

## ON THE LIMIT PROPERTIES OF THE PICARD SINGULAR INTEGRAL

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**Abstract.** We present some direct and inverse approximation theorems for the Picard singular integral in the spaces  $L^p$  ( $1 \leq p \leq \infty$ ) and the generalized Holder spaces. Those theorems extend and improve the results for the Picard integral given in [1]–[3].

**1. Notations 1.1.** Let  $L^p \equiv L^p(R)$ ,  $1 \leq p < \infty$  be the space of realvalued functions Lebesgue-integrable with  $p$ -th power over  $R := (-\infty, +\infty)$  and let  $L^\infty \equiv L^\infty(R)$  be the space of real-valued functions uniformly continuous and bounded on  $R$ . The norm in  $L^p$  we define as usual

$$\|f\|_{L^p} \equiv \|f(\cdot)\|_{L^p} := \begin{cases} \left( \int_R |f(x)|^p dx \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup_{x \in R} |f(x)| & \text{if } p = \infty. \end{cases} \quad (1)$$

Let  $X$  be one of the spaces  $L^p$ ,  $1 \leq p \leq \infty$ , with the norm (1). For a given  $f \in X$  denote by  $\omega_2(\cdot; f, X)$  the modulus of smoothness of order 2, i.e.

$$\omega_2(t; f, X) := \sup_{|h| \leq t} \|\Delta_h^2 f\|_X \quad (2)$$

for  $t \geq 0$ , where

$$\Delta_h^2 f(x) := f(x+h) + f(x-h) - 2f(x). \quad (3)$$

Denote as in [4] by  $\Omega^2$  the set of functions of the type of modulus of smoothness of order 2 [5, p. 116], i.e.  $\Omega^2$  is the set of all functions  $\omega$  satisfying the following conditions:

- (a)  $\omega$  is defined and continuous on  $(0, +\infty)$ .
- (b)  $\omega$  is monotonically increasing and  $\omega(0) = 0$ .
- (c)  $\omega(h)h^{-2}$  is monotonically decreasing for  $h > 0$ .

It is easy to verify that for every  $\omega \in \Omega^2$ , there exist positive constants  $M_1$  and  $M_2$  ( $M_i = M_i(\omega), i = 1, 2$ ) such that

$$M_1 t^2 \leq \omega(t) \leq M_2 t^2 \int_t^1 \omega(z) z^{-3} dz \quad (4)$$

for all  $0 \leq t \leq 1/2$ .

Similarly as in [4], for a given  $\omega \in \Omega^2$ , denote by  $X^\omega$  the class of all functions  $f \in X$  for which

$$\|f\|_{X^\omega}^* := \sup_{h>0} \|\Delta_h^2 f\|_X / \omega(h) < +\infty. \quad (5)$$

In  $X^\omega$  we define the norm

$$\|f\|_{X^\omega} := \|f\|_X + \|f\|_{X^\omega}^*. \quad (6)$$

Denote as in [4] by  $\tilde{X}^\omega$ ,  $\omega \in \Omega^2$ , the class of all functions  $f \in X$  such that  $\lim_{h \rightarrow 0+} \|\Delta_h^2 f\|_X / \omega(h) = 0$ . The norm in  $\tilde{X}^\omega$  we define by (6).  $X^\omega$  and  $\tilde{X}^\omega$  with the norm (6) are called the generalized Hölder spaces. If  $\omega, \mu \in \Omega^2$  and the function

$$q(h) := \omega(h) / \mu(h), \quad h > 0, \quad (7)$$

is monotonically increasing, then

$$X^\omega \subset X^\mu \quad \text{and} \quad \tilde{X}^\omega \subset \tilde{X}^\mu. \quad (8)$$

If  $f \in X^\omega$ , then

$$\omega_2(t; f, X) \leq \omega(t) \|f\|_{X^\omega}^*, \quad t > 0. \quad (9)$$

If  $f \in \tilde{X}^\omega$ , then

$$\omega_2(t; f, X) = o(\omega(t)) \quad \text{as } t \rightarrow 0+. \quad (10)$$

**1.2.** In the papers [1]–[3] are examined the limit properties of the Picard singular integral

$$P(x, r; f) := (2r)^{-1} \int_R f(x+t) \exp(-|t|/r) dt, \quad (11)$$

$x \in \mathcal{R}$ ,  $r \in I := (0, 1 >$  and  $r \rightarrow 0+$ , for the functions  $f$  belonging to  $L^p$  ( $1 \leq p \leq \infty$ ) or the classical Hölder spaces  $H^\alpha$  ( $0 < \alpha \leq 1$ ) of continuous functions.

The purpose of this note is to generalize the result given in [1]–[3].

By  $M_k(\cdot)$ ,  $k = 1, 2, \dots$ , we shall denote suitable positive constants depending only on the indicated parameters.

**2. Auxillary results.** In this section we shall give some auxiliary inequalities.

LEMMA 1. *If  $f \in X$ , then  $\|P(\cdot, r; f)\|_X \leq \|f\|_X$ ,  $r \in I$ .*

*Proof.* By (1) and (11) we have

$$\|P(\cdot, r; f)\|_X \leq \|f\|_X (2r)^{-1} \int_R \exp(-|t|/r) dt$$

for all  $r \in I$ . We also have

$$\frac{1}{2r} \int_R \exp(-|t|/r) dt = \frac{1}{r} \int_0^{+\infty} \exp(-t/r) dt = 1, \quad (12)$$

for all  $r \in I$ . Hence the proof is completed.

LEMMA 2. *If  $f \in X^\omega$ , then  $\|P(\cdot, r; f)\|_{X^\omega}^* \leq \|f\|_{X^\omega}^*$  for  $r \in I$ , which proves that, for every fixed  $r \in I$ , the function  $P(\cdot, r; f)$  also belongs to  $X^\omega$ .*

*Proof.* By (5) we have

$$\|P(\cdot, r; f)\|_{X^\omega}^* = \sup_{h>0} \|\Delta_h^2 P(\cdot, r; f)\|_X / \omega(h), \quad r \in I.$$

From (11) and (3) we get

$$\Delta_h^2 P(x, r; f) = P(x, r; \Delta_h^2 f) \quad (x \in R, h \in R, r \in I), \quad (13)$$

which by Lemma 1 gives

$$\|\Delta_h^2 P(\cdot, r; f)\|_X \leq \|\Delta_h^2 f\|_X,$$

$r \in I, h \in R$ . From this and by (5) we obtain our assertion.

Lemma 1 and Lemma 2 imply the following

COROLLARY 1. *If  $f \in X^\omega$ , then  $\|P(\cdot, r; f)\|_{X^\omega} \leq \|f\|_{X^\omega}$  for all  $r \in I$ .*

LEMMA 3. *If  $f \in \tilde{X}^\omega$ , then  $P(\cdot, r; f) \in \tilde{X}^\omega$ , for every fixed  $r \in I$ .*

*Proof.* By (13) and Lemma 1,

$$0 \leq \frac{\|\Delta_h^2 P(\cdot, r; f)\|_X}{\omega(h)} = \frac{P(\cdot, r; \Delta_h^2 f)\|_X}{\omega(h)} \leq \frac{\|\Delta_h^2 f\|_X}{\omega(h)}$$

( $h > 0, r \in I$ ). Hence, by the assumption  $f \in \tilde{X}^\omega$ , we obtain

$$\lim_{h \rightarrow 0+} \|\Delta_h^2(\cdot, r; f)\|_X / \omega(h) = 0,$$

which proves that  $P(\cdot, r; f) \in \tilde{X}^\omega$ .

It is easy to verify that

LEMMA 4. For  $k = 0, 1, 2, \dots$  and  $r > 0$  we have

$$\int_0^{+\infty} t^k \exp(-t/r) dt = k! r^{k+1}.$$

LEMMA 5. If  $f \in X$ , then  $\|\partial^2 P(\cdot, r; f)/\partial x^2\|_X \leq \|f\|_X r^{-2}$  for every  $r \in I$ .

*Proof.* From (11) we get

$$\begin{aligned} \frac{\partial^2}{\partial x^2} P(x, r; f) &= \frac{1}{2r} \int_R f(t) \left( \frac{\partial^2}{\partial x^2} \exp(-|x-t|/r) \right) dt \\ &= (2r^3)^{-1} \int_R f(x+t) \exp(-|t|/r) dt = r^{-2} P(x, r; f) \end{aligned}$$

for  $x \in R$  and  $r \in I$ . From this and by Lemma 1 we obtain the desired inequality.

### 3. Approximation theorems

Let

$$U(r, x; f) := P(x, r; f) - f(x) \quad (14)$$

for  $x \in R$  and  $r \in I$ .

**3.1.** First we shall consider direct approximation problem in the norm of the space  $X$ .

THEOREM 1. If  $f \in X$ , then  $\|U(\cdot, r; f)\|_X \leq \frac{5}{2} \omega_2(r; f, X)$  for all  $r \in I$ .

*Proof.* By (11), (12), (14) and (3), we have

$$U(x, r; f) = (2r)^{-1} \int_0^{+\infty} (\Delta_t^2 f(x)) \exp(-t/r) dt$$

for all  $x \in R$  and  $r \in I$ . From this and by (1), (2) and the properties of  $\omega_2(\cdot; f, x)$  [5, p. 116] we get

$$\begin{aligned} \|U(\cdot, r; f)\|_X &\leq (2r)^{-1} \int_0^{+\infty} \omega_2(t; f, X) \exp(-t/r) dt \\ &\leq (2r)^{-1} \omega_2(r; f, X) \int_0^{+\infty} \left(\frac{t}{r} + 1\right)^2 \exp(-t/r) dt \end{aligned}$$

for all  $r \in I$ . Now using Lemma 4, we obtain our thesis.

From Theorem 1 and by (9) we obtain the following

COROLLARY 2. *If  $f \in X^\omega$ , then  $\|U(\cdot, r; f)\|_X \leq \frac{5}{2}\|f\|_{X^\omega}^* \omega(r)$  for all  $r \in I$ . In the case  $\omega(t) = t^\alpha$ ,  $0 < \alpha \leq 2$ , we have  $\|U(\cdot, r; f)\|_X \leq \frac{5}{2}\|f\|_{X^\omega}^* r^\alpha$  for all  $r \in I$ .*

**3.2.** In this part we shall give an inverse approximation theorem in the norm  $X$ . We shall use the notation (14).

THEOREM 2. *Suppose that  $f \in X$  and*

$$\|U(\cdot, r; f)\|_X \leq \omega(r) \quad (15)$$

for all  $r \in I$ , where  $\omega$  is given function belonging to  $\Omega^2$ . Then

$$\omega_2(t; f, X) \leq M_1^* t^2 \int_t^1 \omega(z) z^{-3} dz \quad (16)$$

for all  $t \in (0, 1/2)$ , where  $M_1^* = M_1(\|f\|_X, \omega)$ .

*Proof.* We shall apply the Bernstein method [5, p.345]. For every fixed integer  $m \geq 2$  we can write

$$\begin{aligned} f(x) &= P(x, 2^{-1}; f) + \sum_{k=1}^{m-1} \{P(x, 2^{-k-1}; f) - P(x, 2^{-k}; f)\} \\ &\quad + f(x) - P(x, 2^{-m}; f), \quad x \in R. \end{aligned} \quad (17)$$

Let

$$v_n(x; f) := P(x, 2^{-n-1}; f) - P(x, 2^{-n}; f) \quad (n = 1, 2, \dots). \quad (18)$$

From (17) and by (3), (14) and (18) we get

$$\begin{aligned} \Delta_h^2 f(x) &= \Delta_h^2 P(x, 2^{-1}; f) + \sum_{k=1}^{m-1} \Delta_h^2 v_k(x; f) + \Delta_h^2 U(x, 2^{-m}; f) \\ &=: A_1(x, h) + A_2(x, h) + A_3(x, h). \end{aligned}$$

for  $x \in R$  and  $h \in R$ . We notice that

$$\Delta_h^2 P(x, r; f) = \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} \frac{\partial^2}{\partial x^2} P(x + t_1 + t_2, r; f) dt_1 dt_2.$$

Hence, by Hölder-Minkowski inequality and by Lemma 5, we get

$$\begin{aligned} \|A_1(\cdot, h)\|_X &= \left\| \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} \frac{\partial^2}{\partial x^2} P(x + t_1 + t_2, 1/2; f) dt_1 dt_2 \right\|_X \\ &\leq h^2 \left\| \frac{\partial^2}{\partial x^2} P(x, 1/2; f) \right\|_X \leq 4\|f\|_X h^2. \end{aligned} \quad (20)$$

From the assumption (15) we have

$$\|A_3(\cdot, h)\|_X \leq 4\omega(2^{-m}). \quad (21)$$

From the Fubini theorem it follows that

$$P(x, 2^{-n-1}; P(\cdot; 2^{-n}; f)) = P(x, 2^{-n}; P(\cdot; 2^{-n-1}; f))$$

for  $n = 1, 2, \dots$  and  $x \in R$ . Hence and by (18) we get

$$v_n(x; f) = P(x, 2^{-n-1}; f(\cdot) - P(\cdot; 2^{-n}; f)) - P(x, 2^{-n}; f(\cdot) - P(\cdot; 2^{-n-1}; f)).$$

Further, we find

$$\Delta_h^2 v_n(x; f) = \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} \frac{\partial^2}{\partial x^2} v_n(x + t_1 + t_2; f) dt_1 dt_2,$$

Hence, by (14) and Lemma 5, we get

$$\begin{aligned} \|A_2(\cdot, h)\|_X &\leq \sum_{k=1}^{m-1} \|\Delta_h^2 v_k(\cdot; f)\|_X \\ &\leq h^2 \sum_{k=1}^{m-1} \{ \|P(\cdot, 2^{-k-1}; U(\cdot, 2^{-k}; f))\|_X + \|P(\cdot, 2^{-k}; U(\cdot, 2^{-k-1}; f))\|_X \} \\ &\leq h^2 \sum_{k=1}^{m-1} \{ \|U(\cdot, 2^{-k}; f)\|_X \cdot 2^{2k+2} + \|U(\cdot, 2^{-k-1}; f)\|_X \cdot 2^{2k} \}. \end{aligned}$$

Now applying the assumptions (14) and (15), and by properties of the function  $\omega \in \Omega^2$ , we obtain

$$\begin{aligned} \|A_2(\cdot, h)\|_X &\leq h^2 \sum_{k=1}^{m-1} \{ 2^{2k+2} \omega(2^{-k}) + 2^{2k} \omega(2^{-k-1}) \} \\ &\leq 5h^2 \sum_{k=1}^{m-1} 2^{2k} \omega(2^{-k}) \leq 10h^2 \int_{2^{-m}}^{1/2} \omega(z) z^{-3} dz. \end{aligned} \quad (22)$$

Using (20)–(22) and (19), we get

$$\|\Delta_h^2 f\|_X \leq 4\|f\|_X h^2 + 4\omega(2^{-m}) + 10h^2 \int_{2^{-m}}^{1/2} \omega(z) z^{-3} dz \quad (23)$$

for every integer  $m \geq 2$  and  $r \in I$ . Now let  $0 < t < 1/2$ ,  $|h| \leq t$  and let  $m$  be an integer such that  $2^{-m} \leq t < 2^{-m+1}$ . Then, by properties of  $\omega \in \Omega$ , from (23) and (2) it follows that

$$\omega_2(t; f, X) \leq 4\|f\|_X t^2 + 4\omega(t) + \frac{5}{2} t^2 \int_t^1 \omega(z) z^{-3} dz.$$

Now using (4), we obtain (16).

In the case  $\omega(h) = M_5(\alpha)h^\alpha$ ,  $0 < \alpha \leq 2$ , from Theorem 2 we get

**COROLLARY 3.** *If  $f \in X$  and  $\|U(\cdot, r; f)\|_X \leq M_5 r^\alpha$ ,  $r \in I$ , where  $0 < \alpha \leq 2$ , then*

$$\omega(t; f, X) \leq M_6(\alpha) \begin{cases} t^\alpha & \text{if } 0 < \alpha < 2 \\ t^2 |\log t| & \text{if } \alpha = 2, \end{cases}$$

for all  $0 < t < 1/2$ .

**3.3.** Now we shall examine the limit properties of  $P(\cdot, r; f)$  in the Hölder norms.

**THEOREM 3.** *Suppose that  $\omega, \mu \in \Omega^2$  and the function  $q(\cdot)$  defined by (7) is monotonically increasing for  $h > 0$ . If  $f \in X^\omega$ , then*

$$\|U(\cdot, r; f)\|_{X^\mu} \leq M_6(\mu) \|f\|_{X^\omega}^* q(r) \quad (24)$$

for all  $r \in I$ , where  $M_6(\mu) = \frac{5}{2}\mu(1) + 12$ .

*Proof.* By our assumptions and (8) we have  $P(\cdot, r; f) \in X^\mu$  for every  $r \in I$ . Hence the function  $U(\cdot, r; f)$  ( $r \in I$ ) defined by (14), belongs to  $X^\mu$  and by (6) we have

$$\|U(\cdot, r; f)\|_{X^\mu} = \|U(\cdot, r; f)\|_X + \|U(\cdot, r; f)\|_{X^\mu}^*$$

for every  $r \in I$ . Using Corollary 2, we get

$$\|U(\cdot, r; f)\|_X \leq \frac{5}{2} \|f\|_{X^\omega}^* \mu(1) q(r), \quad r \in I.$$

By (5), for every  $r \in I$ , we have

$$\|U(\cdot, r; f)\|_{X^\mu}^* \leq \sup_{h>r} \frac{\|\Delta_h^2 U(\cdot, r; f)\|_X}{\mu(h)} + \sup_{0<h\leq r} \frac{\|\Delta_h^2 U(\cdot, r; f)\|_X}{\mu(h)} =: S_1(r) + S_2(r).$$

Since  $\|\Delta_h^2 U(\cdot, r; f)\|_X \leq 4 \|U(\cdot, r; f)\|_X$  and by Corollary 2, we get

$$S_1(r) \leq 4(\mu(r))^{-1} \|U(\cdot, r; f)\|_X \leq 10 \|f\|_{X^\omega}^* q(r), \quad r \in I.$$

From (3), (14) and (13) it follows that  $\Delta_h^2 U(x, r; f) = P(x, r; \Delta_h^2 f) - \Delta_h^2 f(x)$  for  $x \in R$ ,  $h \in R$  and  $r \in I$ . Hence, by Lemma 1, we get

$$\|\Delta_h^2 U(\cdot, r; f)\|_X \leq \|P(\cdot, r; \Delta_h^2 f)\|_X + \|\Delta_h^2 f\|_X \leq 2 \|\Delta_h^2 f\|_X, \quad r \in I.$$

Consequently,

$$\begin{aligned} S_2(r) &\leq 2 \sup_{0<h\leq r} \frac{\|\Delta_h^2 f\|_X}{\mu(h)} = 2 \sup_{0<h\leq r} q(h) \frac{\|\Delta_h^2 f\|_X}{\omega(h)} \\ &\leq 2 \|f\|_{X^\omega}^* q(r) \quad \text{for } r \in I. \end{aligned}$$

Summing up, we obtain (24).

Similarly, we obtain the following

**THEOREM 4.** *Suppose that the functions  $\omega, \mu, q$  satisfy the assumptions of Theorem 3. If  $f \in \tilde{X}^\omega$ , then*

$$\|U(\cdot, r; f)\|_{\tilde{X}^\mu} = o(q(r)) \text{ as } r \rightarrow 0+.$$

Theorem 3 and Theorem 4 imply

**COROLLARY 4.** *Let the assumptions of Theorem 3 be satisfied and let  $q(h) \leq M_7(\gamma)h^\gamma$ ,  $h > 0$ ,  $0 < \gamma < 2$ . If  $f \in X^\omega$ , then*

$$\|U(\cdot, r; f)\|_{X^\mu} \leq M_8(\mu, \gamma) \|f\|_{X^\omega}^* r^\gamma$$

for all  $r \in I$ , where  $M_8(\mu, \gamma) = (\frac{5}{2}\mu(1) + 12) M_7(\gamma)$ . If  $f \in \tilde{X}^\omega$ , then

$$\|U(\cdot, r; f)\|_{\tilde{X}^\mu} = o(r^\gamma) \text{ as } r \rightarrow 0+.$$

In the case  $\omega(h) = h^\alpha$ ,  $\mu(h) = h^\beta$ ,  $0 < \beta < \alpha \leq 2$  and  $f \in X^\omega$  we have

$$\|U(\cdot, r; f)\|_{X^\mu} \leq \frac{29}{2} r^{\alpha-\beta}$$

for all  $r \in I$ . If moreover  $f \in \tilde{X}^\omega$ , then

$$\|U(\cdot, r; f)\|_{\tilde{X}^\mu} = o(r^{\alpha-\beta}) \text{ as } r \rightarrow 0+.$$

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