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TOEPLITZ OPERATORS ON M-HARMONIC HARDY SPACE $H^p_m(S)$ WITH $0 < p \le 1$

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Abstract. Let B^n be the unit ball in C^n, S is the boundary of B^n. Let $L^p(S)$ denote the usual Lebesgue spaces over S with respect to the normalized surface measure, $H^p_m(B^n)$ is the Hardy space of M-harmonic functions and $H^p_{at}(S)$ denotes the atomic Hardy spaces defined in [**4**]. Let $P : L^2(S) \to H^2_m(B^n)$ denote the Poisson–Szëgo projection. We use $M_f : L^p(S) \to L^p(S)$ to denote the multiplication operator, and we define the Toeplitz operator $T_f = PM_f$. The paper gives characterization theorems on f such that the Toeplitz operator T_f is bounded from $H^p_{at}(S) \to H^p_m(B^n)$ with $0 < p \le 1$.

1. Introduction

Let B^n denote the unit ball in C^n. Set S as the boundary of B^n. Let σ be the normalized surface measure over S. Here $L^p(S)$ denotes the usual Lebesgue space over S with respect to σ for $0 < p \le \infty$, $H^p_a(B^n)$ denotes the usual Hardy space of holomorphic functions, $H^p_m(B^n)$ denotes the Hardy space of M-harmonic functions and $H^p_{at}(S)$ denotes the atom Hardy space of complex valued functions on S defined in [**4**] ($H^p_{at}(S) = L^p(S)$, for $1 < p < \infty$). Also we let $\mathrm{BMO}(S)$ denote the usual bounded mean oscillation function space with norm $\|\cdot\|_*$ and $BMOA$ be its holomorphic subspace.We define another function space $\mathrm{LMO}(S)$. We say a function $f \in \mathrm{LMO}(S)$ if $f \in L^1(S)$ and $\|f\|_{\mathrm{LMO}} < \infty$ where

$$\|f\|_{\mathrm{LMO}} = \|f\|_{L^1(S)} + \sup_{\xi\in S,\delta>0} \left\{ \frac{\log(\frac{2}{\sigma(B(\xi,\delta))})}{\sigma(B(\xi,\delta))} \int_{B(\xi,\delta)} |f - f_B| d\sigma \right\}.$$

and

$$f_B = \frac{1}{\sigma(B(\xi,\delta))} \int_{B(\xi,\delta)} f(\eta) d\sigma(\eta).$$

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Here $B = B(\xi, \delta) = B_\delta(\xi) = \{\eta \in S : |1 - \langle \eta, \xi \rangle| < \delta\}$ is nonisotropic ball on S.

Let $C : L^2(S) \to H^2_a(B^n)$ denote the orthogonal projection, i.e.

$$C[f](z) = \int_S f(\xi) C(z, \xi) d\sigma(\xi), z \in B^n, f \in L^2(S),$$

where $C(z, \xi) = (1 - \langle z, \xi \rangle)^{-n}$ is the Cauchy kernel for the unit ball B^n.

The Poisson–Szegö projection is an integral operator

$$P[f](z) = \int_S f(\xi) P(z, \xi) d\sigma(\xi), z \in B^n, f \in L^2(S).$$

The integral kernel $P(z, \xi) = \frac{(1-|z|^2)^n}{|1-\langle z, \xi \rangle|^{2n}}$ is called the Poisson–Szegö kernel of B^n.

We use M_f to denote the multiplication operator, and we define the Toeplitz operators as

$$T^C_f = CM_f \quad \text{and} \quad T_f = PM_f.$$

We begin with a characterization theorem on f such that the Toeplitz operator T^C_f is bounded from $H^1_{at}(S)$ to $H^1_a(B^n)$.

Theorem 0. *Let $f \in L^2(S)$. Then the following statements are equivalent:*

(i) $f \in L^\infty(S) \cap \mathrm{LMO}(S)$,

(ii) *The Toeplitz operators T^C_f and $T^C_{\bar f}$ are bounded from $H^1_{at}(S)$ to $H^1_a(B^n)$,*

(iii) $M_f : \mathrm{BMO}(S) \to \mathrm{BMO}(S)$ *is bounded,*

(iv) $M_f : BMOA \to \mathrm{BMO}(S)$ *is bounded,*

(v) *The Toeplitz operator T^C_f is bounded from $H^1_{at}(S)$ to $H^1_a(B^n)$.*

This is Theorem 1 in [**7**]. We note that in the proof of the implication (iv) $\Rightarrow$ (i) it is not shown that $M_f : BMOA \to \mathrm{BMO}(S)$ implies $f \in L^\infty$. Here is a simple proof.

For $\xi_0 \in S$ any Lebesgue point of f, we consider $g_{\xi_0}(z) = \log(1 - \langle z, \xi_0 \rangle) \in BMOA$ with $||g_{\xi_0}||_\star \le C$, where C is a constant depending only on n. By assumption we have $||fg_{\xi_0}||_\star \le C$ (In this paper constants are denoted by C which may indicate a different constant from one occurrence to the next). Therefore, for any nonisotropic ball $B_\delta(\xi_1) \subset S$ we have

$$\frac{1}{\sigma(B_\delta(\xi_1))} \int_{B_\delta(\xi_1)} |f(\eta) g_{\xi_0}(\eta)| \quad d\sigma(\eta) \le C \log \frac{1}{\sigma(B_\delta(\xi_1))}$$

(see [**7**, Lemma 1]).

In particular, if we let $\xi_1 = \xi_0$ and $0 < \delta < 1/4$, we have

$$C|g_{\xi_0}(\eta)| \ge \log \frac{1}{\sigma(B_\delta(\xi_0))}, \quad \eta \in B_\delta(\xi_0)$$

Thus

$$C\log\frac{1}{\sigma(B_\delta(\xi_0))} \ge \frac{1}{\sigma(B_\delta(\xi_0))}\int_{B_\delta(\xi_0)} |f(\eta)g_{\xi_0}(\eta)|\,d\sigma(\eta)$$
$$\ge C^{-1}\left(\log\frac{1}{\sigma(B_\delta(\xi_0))}\right)\frac{1}{\sigma(B_\delta(\xi_0))}\int_{B_\delta(\xi_0))} |f(\eta)|\,d\sigma(\eta)$$

Therefore

$$\frac{1}{\sigma(B_\delta(\xi_0))}\int_{B_\delta(\xi_0)} |f(\eta)|\,d\sigma(\eta) \le C^2$$

for all $\delta > 0$. Since ξ_0 is Lebesgue point, we have $|f(\xi_0)| \le C^2$. By Lebesgue theorem, we have $|f(\xi)| \le C^2$ for a.e. $\xi \in S$. This completes the proof of Theorem 0.

The main purpose of this paper is to give characterization theorems on f such that the Toeplitz operator T_f is bounded from $H^p_{at}(S) \to H^p_m(B^n)$ with $0 < p \le 1$. More precisely we prove the following:

THEOREM 1. *Let $f \in L^2(S)$. Then the following statements are equivalent:*

(i) $f \in L^\infty(S) \cap \mathrm{LMO}(S)$

(ii) *The Toeplitz operator T_f is bounded from $H^1_{at}(S)$ to $H^1_m(B^n)$.*

(iii) $M_f : \mathrm{BMO}(S) \to \mathrm{BMO}(S)$ *is bounded.*

THEOREM 2. *Let $f \in L^2(S)$, and $0 < p < 1$. Then the following statements are equivalent:*

(i) $f \in (H^p_{at}(S))^*$

(ii) $T_f : H^p_{at}(S) \to H^p_m(B^n)$ *is bounded.*

2. Some basic notations and known results

Let $\tilde\Delta = (1-|z|^2)\sum_{j,k}(\delta_{jk} - z_j\overline{z}_k)D_j\overline{D}_k$ be the invariant or Bergman laplacian. The functions anihilated by $\tilde\Delta$ are called M-harmonic functions, $f \in M$, (see [**8**, Chapter 4], for general properties of these functions).

We will also use the following expressions, defined for a smooth function u in B^n:

(a) The radial maximal function

$$u^+(\xi) = \sup\{|u(r\xi)|; 0 \le r < 1\}.$$

(b) The admissible maximal function

$$M_\delta[u](\xi) = M[u](\xi) = \sup\Big\{|u(z)| : z \in D_\delta(\xi)\Big\}.$$

(c) The admissible area function

$$S[u](\xi) = \Big\{ \int_{D_\delta(\xi)} \|\nabla_{B^n} u(z)\|^2_{B^n} d\tau(z) \Big\}^{1/2}.$$

Here, in (b) and (c), $D_\delta(\xi) = D(\xi)$ is the admissible approach region given by

$$D_\delta(\xi) = \{z \in B^n : |1 - \langle z, \xi\rangle| < \delta(1 - |z|^2)\},$$
$$d\tau(z) = \frac{1}{(1 - |z|^2)^{n+1}} dV(z),$$

dV denoting Lebesgue measure, and $\|\nabla_{B^n} u\|_{B^n}$ is the Bergman length of the Bergman gradient given in coordinates by

$$\|\nabla_{B^n} u\|^2_{B^n} = (1 - |z|^2)\Big\{ \sum_{i=1}^n |D_i u|^2 - \Big|\sum_{i=1}^n z_i D_i u\Big|^2 + \sum_{i=1}^n |\overline{D}_i u|^2 - \Big|\sum_{i=1}^n \overline{z}_i \overline{D}_i u\Big|^2 \Big\}.$$

A function $f \in M$ is said to belong $H^p_m(B^n)$, $0 < p < \infty$, if $M_\delta[f] \in L^p(S)$.

For the proof of Theorem 1 and Theorem 2 the following two lemmas will be needed.

Lemma 2.1. [**1**] *Let $u \in M$. Then the following are equivalent:*

(i) $u \in H^1_m(B^n)$

(ii) *The radial maximal function $u^+ \in L^1(S)$.*

(iii) *The area function $S[u] \in L^1(S)$.*

(iv) *There exists $f \in H^1_{at}(S)$ such that $u = P[f]$.*

Lemma 2.2. [**1**], [**4**], [**5**], [**8**] *The following statements hold:*

(i) $P : H^p_{at}(S) \to H^p_m(B^n)$, $0 < p < \infty$ *is bounded and onto.*

(ii) *The dual space of $H^p_{at}(S)$, $0 < p < 1$, is $\mathcal{L}^\gamma(S)$, $\gamma = n(1/p - 1)$.*

(iii) *The dual space of $H^1_{at}(S)$ is* BMO(S).

See [**4**] and [**7**] for the definition of $\mathcal{L}^\gamma(S)$ spaces.

3. Proof of Theorem 1

First, we prove (ii) $\Rightarrow$ (iii).

By Lemma 2.1 every function in the space $H^1_m(B^n)$ is the Poisson integral of a function in the space $H^1_{at}(S)$ and so the hypotheses that T_f is bounded from $H^1_{at}(S)$ to $H^1_m(B^n)$ is equivalent to the hypotheses that M_f is bounded from $H^1_{at}(S)$

to itself. But then by duality $(H^1_{at}(S))^\star = \mathrm{BMO}(S)$ (Lemma 2.2) M_f is bounded from $\mathrm{BMO}(S)$ to itself.

By Theorem 0 we have that (iii) $\Leftrightarrow$ (i).

The proof of the implication (i) $\Rightarrow$ (ii).

By the atomic decomposition theorem [**4**], [**7**], it suffices to show that for every atom a on S with support $B_0 = B(\xi_0, \delta)$, we have

$$\|T_f(a)\|_{H^1_m(B^n)} \le C(\|f\|_{\mathrm{LMO}} + \|f\|_\infty).$$

Now we let $B_1 = 2B_0 = B(\xi_0, 2\delta)$. Since $f \in L^\infty(S)$, $T_f : L^2(S) \to H^2_m(B^n)$ is bounded (Lemma 2.2). So we have that

$$\begin{aligned}
&\int_{B_1} \Big(\sup_{z\in D(\eta)} \int_{B_0} |a(\xi)||f(\xi)|P(z,\xi)\, d\sigma(\xi)\Big)\, d\sigma(\eta)\\
&\quad \le \|f\|_\infty \bigg(\int_{B_1} \Big(\sup_{z\in D(\eta)} \int_{B_0} |a(\xi)|P(z,\xi) d\sigma(\xi)\Big)^2 d\sigma(\eta)\bigg)^{1/2} (\sigma(B_1))^{1/2}\\
&\quad \le \|f\|_\infty \|a\|_{L^2(S)} (\sigma(B_1))^{1/2} \le C\|f\|_\infty
\end{aligned}$$

(Here we used Lemma 2.2 and the estimate $|a(\xi)| \le (\sigma(B_0))^{-1}$, $\xi \in B_0$). Let $\eta \in S\backslash B_1$ and $z \in D_\delta(\eta)$. Then

$$\begin{aligned}
T_f(a)(z) &= \int_{B_0} a(\xi) f(\xi) P(z,\xi)\, d\sigma(\xi)\\
&= \int_{B_0} a(\xi)(f(\xi) - f_{B_0}) P(z,\xi)\, d\sigma(\xi) + f_{B_0} \int_{B_0} a(\xi) P(z,\xi)\, d\sigma(\xi)\\
&= I_1(z) + I_2(z)
\end{aligned}$$

Now

$$\begin{aligned}
\int_{S\backslash B_1} \sup_{z\in D_\delta(\eta)} |I_2(z)|\, d\sigma(\eta) &\le |f_{B_0}| \int_S \Big(\sup_{z\in D_\delta(\eta)} \int_S |a(\xi)| P(z,\xi)\, d\sigma(\xi)\Big) d\sigma(\eta)\\
&\le C|f_{B_0}|\, \|a\|_{H^1_{at}(S)} \le C\|f\|_\infty, \quad \text{by Lemma 2.1 (or Lemma 2.2).}
\end{aligned}$$

We have

$$\begin{aligned}
I_1(z) &= \int_{B_0} (f(\xi) - f_{B_0}) a(\xi) P(z,\xi)\, d\sigma(\xi)\\
&= \int_{B_0} (f(\xi) - f_{B_0}) a(\xi) [P(z,\xi) - P(z,\xi_0)]\, d\sigma(\xi)\\
&\quad + \int_{B_0} (f(\xi) - f_{B_0}) a(\xi) P(z,\xi_0)\, d\sigma(\xi) = I_{11}(z) + I_{12}(z)
\end{aligned}$$

So

$$\begin{aligned}
&\int_{S\setminus B_1} \big(\sup_{z\in D_\delta(\eta)} |I_{12}(z)|\big)\, d\sigma(\eta) \\
&\quad\le \int_{S\setminus B_1} \Big(\sup_{z\in D_\delta(\eta)} \int_{B_0} |f(\xi) - f_{B_0}|\, |a(\xi)| P(z,\xi_0) d\sigma(\xi)\Big)\, d\sigma(\eta) \\
&\quad\le C \frac{\|f\|_{\mathrm{LMO}}}{\log \frac{2}{\sigma(B_0)}} \int_{S\setminus B_1} \big(\sup_{z\in D_\delta(\eta)} P(z,\xi_0)\big)\, d\sigma(\eta) \\
&\quad\le C \frac{\|f\|_{\mathrm{LMO}}}{\log \frac{1}{\delta}} \int_{S\setminus B_1} \frac{d\sigma(\eta)}{|1 - \langle \eta, \xi_0\rangle|^n} \le C \|f\|_{\mathrm{LMO}}
\end{aligned}$$

Now we turn to estimate

$$\begin{aligned}
|I_{11}(z)| &\le \frac{C}{\sigma(B_0)} \int_{B_0} |f(\xi) - f_{B_0}| |P(z,\xi) - P(z,\xi_0)|\, d\sigma(\xi) \\
&\le \frac{C}{\sigma(B_0)} \int_{B_0} |f(\xi) - f_{B_0}| \frac{\delta^{1/2}}{|1 - \langle z, \xi_0\rangle|^{n+1/2}}\, d\sigma(\xi) \\
&\le C \|f\|_* \frac{\delta^{1/2}}{|1 - \langle \eta, \xi_0\rangle|^{n+1/2}}.
\end{aligned}$$

(See [**8**]) (Note that $z \in D_\delta(\eta)$ and $\eta \in S\setminus B_1$).

Therefore

$$\int_{S\setminus B_1} \sup_{z\in D_\delta(\eta)} |I_{11}(z)|\, d\sigma(\eta) \le C\|f\|_* \le C\|f\|_\infty.$$

4. Proof of Theorem 2

First, we prove that (ii) $\Rightarrow$ (i). Let $g \in H^p_{at}(S) \cap L^2(S)$. Then

$$\begin{aligned}
\Big| \int_S f(\xi) g(\xi)\, d\sigma(\xi) \Big| &= |P[fg](0)| = |T_f(g)(0)| \\
&\le \|T_f g\|_{H^p_m(B^n)} \le \|T_f\|\, \|g\|_{H^p_{at}(S)}
\end{aligned}$$

Since $H^p_{at}(S) \cap L^2(S)$ is dense in $H^p_{at}(S)$, then we have that

$$\Big| \int_S f(\xi) g(\xi)\, d\sigma(\xi) \Big| \le \|T_f\|\, \|g\|_{H^p_{at}(S)}$$

for all $g \in H^p_{at}(S)$. Therefore $f \in (H^p_{at}(S))^*$, i.e. (i) holds.

Now we prove that (i) $\Rightarrow$ (ii). By the atom decomposition theorem, we need only to show that

$$\|T_f(a)\|_{H^p_m(B^n)} \le C(f),$$

for all p-atom a will support $B(\xi_0, \delta) \subset S$ for some $\xi_0 \in S$ and $0 < \delta < 1$.

Now we proceed as in the proof of the implication (i) $\Rightarrow$ (iv), Theorem 2 [**7**]. The contributions of $B(\xi_0, 2\delta)$ to the integral $\int_S (\sup_{z\in D(\eta)} |T_f(a)(z)|)^p \, d\sigma(\eta)$ is estimated using Hölder's inequality and the boundedness of the map $P : L^2(S) \to H^2_m(B^n)$.

For points $\eta \notin B(\xi_0, 2\delta)$ one uses the cancellation of the atom and boundedness $P : H^p_{at}(S) \to H^p_m(B^n)$ (Lemma 2.2) to obtain the desired estimate.

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