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# ON TOPOLOGICAL SPACES WITH DENSE COMPLETELY METRIZABLE SUBSPACES

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*Dedicated to Professor Akihiro Okuyama on his 60th birthday*

**Abstract**. We obtain some characterizations for the spaces which have a dense completely metrizable subspace and some results related with these spaces.

**1. Introduction.** In 1991, Michael [**7**] gave some characterizations for the spaces called *almost Čech-complete* which are simply called *almost complete*.

We know that for a metrizable space $X$, the following statements are equivalent [**7**, Proposition 4.4]:

(1) $X$ is a almost complete space,

(2) $X$ has a dense completely metrizable subspace.

The first purpose of this paper is to obtain some characterizations for the spaces which have a dense completely metrizable subspace and some results related with these spaces.

Arhangel'skiĭ and Kočinac asked several questions on weakly perfect spaces and spaces with dense $G_\delta$-diagonal [**1**]. The second purpose of this paper is to give answers to their Questions 8 and 9.

**2. Definitions and notations.** All considered spaces are completely regular. A sequence $\{U_n| \ n \in \mathbf{N}\}$ of subsets of a space $X$ is said to be *complete* if every filter base $\mathcal{F}$ on $X$ which is controlled* by $\{U_n| \ n \in \mathbf{N}\}$ clusters at some $x \in X$.

A sequence $\{\mathcal{U}_n| \ n \in \mathbf{N}\}$ of collections of subsets of $X$ is said to be *complete* if $\{U_n| \ n \in \mathbf{N}\}$ is a complete sequence whenever $U_n \in \mathcal{U}_n$ for all $n \in \mathbf{N}$.

A collection $\mathcal{U}$ of subsets of a space $X$ is said to be an *almost cover* if $\bigcup \mathcal{U}$ is dense in $X$. Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of $X$. We say that $\mathcal{V}$ is a *strong*

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*$\mathcal{F}$ is controlled by $\{U_n\}$ if each $U_n$ contains some $F \in \mathcal{F}$.

*refinement of* $\mathcal{U}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$ and for each element $V \in \mathcal{V}$ there exists an element $U \in \mathcal{U}$ with $\mathrm{Cl}(V) \subset U$.

The following lemma is proved in [**7**, Lemma 4.6].

LEMMA 2.1. *If $X$ has a complete sequence $\{\mathcal{U}_n |\ n \in \mathbf{N}\}$ of open almost covers, then there exists a complete sequence $\{\mathcal{V}_n |\ n \in \mathbf{N}\}$ of disjoint open almost covers of $X$ such that $\mathcal{V}_{n+1}$ is a strong refinement of $\mathcal{V}_n$ for each $n \in \mathbf{N}$.*

Let $\mathcal{U}$ be a collection of subsets of $X$. $\mathcal{U}$ is said to *separate points of* $X$ if $x$ and $y$ are distinct points of $X$, then there exists different elements $U_x$ and $U_y$ in $\mathcal{U}$ such that $x \in U_x$ and $y \in U_y$. $\mathcal{U}$ is said to have a *finite intersection property* (f.i.p.) if every finite subcollection of $\mathcal{U}$ have a nonempty intersection.

**3. Characterizations.** The main purpose in this section is to prove Theorem 3.4.

LEMMA 3.1. *Let $X$ has a complete sequence $\{\mathcal{U}_n | n \in \mathbf{N}\}$ of open almost covers such that for each sequence $\{U_n |\ U_n \in \mathcal{U}_n,\ n \in \mathbf{N}\}$ with f.i.p., the set $\bigcap\{\mathrm{Cl}(U_n) |\ n \in \mathbf{N}\}$ is a singleton. Then there exists a complete sequence $\{\mathcal{V}_n |\ n \in \mathbf{N}\}$ of disjoint open almost covers of $X$ such that*

(i) *$\mathcal{V}_{n+1}$ is a strong refinement of $\mathcal{V}_n$ for each $n \in \mathbf{N}$, and*

(ii) *for each decreasing sequence $\{V_n |\ V_n \in \mathcal{V}_n,\ n \in \mathbf{N}\}$, the set $\bigcap\{\mathrm{Cl}(V_n) |\ n \in \mathbf{N}\}$ is a singleton.*

*Proof.* From Lemma 2.1, there exists a complete sequence $\{\mathcal{V}_n |\ n \in \mathbf{N}\}$ of disjoint open almost covers of $X$ such that $\mathcal{V}_{n+1}$ is a strong refinement of $\mathcal{V}_n$ and $\mathcal{U}_n$ for each $n \in \mathbf{N}$.

Let $\{V_n |\ n \in \mathbf{N}\}$ is a decreasing sequence where $V_n \in \mathcal{V}_n$ for each $n \in \mathbf{N}$. By the construction of $\mathcal{V}_n$, for each $n \in \mathbf{N}$, there exists $U_n \in \mathcal{U}_n$ such that $\mathrm{Cl}(V_{n+1}) \subset V_n \cap U_n$. Since $\{V_n |\ n \in \mathbf{N}\}$ is decreasing, $\{U_n |\ n \in \mathbf{N}\}$ has f.i.p.

For each $n \in \mathbf{N}$, we put $F_n = \mathrm{Cl}(V_{n+1})$. By completeness of $\{\mathcal{V}_n |\ n \in \mathbf{N}\}$, we have:

$$\oslash \neq \bigcap_{n\in\mathbf{N}} F_n = \bigcap_{n\in\mathbf{N}} V_n = \bigcap_{n\in\mathbf{N}} \mathrm{Cl}(V_n) = \bigcap_{n\in\mathbf{N}} \mathrm{Cl}(U_n).$$

Since $\bigcap_{n\in\mathbf{N}} \mathrm{Cl}(U_n)$ is a singleton, $\bigcap_{n\in\mathbf{N}} \mathrm{Cl}(V_n)$ is also a singleton. □

THEOREM 3.2. *Let $X$ have a complete sequence $\{\mathcal{U}_n |\ n \in \mathbf{N}\}$ of disjoint open almost covers such that*

(i) *$\mathcal{U}_{n+1}$ is a strong refinement of $\mathcal{U}_n$ for each $n \in \mathbf{N}$, and*

(ii) *the set $\bigcap_{n\in\mathbf{N}} \mathrm{Cl}(U_n)$ is a singleton for each decreasing sequence $\{U_n |\ U_n \in \mathcal{U},\ n \in \mathbf{N}\}$.*

*Then $X$ has a dense $G_\delta$ completely metrizable subspace.*

*Proof.* By [**7**, Proposition 4.5], $X$ is a Baire space. Since $G_n = \bigcup \mathcal{U}_n$ is an open dense subset in $X$ for each $n \in \mathbf{N}$, then $M = \bigcap_{n\in\mathbf{N}} G_n$ is a dense $G_\delta$ set in $X$.

By the condition (ii), if $x$ and $y$ are two distinct points of $M$, then there exists $n \in \mathbf{N}$ such that $\mathcal{U}_n$ separates $x$ and $y$.

Let us define the metric $\rho$ on $M$ by

$$\rho(x,y) = \begin{cases} 0, & x = y \\ \min\{n|\ \mathcal{U}_n \text{ separates } x \text{ and } y\})^{-1}, & \text{otherwise.} \end{cases}$$

It is easy to check that $\rho$ is a complete metric on $M$, by the condition (i) and (ii). Moreover, $U_n \cap M$ is a $1/n$-open ball at $x$ for each $U_n \in \mathcal{U}_n$ and $x \in U_n \cap M$. Hence the original topology on $M$ is stronger than $\rho$-topology.

Now we show the next claim.

CLAIM. *Let $F$ be a closed subset of $M$ and $x \in M \backslash F$. Then there exist $n \in \mathbf{N}$ and $U_n \in \mathcal{U}_n$ such that $x \in U_n$ and $U_n \cap F = \oslash$.*

Proof of the claim. Suppose that $U_n \cap F \neq \oslash$ whenever $x \in U_n$ for each $n \in \mathbf{N}$. Pick a point $x_n$ in $U_n \cap F$, and put $F_n = \mathrm{Cl}\{x_m|\ m \geq n+1\}$ for each $n \in \mathbf{N}$. Then by the condition (i), $\{U_n|\ n \in \mathbf{N}\}$ is a decreasing sequence, and $\{x\} = \bigcap_{n\in\mathbf{N}} U_n$, by the condition (ii). It follows that

$$\oslash \neq \bigcap_{n\in\mathbf{N}} F_n \subset \bigcap_{n\in\mathbf{N}} U_n = \{x\}.$$

Hence $x \in F$. This is a contradiction, and the claim is proved.

By the claim, $\rho$-topology is stronger than the original topology. It follows that $M$ is a dense $G_\delta$ completely metrizable subspace. The proof is complete. □

THEOREM 3.3. *Let $X$ be a space with a dense completely metrizable subspace. Then there exists a complete sequence $\{\mathcal{U}_n|\ n \in \mathbf{N}\}$ of open almost covers of $X$ such that for each sequence $\{U_n|\ U_n \in \mathcal{U}_n,\ n \in \mathbf{N}\}$ with the f.i.p., the set $\bigcap_{n\in\mathbf{N}} \mathrm{Cl}(U_n)$ is a singleton.*

*Proof.* Let $M$ be a dense completely metrizable subspace of $X$ and $\rho$ a compatible metric on $M$. Let $U(x,n)$ be an open subset of $X$ such that $B(x,1/n) = U(x,n) \cap M$ for each $x \in M$ and $n \in \mathbf{N}$, where $B(x,1/n) = \{y \in M|\ \rho(x,y) < 1/n\}$ be a $1/n$-open ball in $M$. Then for each $n \in \mathbf{N}, \mathcal{U}_n = \{U(x,n)|\ x \in M\}$ is an open almost cover of $X$.

Now we show that $\{\mathcal{U}_n|\ n \in \mathbf{N}\}$ is a complete sequence. Let $\{U_n|\ U_n \in \mathcal{U}_n,\ n \in \mathbf{N}\}$ be a sequence and $\mathcal{F}$ a filter base on $X$ which is controlled by $\{U_n|\ n \in \mathbf{N}\}$. Then for each $n \in \mathbf{N}$, there exists $F_n \in \mathcal{F}$ such that $F_n \subset U_n$. By the construction of $\mathcal{U}_n$, there exists $x_n$ such that $x_n \in M$, $U_n = U(x_n, n)$ for each $n \in \mathbf{N}$. Since $\{U_n|\ n \in \mathbf{N}\}$ has the f.i.p., it follows that $\{x_n|\ n \in \mathbf{N}\}$ is a $\rho$-Cauchy sequence. Then there exists $x_0 \in M$ such that $\{x_n|\ n \in \mathbf{N}\}$ converges to $x_0$. Therefore we have that $x_0 \in \bigcap\{\mathrm{Cl}(F)|\ F \in \mathcal{F}\}$. Hence $\{U_n|\ n \in \mathbf{N}\}$ is a complete sequence.

In the same way, it is easy to see that $\bigcap_{n\in\mathbf{N}} \mathrm{Cl}(U_n) = \{x_0\}$. The proof is complete. □

These results lead to the following theorem.

THEOREM 3.4. *For the space $X$, the following conditions are equivalent.*

(1) *$X$ has a complete sequence $\{\mathcal{U}_n \mid n \in \mathbf{N}\}$ of open almost covers such that for each sequence $\{U_n \mid U_n \in \mathcal{U}_n,\ n \in \mathbf{N}\}$ with f.i.p., the set $\bigcap_{n\in\mathbf{N}} \mathrm{Cl}(U_n)$ is a singleton.*

(2) *$X$ has a complete sequence $\{\mathcal{U}_n \mid n \in \mathbf{N}\}$ of disjoint open almost covers such that*

(i) *$\mathcal{U}_{n+1}$ is a strong refinement of $\mathcal{U}_n$ for each $n \in \mathbf{N}$, and*

(ii) *for each decreasing sequence $\{U_n \mid U_n \in \mathcal{U}_n,\ n \in \mathbf{N}\}$, the set $\bigcap_{n\in\mathbf{N}} \mathrm{Cl}(U_n)$ is a singleton.*

(3) *$X$ has a dense $G_\delta$ completely metrizable subspace.*

(4) *$X$ has a dense completely metrizable subspace.*

A space $X$ is said to be a *Namioka space* if the following condition is satisfied:

(∗) for any compact space $Y$ and any separately continuous function $f\colon X \times Y \to \mathbf{R}$, there exists a dense $G_\delta$ subset $A \subset X$ such that $f$ is jointly continuous at each point of $A \times Y$.

Next we consider the following game. Let $\alpha$ and $\beta$ be two players with $\beta$ the first to move. $\beta$ starts by choosing a nonempty open subset $U_1 \subset X$. Then $\alpha$ chooses an open subset $V_1 \subset U_1$ and a point $x_1 \in V_1$. $\beta$ then chooses a nonempty open subset $U_2 \subset V_1$ (he may choose as he wishes but is expected to escape from $x_1$). Next $\alpha$ chooses an open subset $V_2 \subset U_2$ and a point $x_2 \in V_2$, and so on. $\alpha$ wins if any subsequence $\{x_{n_p} \mid p \in \mathbf{N}\}$ of the sequence $\{x_n \mid n \in \mathbf{N}\}$ accumulates to at least one point of the set $\bigcap_{i=1}^{\infty} V_i = \bigcap_{i=1}^{\infty} U_i$. Then $X$ is said to be *$\sigma$-well $\alpha$-favorable* if $\alpha$ has a winning strategy in the game above.

It is well known that $\sigma$-well $\alpha$-favorable spaces are Namioka [**9**, Theorem 6.3].

THEOREM 3.5. *Let $X$ be a space with a dense completely metrizable subspace $M$. Then $X$ is a $\sigma$-well $\alpha$-favorable space. Hence $X$ is a Namioka space.*

*Proof.* Let $U_1$ be a nonempty open subset of $X$. Since $M$ is a dense subspace, we can pick a point $x_1$ in $M \cap U_1$. Then there exists a nonempty open subset $V_1$ of $X$ such that $x_1 \in V_1 \subset \mathrm{Cl}(V_1) \subset U_1$ and $\mathrm{d}_M - \mathrm{diam}(V_1 \cap M) \leq 1/2$, where $\mathrm{d}_M$ is a compatible metric on $M$. By induction, there exists a sequence $\{x_n \mid n \in \mathbf{N}\}$ in $X$ and sequences $\{V_n \mid n \in \mathbf{N}\}$, $\{U_n \mid n \in \mathbf{N}\}$ of subsets of $X$ such that

$$x_n \in V_n \cap M,\ \ U_{n+1} \subset V_n \subset \mathrm{Cl}(V_n) \subset U_n,\ \text{ and }\ \mathrm{d}_M - \mathrm{diam}(V_n \cap M) \leq 1/n+1$$

for each $n \in \mathbf{N}$. Since $\{x_n \mid n \in \mathbf{N}\}$ is a $\mathrm{d}_M$-Cauchy sequence in $M$, there exists $x_0$ in $M$ such that $\{x_n \mid n \in \mathbf{N}\}$ converges to $x_0$. By the construction of $\{V_n \mid n \in \mathbf{N}\}$, we have $x_0 \in \bigcap_{n\in\mathbf{N}} V_n = \bigcap_{n\in\mathbf{N}} \mathrm{Cl}(V_n)$. The proof is complete. □

THEOREM 3.6. *Let $X$ be a space with a dense completely metrizable subspace, $Y$ a space and $f\colon X \to Y$ an irreducible, closed, continuous and onto map. Then $Y$ has a dense completely metrizable subspace.*

*Proof.* By Theorem 3.4, there exists a complete sequence $\{\mathcal{U}_n | \; n \in \mathbf{N}\}$ of disjoint open almost covers of $X$, which satisfies the conditions (i) and (ii) of (2). For each $U \in \mathcal{U}_n$, put $W(U) = Y \setminus f(X \setminus U)$. Then each $W(U)$ is a nonempty open subset of $Y$. Now put $\mathcal{V}_n = \{W(U) | \; U \in \mathcal{U}_n\}$ for each $n \in \mathbf{N}$. It is easy to see that $\{\mathcal{V}_n | \; n \in \mathbf{N}\}$ is a complete sequence of open almost covers of $Y$, which satisfies the condition (1) of Theorem 3.4. The proof is complete. □

**4. countable dense Δ-base.** Here $\Delta_X = \{(x,x) | \; x \in X\}$ is the diagonal in $X \times X$. Arhangel'skiĭ and Kočinac [**1**] asked the following questions:

*Question* 1. When there exist a countable family $\mathcal{U}$ of open sets in $X \times X$ such that $\bigcap \mathcal{U} \cap \Delta_X$ is dense in $\Delta_X$ and for each open neighborhood $V$ of $\Delta_X$ in $X \times X$ one can find $U \in \mathcal{U}$ such that $U \subset V$? Such $\mathcal{U}$ will be called a *dense* Δ*-base* of $X$.

*Question* 2. Let $X$ be a compact space with a countable dense Δ-base. Does there exist a dense open metrizable subspace $Y \subset X$? A dense separable subspace $Z \subset X$?

It is clear that if $X$ has a dense discrete subspace, then $X$ has a countable dense Δ-base. Now we prove the following theorem.

THEOREM 4.1. *Let $X$ be a compact space. If $X$ has a dense completely metrizable subspace, then $X$ has a countable dense Δ-base.*

*Proof.* Let $M$ be a completely metrizable subspace of $X$ and $\rho$ a compatible metric on $M$. For each $n \in \mathbf{N}$, put $V_n = \{(x,y) \in M \times M | \; \rho(x,y) < 1/n\}$. Since each $V_n$ is open set in $M \times M$, there exists an open set $U_n$ in $X \times X$ such that $V_n = U_n \cap (M \times M)$. We show that $\mathcal{U} = \{U_n | \; n \in \mathbf{N}\}$ is a countable dense Δ-base of $X$.

Let $V$ be an open neighborhood of $\Delta_X$ in $X \times X$. Then we prove that there exists $n \in \mathbf{N}$ such that $U_n \subset V$. By normality of $X \times X$, it is enough to show that $U_n \subset \mathrm{Cl}(V)$.

Indeed, suppose that $U_n \not\subseteq \mathrm{Cl}(V)$ for each $n \in \mathbf{N}$. Then there exists $(x_n, y_n) \in V_n \setminus \mathrm{Cl}(V)$ for each $n \in \mathbf{N}$. By the definition of $V_n$, $\rho(x_n, y_n) < 1/n$ and $\{(x_n, y_n) | \; N \in \mathbf{N}\} \subset (X \times X) \setminus \mathrm{Cl}(V) \subset (X \times X) \setminus V$. Since $(X \times X) \setminus V$ is compact, there exists a cluster point $(x_0, y_0)$ of $\{(x_n, y_n) | \; n \in \mathbf{N}\}$ such that $(x_0, y_0) \in (X \times X) \setminus V$. Hence $x_0 \neq y_0$. Then there exist open subsets $V_{x_0}$ and $V_{y_0}$ such that $x_0 \in V_{x_0}$, $y_0 \in V_{y_0}$ and $\mathrm{Cl}(V_{x_0}) \cap \mathrm{Cl}(V_{y_0}) = \emptyset$. By the completeness, it follows that dist $(\mathrm{Cl}(V_{x_0}) \cap M, \mathrm{Cl}(V_{y_0}) \cap M) > 0$. But $\rho(x_n, y_n) < 1/n$ for each $n \in \mathbf{N}$, a contradiction.

Finally, since $\Delta_M \subset (\bigcap \mathcal{U}) \cap \Delta_X$, the set $(\bigcap \mathcal{U}) \cap \Delta_X$ is dense in $\Delta_X$. The proof is complete. □

Next we consider Question 2. We remark the following proposition.

PROPOSITION 4.2. *Let $X$ be a space and $M$ a dense completely metrizable subspace of $X$. Then the following conditions are equivalent.*

(1) *$X$ is separable.*

(2) *M is separable.*

(3) *X satisfies the countable chain condition.*

We have the negative answer of the second part of Question 2.

*Example* 4.3. Let $X$ be the closed ordinal space $[0, \Omega]$, where $\Omega$ is the first uncountable ordinal. Since $X$ is a compact scattered space, it has a dense uncountable discrete subspace. Therefore $X$ has a countable dense $\Delta$-base. But it is clear that $X$ does not have any dense separable metrizable subspaces.

Let us note that if $M$ is a dense open metrizable subspace of a compact space $X$, then $M$ is a completely metrizable.

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