mathcal{S}'$ are defined in a usual way ([**7**]). The connection between the quasiasymptotics at 0 and $\pm\infty$ is given in the theorem which follows. Note that the definition of the quasiasymptotic behaviour at 0 (in $\mathcal{S}'$) is given in Part II (Definition 2).

THEOREM 6. *Let $f\in\mathcal{D}'$ and $\nu\in R\backslash(-N)$. If*

$$f\sim^q\ \text{ at }\ \pm\infty\ \text{ with respect to }\ k^\nu L(k)\tag{2}$$

then

$$\hat f\sim^q\hat g\ \text{ at } 0\ \text{ with respect to }\ (1/k)^{-\nu-1}L_1(1/k)\ (\text{in }\mathcal{S}')\tag{3}$$

where $L_1(\cdot)=L(1/\cdot)$ is slowly varying at 0^+.

Conversely, if $f\in\mathcal{S}'$ and (3) holds with $\nu\in R$, then (2) holds.

Proof. Let $\Phi\in\mathcal{S}$. We have

$$\begin{aligned}\Big\langle\frac{f(kx)}{k^\nu L(k)},\quad \hat\Phi(x)\Big\rangle&=\Big\langle\widehat{\frac{f(kx)}{k^\nu L(k)}},\quad \Phi(x)\Big\rangle\\ &=\Big\langle\frac{\hat f(x/k)}{(1/k)^{-\nu-1}L_1(1/k)},\quad \Phi(x)\Big\rangle,\quad k>0.\end{aligned}$$

This implies the assertion.

We studied in [**6**] the notion of the S-asymptotic at ∞ of Schwartz distributions. The relation between this notion and the quasiasymptotic at $+\infty$ can be deduced from the following theorem.

THEOREM 7. *Let $f \in \mathcal{S}'$ and $\Phi = \mathcal{F}^{-1}(\psi)$ where $\psi \in \mathcal{S}$ such that $\psi = 1$ in some neighbourhood of U. If*

$$\lim_{k\to+\infty} \frac{(f*\Phi)(k)}{k^\nu L(k)} = C_+ \quad \textit{and} \quad \lim_{k\to+\infty} \frac{(f*\Phi)(k)}{k^\nu L(k)} = C_-$$

where $\nu > -1$ and $(C_+, C_-) \neq (0,0)$, then $f \sim^q g$ at $\pm\infty$ with respect to $k^\nu L(k)$ where $g \neq 0$ is a suitable distribution from $\mathcal{S}'$.

(For the convolution in $\mathcal{S}'$ see [**7**]).

Proof. Theorem 3 implies that $f*\Phi \sim^q g$ at $\pm\infty$ with respect to $k^\nu L(k)$ $(g \neq 0)$. From the well-known exchange formula and Theorem 6 it follows that $\hat{f}\psi \sim^q \hat{g}(x)$ at 0 (in $\mathcal{S}'$) with respect to $(1/k)^{-\nu-1}L_1(1/k)$, where $L_1(\cdot) = L(1/\cdot)$.

Since the quasiasymptotic at 0 is the local property of a distribution (see Part II) we obtain that $\hat{f} \sim^q C\hat{g}$ at 0 with respect to $(1/k)^{-\nu-1}L_1(1/k)$ $(C \neq 0)$. Now, Thorem 6 implies the assertion.

5. The convolution and the quasiasymptotic

THEOREM 8. *Let $T \in \mathcal{E}'$ and $T \sim^q g_1$, at $\pm\infty$ with recpect to k^ν, $\nu \in -N$ (see Theorem 2). Let $f \in \mathcal{D}$ and $f \sim^q g$ at $\pm\infty$ with respect to $k^\alpha L(k)$, $\alpha \in R\backslash(-N)$. Then $T * f \sim^q g_1 * g$ with respect to $k^{\alpha+\nu+1}L(k)$.*

Proof. Let $\Phi \in \mathcal{S}$. Using the properties of the Fourier transformation we have (with $L_1(\cdot) = L(1/\cdot)$)

$$\left\langle \frac{(T*f)(kx)}{k^{\alpha+\nu+1}L(x)}, \quad \hat{\Phi}(x)\right\rangle = \left\langle \frac{\hat{T}(x/k)\hat{f})(x/k)}{k^{\alpha+\nu+2}L(k)}, \quad \Phi(x)\right\rangle \tag{4}$$

$$= \left\langle \frac{\hat{f}(x/k)}{(1/k)^{-\alpha-1}L_1(1/k')}, \quad \frac{\hat{T}(x/k)}{(1/k)^{\nu-1}}\Phi(x)\right\rangle.$$

Since $\hat{T}$ is an entire function of polynomial growth when $|\, x \,|\to \infty$, it must be of the form $\hat{T}(x) = x^{-\nu-1}T_1(x)$, $x \in R$, where T_1 is an entire function of polynomial growt such that $T_1(0) = C \neq 0$. All the derivatives of $\hat{T}$ are of polynomial growth when $|\, x \,|\to \infty$. So, the same holds for T_1. This implies that for any $\Phi \in \mathcal{S}$.

$$\left\langle \frac{1}{k^{\nu+1}}(x/k)^{-\nu-1}T_1(x/k), \quad \Phi(x)\right\rangle = \langle x^{\nu-1}T_1(x/k), \quad \Phi(x)\rangle \tag{5}$$

$$\to \langle x^{\nu-1}T_1(0), \quad \Phi(x)\rangle, \quad k\to\infty,$$

in the sense of convergence in $\mathcal{S}$. Let us note that $\hat{g}_1(x) = x^{-\nu-1}T_1(0)$, $x \in R$.

In the spaces $\mathcal{S}$ and $\mathcal{S}'$ the strong and weak sequential convergence are equaivalent. This implies that

$$\Big\langle \frac{\hat{f}(x/k)}{(1/k)^{-\alpha-1}L_1(1/k)}, \quad \frac{\hat{T}(x/k)}{(1/k)^{-\nu-1}}, \Phi(x) \Big\rangle$$

$$\to \langle \hat{g}(x), \quad \hat{g}_1(x)\Phi(x)\rangle = \langle (g_1 * g)(x), \quad \hat{\Phi}(x)\rangle, \quad k \to \infty.$$

By (4) we obtain the assertion.

THEOREM 9. *Let $f \in \mathcal{D}'$ and $\{f(kx)/k^\alpha L(k), \ k > a\}$ $\alpha \in R\backslash(-N)$, be a bounded subset of $\mathcal{D}'$. Let $T \in \mathcal{E}'$ and $T \sim^q g_1$ at $\pm\infty$ with respect to k^{-1}. If $T * f \sim^q g_2$ at $\pm\infty$ with respect to $k^\alpha L(k)$, then $f \sim^q g$ at $\pm\infty$ with respect to $k^\alpha L(k)$ and $g_2 = g$.*

Proof. The same arguments, as in Theorem I imply that $f \in \mathcal{S}'$ and that $\{f(k\cdot)/k^\alpha L(k), \ k > a\}$ is a bounded subset of $\mathcal{S}'$. With the same arguments as above we have ($\Phi \in \mathcal{S}$)

$$\Big\langle \frac{\hat{f}(x/k)}{(1/k)^{-\alpha-1}L_1(1/k)}, \quad \Phi(x)(1-\hat{T}(x/k)) \Big\rangle \to 0 \text{ as } k \to \infty.$$

This implies the assertion.

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