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ON THE LOGARITHMIC DERIVATIVE OF SOME BAZILEVIC FUNCTIONS

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Abstract. For $\alpha > 0$, $0 \leq \beta < 1$, let $B_0(\alpha, \beta)$ be the class of normalised analytic functions f defined in the open unit disc D such that

$$\operatorname{Re} e^{i\psi} \left(f'(z)\big(f(z)/z\big)^{\alpha-1} - \beta\right) > 0$$

for $z \in D$ and for some $\psi = \psi(f) \in \mathbf{R}$. Upper and lower bounds for the logarithmic derivative zf'/f for $f \in B_0(\alpha, \beta)$ are obtained.

Introduction

For $\alpha > 0$, denote by $B_0(\alpha)$ the class of normalised analytic functions f defined in the unit disc $D = \{z \colon |z| < 1\}$ satisfying the condition

$$\operatorname{Re}\, e^{i\psi} f'(z)\big(f(z)/z\big)^{\alpha-1} > 0$$

for $z \in D$ and for some $\psi = \psi(f) \in \mathbf{R}$.

It is clear that $B_0(\alpha) \subset B(\alpha)$, the class of Bazilevic functions [**1**], [**5**]. Thus each $f \in B_0(\alpha)$ is univalent in D.

In [**3**], sharp upper and lower bounds for $|zf'(z)/f(z)|$ were obtained for $f \in B_0(\alpha)$ (see also [**2**]). In this paper, we consider the same problem for the wider class $B_0(\alpha, \beta)$ defined as follows:

Definition. For $\alpha > 0$ and $0 \leq \beta < 1$, denote by $B_0(\alpha, \beta)$ the class of normalised analytic functions f defined in D and satisfying the condition

$$\operatorname{Re}\, e^{i\psi} \left(f'(z)\big(f(z)/z\big)^{\alpha-1} - \beta\right) > 0 \tag{1}$$

for $z \in D$ and for some $\psi = \psi(f) \in \mathbf{R}$.

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Statement of results

THEOREM 1. *Let $f \in B_0(\alpha, \beta)$. Then for $z = re^{i\theta} \in D$,*

$$\left|\frac{zf'(z)}{f(z)}\right| \leq \left[(1-\beta)\left(\frac{1+r}{1-r}\right)+\beta\right] \Big/ \left[\alpha(1-\beta)\int_0^1 t^{\alpha-1}\left(\frac{1+tr}{1-tr}\right)dt+\beta\right]. \quad (2)$$

Equality is attained in $B_0(\alpha, \beta)$ for the function f_1 given by

$$f_1(z) = z\left(\alpha(1-\beta)\int_0^1 t^{\alpha-1}\left(\frac{1+tz}{1-tz}\right)dt+\beta\right)^{1/\alpha}, \quad \text{when } z = r.$$

THEOREM 2. *Let $f \in B_0(\alpha, \beta)$ and $\beta \neq 0$. Then for $z = re^{i\theta} \in D$,*

$$\left|\frac{zf'(z)}{f(z)}\right| \geq \left[\left(\frac{1-\beta r^2}{\beta(1-r^2)}\right)^{1/2}+1\right]^{-1}.$$

In the opposite direction we have

THEOREM 3. *Suppose $\alpha > 0$, $0 < \beta < 1$, $\mu > 1$ and $0 < \rho < 1$. Then there exists $f \in B_0(\alpha, \beta)$ and r satisfying $\rho < r < 1$ such that*

$$\left|\frac{zf'(z)}{f(z)}\right| < \mu\left[\frac{\beta(1-r^2)}{1-\beta r^2}\right]^{1/2}, \quad \text{for } |z| = r.$$

Remark. We note that when $\psi = 0$, the upper bound (2) is sharp in this subclass. Theorem 3 shows that the expected lower bound $\left|\big(zf'(z)\big)/\big(f(z)\big)\right| \geq \big(rf'_1(-r)\big)/\big(f_1(-r)\big)$ is false for the wider class $B_0(\alpha, \beta)$, $\beta \neq 0$. The methods of this paper appear to indicate that the case $\beta \neq 0$ is significantly more difficult than the case $\beta = 0$.

Proof of Theorems

In order to prove Theorems 1 and 2, we modify the method of Gray and Ruscheweyh [**2**], and require the following lemmas:

LEMMA 1. *Let $F(z) = 1 - z^\alpha \big/ \left(\alpha\int_0^z \xi^{\alpha-1}(1-\beta\xi)/(1-\xi)\,d\xi\right)$, where $\alpha > 0$ and $0 \leq \beta < 1$. Then $F(z)$ has non-negative Taylor coefficients about $z = 0$ and in particular for $|z| \leq r$,*

$$|F(z)| \leq F(r) < \lim_{t\to 1} F(t) = 1 \quad \text{and} \quad |F'(z)| \leq F'(r).$$

Proof. It is easily seen that

$$\frac{\alpha}{z^\alpha}\int_0^z \xi^{\alpha-1}\frac{(1-\beta\xi)}{1-\xi}\,d\xi = 1 + \sum_{k=1}^{\infty}\frac{\alpha(1-\beta)}{k+\alpha}z^k.$$

Now let $H(z) = F(z) - 1 = \sum_{k=0}^{\infty} c_k z^k$. Then $\left(\sum_{k=0}^{\infty} c_k z^k\right)\left(1 + \sum_{k=1}^{\infty} \frac{\alpha(1-\beta)}{k+\alpha} z^k\right) = -1$.
Equating coefficients of z^k we have $c_0 = -1$ and for $k \geq 1$

$$c_k + d_k = \alpha(1-\beta)/(k+\alpha) \tag{3}$$

where $d_1 = 0$ and $d_k = \sum_{j=1}^{k-1} \frac{\alpha(1-\beta)}{j+\alpha} c_{k-j}$ $(k \geq 2)$.

Now let $k \geq 2$. Replace k by $k-1$ in (3), multiply by $(k-1+\alpha)/(k+\alpha)$ and substract from (3) to obtain

$$c_k + \left(\frac{\alpha(1-\beta)}{1+\alpha} - \frac{k-1+\alpha}{k+\alpha}\right) c_{k-1} + e_k = 0,$$

where $e_2 = 0$ and for $k \geq 3$,

$$e_k = \sum_{j=2}^{k-1} \alpha(1-\beta) c_{k-j} \left[\frac{1}{j+\alpha} - \frac{k-1+\alpha}{(j-1+\alpha)(k+\alpha)}\right].$$

Thus for $k \geq 2$

$$c_k = \frac{\beta(k-1+\alpha)}{k+\alpha} c_{k-1} + \sum_{j=1}^{k-1} \alpha(1-\beta) c_{k-j} \left[\frac{k-1+\alpha}{(j-1+\alpha)(k+\alpha)} - \frac{1}{j+\alpha}\right].$$

Also $c_1 > 0$ from (3) and

$$\frac{k-1+\alpha}{(j-1+\alpha)(k+\alpha)} - \frac{1}{j+\alpha} = \frac{k-j}{(j-1+\alpha)(k+\alpha)(j+\alpha)} > 0$$

for $1 \leq j \leq k-1$. Hence $c_k > 0$ for $k \geq 1$ by induction. Thus $F(z)$ has positive coefficients and the lemma follows.

LEMMA 2. *Let V be a compact and complete subspace of the space A of analytic functions f defined in D with $f(0) = 1$ and let Λ be the space of all continuous linear functionals on A. Suppose $\lambda_1, \lambda_2 \in \Lambda$ with $0 \notin \lambda_2(V) \oplus d$, where $\oplus$ denotes direct sum and d is constant. Let V^{**} be the dual space of V. Then for $f \in V^{**}$, there exists $f_0 \in V$ such that*

$$\frac{\lambda_1(f) + d}{\lambda_2(f) + d} = \frac{\lambda_1(f_0) + d}{\lambda_2(f_0) + d}.$$

Proof. Let $f \in V^{**}$ and put

$$\lambda(F) = \big(\lambda_1(f) + d\big)\lambda_2(F) - \big(\lambda_2(f) + d\big)\lambda_1(F). \tag{4}$$

Then $\lambda \in \Lambda$ and $\lambda(f) = d\big(\lambda_2(f) - \lambda_1(f)\big)$. Now by the duality principle [**4**, Theorem 1.1], $\lambda(V^{**}) = \lambda(V)$ and so there exists $f_0 \in V$ such that $\lambda(f_0) = d\big(\lambda_2(f) - \lambda_1(f)\big)$. Hence using (4) with F replaced by f_0 gives

$$\big(\lambda_1(f) + d\big)\big(\lambda_2(f_0) + d\big) = \big(\lambda_2(f) + d\big)\big(\lambda_1(f_0) + d\big).$$

By hypothesis $0 \notin \lambda_2(V) \oplus d$ and $0 \notin \lambda_2(V^{**}) \oplus d$ by duality. Thus

$$\frac{\lambda_1(f)+d}{\lambda_2(f)+d} = \frac{\lambda_1(f_0)+d}{\lambda_2(f_0)+d}.$$

Proof of Theorem 1. From (1) we have

$$\frac{zf'(z)}{f(z)} = \frac{z^{\alpha}\big((1-\beta)h(z)+\beta\big)}{\alpha\int_0^z \left[(1-\beta)h(\xi)+\beta\right]\xi^{\alpha-1}d\xi}, \tag{5}$$

where Re $e^{i\psi}h(z) > 0$ for $z \in D$ and $h(0) = 1$. Thus

$$\frac{zf'(z)}{f(z)} = \frac{(1-\beta)h(z)+\beta}{\alpha\int_0^1 (1-\beta)h(tz)t^{\alpha-1}\,dt+\beta}. \tag{6}$$

It follows from Lemma 2 and Theorem 1.6 in [**4**] that any value assumed by the right-hand side of (6) for some $z \in D$, is also assumed for this z when $h(z)$ is a function of the form $(1+xz)/(1+yz)$ where $|x|$, $|y| = 1$. So we may write

$$h(z) = \frac{1+xz}{1-z}, \quad \text{where } |x| = 1 \tag{7}$$

when obtaining upper or lower bounds for $|zf'(z)/f(z)|$.

Using (5) and (7), we have

$$\frac{zf'(z)}{f(z)} = G(z)\left(\frac{1+(1-\beta)xz/(1-\beta z)}{1+xF(z)}\right) \text{ where } G(z) = (1-\beta z)\left(\frac{1-F(z)}{1-z}\right).$$

Since $|F(z)| < 1$ and $(1+ax)/(1+bx)$ maps the closed unit disc onto the circle centre $(1-a\bar{b})/(1-|b|^2)$, radius $|a-b|/(1-|b|^2)$ provided $|b| < 1$, we deduce that

$$\left|\frac{zf'(z)}{f(z)}\right| \le \frac{1}{1-|F(z)|^2}\left(|I_1|+|I_2|\right), \tag{8}$$

where $I_1 = G(z)\Big(\dfrac{(1-\beta)z}{1-\beta z} - F(z)\Big)$ and $I_2 = G(z)\Big(1 - \dfrac{(1-\beta)z\overline{F(z)}}{1-\beta z}\Big)$. Now

$$I_1 = \big(1-F(z)\big)\left(\frac{(1-\beta)z}{1-z} - \frac{(1-\beta z)(F(z))}{1-z}\right) = \big(1-F(z)\big)\big(G(z)-1\big).$$

Also

$$\begin{aligned} I_2 &= \big(1-F(z)\big)\left[\left(\frac{1-\beta z}{1-z}\right)\big(1-\overline{F(z)}\big)+\overline{F(z)}\right] \\ &= \big(1-\overline{F(z)}\big)\big(G(z)-1\big)+1-|F(z)|^2. \end{aligned}$$

From the definition of $F(z)$ and $G(z)$ we have

$$zF'(z) = \alpha\big(1-F(z)\big)\big(G(z)-1\big)$$

and so from (8)

$$\left|\frac{zf'(z)}{f(z)}\right| \le \frac{2|zF'(z)|}{\alpha\,(1-|F(z)|^2)} + 1.$$

Using Lemma 1 we deduce that

$$\left|\frac{zf'(z)}{f(z)}\right| \le \frac{2rF'(r)}{\alpha\big(1-F(r)^2\big)} + 1$$

and the result follows on substituting for $F(r)$.

In order to prove Theorem 2, we require the following:

LEMMA 3. *For* $0<\beta<1$ *and* $z = re^{i\theta} \in D$,

$$\frac{|1-z|}{|1-\beta z| - (1-\beta)r} \le \left[\frac{1-\beta r^2}{\beta(1-r^2)}\right]^{1/2}.$$

Proof. Fix β in $(0,1)$ and put

$$\varphi(z) = \frac{|1-z|}{|1-\beta z| - (1-\beta)r}.$$

Then

$$\frac{\partial}{\partial\theta}|1-\beta z| = |1-\beta z|\,\mathrm{Im}\,\frac{\beta z}{1-\beta z} = \frac{\beta r\sin\theta}{|1-\beta z|},$$

and so, after a simple calculation,

$$\frac{\partial}{\partial\theta}|\varphi(z)| = \frac{(1-\beta)r\sin\theta}{|1-z|}\left(\frac{1-\beta r^2}{|1-\beta z|} - r\right). \tag{9}$$

Let $\lambda = \lambda(r)$ denote any value of z for which

$$|1-\beta z| = r^{-1}(1-\beta r^2). \tag{10}$$

Such values exist for all sufficiently large r in $(0,1)$, since (9) is true if, and only if,

$$2r\beta\cos\theta = 2\beta + 1 - r^{-2}. \tag{11}$$

We now show that

$$\varphi(r) \le \varphi(-r) \le \varphi\big(\lambda(r)\big) = \left[\frac{1-\beta r^2}{\beta(1-r^2)}\right]^{1/2} \tag{12}$$

and this, together with (11) will establish the lemma.

It is easy to verify that $\varphi(r) \le \varphi(-r)$. Now $\varphi(-r) \le \varphi\big(\lambda(r)\big)$ is equivalent (on squaring and subtracting 1 from each side) to the inequality

$$\frac{4\beta r(1+\beta r)}{(1+2\beta r - r)^2} \le \frac{1}{1-r^2}.$$

If $0 < p < 1$, $x(2+x)/(p+x)^2$ assumes its maximum value at $p/(1-p)$ when $x > -p$. Thus with $x = 2\beta r$ and $p = 1-r$, we have

$$\frac{4\beta r(1+\beta r)}{(1+2\beta r-r)^2} \le \frac{x(2+x)}{(p+x)^2} \le \frac{\big((1-r)/r\big)\big(2+(1-r)/r\big)}{\big(1-r+(1-r)/r\big)^2} = \frac{1}{1-r^2}.$$

Finally, using (10) and (11) we obtain

$$\varphi\big(\lambda(r)\big) = \frac{\left[1-\beta^{-1}(2\beta+1-r^{-2})+r^2\right]^{1/2}}{r^{-1}-\beta r-(1-\beta)r} = \left[\frac{1-\beta r^2}{\beta(1-r^2)}\right]^{1/2}.$$

Proof of Theorem 2. As in the proof of Theorem 1, we write $h(z) = (1+xz)/(1-z)$ where $|x| = 1$. Thus we have from (5)

$$\frac{zf'(z)}{f(z)} = \left(\frac{1+x'z}{1-z}\right) \Big/ \left(\alpha\int_0^1 t^{\alpha-1}\frac{1+x'tz}{1-tz}dt\right),$$

where $x' = (1-\beta)x - \beta$ and so

$$\frac{f(z)}{zf'(z)} = \alpha\int_0^1 t^{\alpha-1}\frac{1+x'tz}{1+x'z}\,\frac{1-z}{1-tz}dt = \alpha\int_0^1 t^{\alpha-1}\left(\frac{1-t}{1+x'z}+t\right)\frac{1-z}{1-tz}dt. \quad (13)$$

Hence

$$\begin{aligned}\left|\frac{f(z)}{zf'(z)}\right| &\le \alpha\int_0^1 t^{\alpha-1}\frac{|1-z|}{|1+x'z|}dt + \alpha\int_0^1 t^{\alpha}\left|\frac{1-z}{1-tz}\right|dt\\ &\le \alpha\int_0^1 t^{\alpha-1}\frac{|1-z|}{|1-\beta z-(1-\beta)xz|}\,dt + \alpha\int_0^1 t^{\alpha}\frac{1+r}{1+tr}\,dt\\ &\le \alpha\int_0^1 t^{\alpha-1}\frac{|1-z|}{|1-\beta z|-(1-\beta)r}\,dt + \alpha\int_0^1 \frac{2t^{\alpha}}{1+t}\,dt.\end{aligned}$$

Lemma 3 now gives

$$\left|\frac{f(z)}{zf'(z)}\right| \le \alpha\left[\frac{1-\beta r^2}{\beta(1-r^2)}\right]^{1/2}\int_0^1 t^{\alpha-1}dt + \alpha\int_0^1 t^{\alpha-1}dt = \left[\frac{1-\beta r^2}{\beta(1-r^2)}\right]^{1/2}+1,$$

which completes the proof of Theorem 2.

Proof of Theorem 3. We use the function $\lambda(r)$ defined in Lemma 3 and in particular the fact that as $r \to 1$, $\varphi\big(\lambda(r)\big) \to \infty$ and $\lambda(r) = re^{i\theta} \to 1$, which follows from (12) and (11) respectively. These properities allow us to choose δ in $(0,1)$, and r in $(\rho, 1)$ such that for $\lambda = \lambda(r)$

$$(\delta + 2\delta^{\alpha} - 2 - 1/\mu)\varphi(\lambda) > 1 \quad (14)$$

and

$$\alpha\int_0^{\delta} t^{\alpha}\left|\frac{1-\lambda}{1-t\lambda}\right|dt < 1-\delta. \quad (15)$$

Also choose x_0 so that $|x_0| = 1$ and so that $x_0\lambda$ has the same argument as $\beta\lambda - 1$ and let $x_0' = (1-\beta)x - \beta$. We also note, using Lemma 3 that for $|z| = r$, and $x' = (1-\beta)x - \beta$, $|x| = 1$,

$$\left|\frac{1-z}{1+x'z}\right| \le \varphi(z) \le \varphi(\lambda) = \left|\frac{1-\lambda}{1+x'_0\lambda}\right|. \tag{16}$$

Now let f be given by (5), where h is any function satisfying $\operatorname{Re} e^{i\psi}h(z) > 0$. Then for some x' as above (13) gives

$$\left|\frac{f(z)}{zf'(z)}\right| \ge J_1 - J_2 - J_3, \tag{17}$$

where

$$J_1 = \alpha\left|\int_0^\delta t^{\alpha-1}\frac{1-t}{1+x'z}\frac{1-z}{1-tz}\,dt\right|, \quad J_2 = \alpha\int_\delta^1 t^{\alpha-1}\left|\frac{1-t}{1+x'z}\frac{1-z}{1-tz}\right|dt,$$
$$\text{and} \quad J_3 = \alpha\int_0^1 t^\alpha\left|\frac{1-z}{1-tz}\right|dt.$$

For J_3 we obtain

$$J_3 \le \alpha\int_0^1 t^\alpha\frac{1+r}{1+tr}\,dt \le \alpha\int_0^1\frac{2t^\alpha}{1+t} \le \alpha\int_0^1 t^{\alpha-1}\,dt = 1.$$

Also, using (16)

$$J_2 \le \alpha\varphi(\lambda)\int_\delta^1 t^{\alpha-1}\left|\frac{1-t}{1-tz}\right|\,dt \le \alpha\varphi(\lambda)\int_\delta^1 t^{\alpha-1}dt = (1-\delta^\alpha)\varphi(\lambda).$$

We now choose h specifically so that for $z = \lambda$ the right-hand side of (5) is given by taking $(1+x_0t)/(1-t)$ ($|t| < 1$) in place of h. For this h we define f by putting $h(z) = f'(z)\big(f(z)/z\big)^{\alpha-1} - \beta$ so that we have (5). Then

$$\begin{aligned}J_1 &= \alpha\left|\frac{1-\lambda}{1+x'_0\lambda}\right|\left|\int_0^\delta t^{\alpha-1}\left(1-\frac{t(1-\lambda)}{1-t\lambda}\right)dt\right| \\ &\ge \left|\frac{1-\lambda}{1+x_0'\lambda}\right|\left(\delta^\alpha - \int_0^\delta \alpha t^\alpha\left|\frac{1-\lambda}{1-t\lambda}\right|dt\right).\end{aligned}$$

Thus from (15) and (16) we deduce that

$$J_1 \ge (\delta^\alpha + \delta - 1)\left|\frac{1-\lambda}{1+x'_0\lambda}\right| = (\delta^\alpha + \delta - 1)\varphi(\lambda).$$

The estimates for J_1, J_2, J_3 together with (17) and (14) give

$$\left|\frac{f(\lambda)}{\lambda f'(\lambda)}\right| \ge \varphi(\lambda)(\delta + 2\delta^\alpha - 2) - 1 > \mu^{-1}\varphi(\lambda).$$

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