ON PRE-PARAREFLEXIVE OPERATOR ALGEBRAS

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Abstract. The notions of pre-parareflexivety and pre-algebraic reflexivity are introduced and studied. The conditions under which pre-parareflexivity coincides with pre-reflexivity or pre-algebraic reflexivity, are obtained.

Let H denote an infinite-dimensional separable complex Hilbert space and $B(H)$, the set of all bounded linear operators on H. An operator range is a linear manifold in H that is also the range of an operator in $B(H)$. For any set $S \subseteq B(H)$, Lat S denotes the lattice (with closed linear span as join and intersection as meet) of all the closed linear subspaces of H invariant under every operator in S, $\mathrm{Lat}_{1/2} S$ denotes the lattice (with algebraic sum of linear manifolds as join and intersection as meet) of all operator ranges invariant under every operator in S and $\mathrm{Lat}_0 S$ denotes the lattice of all invariant linear manifolds for S. If S is singleton say $\{T\}$, we write these notions as Lat T, $\mathrm{Lat}_{1/2} T$ and $\mathrm{Lat}_0 T$ respectively. Alg Lat S (Alg $\mathrm{Lat}_{1/2} S$, Alg $\mathrm{Lat}_0 S$) denotes the algebra of all operators leaving invariant every member of Lat S ($\mathrm{Lat}_{1/2} S$, $\mathrm{Lat}_0 S$ respectively). A weakly closed algebra $\mathcal{A}$ containing identity is called reflexive if Alg Lat $\mathcal{A} = \mathcal{A}$ and pre-reflexive [1] if

$$(\mathrm{Alg\,Lat}\,\mathcal{A}) \cap (\mathrm{Alg\,Lat}\,\mathcal{A})^* = \mathcal{A} \cap \mathcal{A}^*$$

where $\mathcal{A}^* = \{T^*: T \in \mathcal{A}\}$. Ong [8] considered an analogous notion of reflexivity with respect to the lattice of invariant operator ranges and the lattice of invariant linear manifolds.

Definition 1. [8]. An algebra $\mathcal{A} \subseteq B(H)$ (not necessarily closed in any topology) is parareflexive if for any $T \in B(H)$ with $\mathrm{Lat}_{1/2} T \supseteq \mathrm{Lat}_{1/2} \mathcal{A}$ we have $T \in \mathcal{A}$. Equivalently Alg $\mathrm{Lat}_{1/2} \mathcal{A} = \mathcal{A}$.

Definition 2. [**8**]. An algebra $\mathcal{A}$ is algebraically reflexive if $\mathcal{A}$ contains all operators $T \in B(H)$ such that $\mathrm{Lat}_0 \, T \supseteq \mathrm{Lat}_0 \, \mathcal{A}$. Equivalently $\mathrm{Alg} \, \mathrm{Lat}_0 \, \mathcal{A} = \mathcal{A}$.

In this paper we define an operator range analogue of prereflexive algebras namely, pre-parareflexive algebras and discuss some properties of such algebras.

Definition 3. An algebra $\mathcal{A}$ is said to be pre-parareflexive if

$$\mathcal{A} \cap \mathcal{A}^* = (\mathrm{Alg} \, \mathrm{Lat}_{1/2} \, \mathcal{A}) \cap (\mathrm{Alg} \, \mathrm{Lat}_{1/2} \, \mathcal{A})^*.$$

Definition 4. An algebra $\mathcal{A}$ is said to be prealgebraically reflexive if

$$\mathcal{A} \cap \mathcal{A}^* = (\mathrm{Alg} \, \mathrm{Lat}_0 \, \mathcal{A}) \cap (\mathrm{Alg} \, \mathrm{Lat}_0 \, \mathcal{A})^*.$$

It is easily seen that every parareflexive algebra is pre-parareflexive and every algebraically reflexive algebra is prealgebraically reflexive. For more details on parareflexive algebras and algebraically reflexive algebras, one may refer to [**4**], [**6**], [**8**].

THEOREM 5. *Every prereflexive algebra is pre-parareflexive and every pre-parareflexive algebra is prealgebraically reflexive.*

Proof. As $\mathrm{Lat} \, \mathcal{A} \subseteq \mathrm{Lat}_{1/2} \, \mathcal{A} \subseteq \mathrm{Lat}_0 \, \mathcal{A}$, we have

$$\mathrm{Alg} \, \mathrm{Lat}_0 \, \mathcal{A} \subseteq \mathrm{Alg} \, \mathrm{Lat}_{1/2} \, \mathcal{A} \subseteq \mathrm{Alg} \, \mathrm{Lat} \, \mathcal{A}.$$

This implies that

$$(\mathrm{Alg} \, \mathrm{Lat}_0 \, \mathcal{A}) \cap (\mathrm{Alg} \, \mathrm{Lat}_0 \, \mathcal{A})^* \subseteq (\mathrm{Alg} \, \mathrm{Lat}_{1/2} \, \mathcal{A}) \cap (\mathrm{Alg} \, \mathrm{Lat}_{1/2} \, \mathcal{A})^*$$
$$\subseteq (\mathrm{Alg} \, \mathrm{Lat} \, \mathcal{A}) \cap (\mathrm{Alg} \, \mathrm{Lat} \, \mathcal{A})^*$$

The result follows.

THEOREM 6. *Any algebra unitarily equivalent to a pre-parareflexive algebra is pre-parareflexive.*

Proof. Let $\mathcal{A}$ be a pre-parareflexive algebra and S, a unitary operator. Let

$$\mathcal{B} = S\mathcal{A}S^* = \{SAS^* \colon A \in \mathcal{A}\}$$

be an algebra unitarily equivalent to $\mathcal{A}$. Let

$$T \in (\mathrm{Alg} \, \mathrm{Lat}_{1/2} \, \mathcal{B}) \cap (\mathrm{Alg} \, \mathrm{Lat}_{1/2} \, \mathcal{B})^*$$

Then
$$\mathrm{Lat}_{1/2} \, \mathcal{B} \subseteq \mathrm{Lat}_{1/2} \, T \quad \text{and} \quad \mathrm{Lat}_{1/2} \, \mathcal{B} \subseteq \mathrm{Lat}_{1/2} \, T^*$$

Let M be an operator range invariant under $\mathcal{A}$. Then

$$\mathcal{B}S(M) = (S\mathcal{A}S^*)S(M) \subseteq S(M)$$

Therefore $S(M)$ is an operator range invariant under $\mathcal{B}$ and thus under T and T^* both. This implies that for $M \in \text{Lat}_{1/2}\,\mathcal{A}$

$$(S^*TS)M \subseteq M \quad \text{and} \quad S^*T^*S(M) \subseteq M.$$

Thus
$$S^*TS \in (\text{Alg}\,\text{Lat}_{1/2}\,\mathcal{A}) \cap (\text{Alg}\,\text{Lat}_{1/2}\,\mathcal{A})^* = \mathcal{A} \cap \mathcal{A}^*,$$

or
$$T \in (S\mathcal{A}S^*) \cap (S\mathcal{A}^*S^*) = \mathcal{B} \cap \mathcal{B}^*.$$

COROLLARY 7. *An algebra unitarily equivalent to a parareflexive algebra is parareflexive.*

For a Hilbert space H, the tensor product of H with itself denoted by $H \otimes H$ is the space $\sum_{n=1}^{\infty} \oplus H_n$ with $H_n = H$ for all n. For an algebra $\mathcal{A}$ the tensor product $\mathcal{A} \otimes B(H)$ is the set of all operators on $H \otimes H$ of the form

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix}$$

such that $A_{ij} \in \mathcal{A}$ for all i and j.

THEOREM 8. *If $\mathcal{A}$ is a preparareflexive algebra then, $\mathcal{A} \otimes B(H)$ is pre-parareflexive.*

Proof. Let $T \in (\text{Alg}\,\text{Lat}_{1/2}\,\mathcal{A} \otimes B(H)) \cap (\text{Alg}\,\text{Lat}_{1/2}\,\mathcal{A} \otimes B(H))^*$. Then

$$\text{Lat}_{1/2}\,\mathcal{A} \otimes B(H) \subseteq \text{Lat}_{1/2}\,T \quad \text{and} \quad \text{Lat}_{1/2}\,\mathcal{A} \otimes B(H) \subseteq \text{Lat}_{1/2}\,T^*.$$

Here T is an operator on $H \otimes H$ which is of the form

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} & \cdots \\ B_{21} & B_{22} & B_{22} & \cdots \\ B_{31} & B_{32} & B_{33} & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix}$$

where $B_{ij} \in B(H)$. Let M be an operator range invariant under $\mathcal{A}$. Let $N = \sum_{k=1}^{\infty} \oplus M_k$ where $M_k = M$ for each k. Then N is an operator range invariant under $\mathcal{A} \otimes B(H)$ and thus invariant under both T and T^*. This implies that M is invariant under every B_{ij} and B_{ij}^*. Thus

$$B_{ij} \in (\text{Alg}\,\text{Lat}_{1/2}\,\mathcal{A}) \cap (\text{Alg}\,\text{Lat}_{1/2}\,\mathcal{A})^* = \mathcal{A} \cap \mathcal{A}^*$$

implying that $T \in (\mathcal{A} \otimes B(H)) \cap (\mathcal{A} \otimes B(H))^*$.

COROLLARY 9. *If $\mathcal{A}$ is a parareflexive algebra, then $\mathcal{A} \otimes B(H)$ is also parar-eflexive.*

Remark. We may likewise prove that for a prealgebraically reflexive algebra $\mathcal{A}$:

(i) $\mathcal{A} \otimes B(H)$ is prealgebraically reflexive,
(ii) $S\mathcal{A}S^*$ is prealgebraically reflexive for any unitary operator S.

THEOREM 10. *Let $\mathcal{A}$ be a weakly closed algebra of operators such that every operator range invariant under $\mathcal{A}$ is the range of an operator of the form*

$$\sum 2^{-k}(E_{k+1} - E_k),$$

where $\{E_k\}$ is an increasing sequence of projections whose ranges are in $\mathrm{Lat}\,\mathcal{A}$. Then $\mathcal{A}$ is pre-parareflexive if and only if $\mathcal{A}$ is prereflexive.

Proof. Let $\mathcal{A}$ satisfy the hypothesis and let

$$\mathcal{A} \cap \mathcal{A}^* = (\mathrm{Alg}\,\mathrm{Lat}_{1/2}\,\mathcal{A}) \cap (\mathrm{Alg}\,\mathrm{Lat}_{1/2}\,\mathcal{A})^*.$$

Let $T \in (\mathrm{Alg}\,\mathrm{Lat}\,\mathcal{A}) \cap (\mathrm{Alg}\,\mathrm{Lat}\,\mathcal{A})^*$. Let R be any operator range invariant under $\mathcal{A}$ and let R be the range of the operator $P = \sum 2^{-k}(E_{k+1} - E_k)$ for some increasing sequence $\{E_k\}$ of projections in $\mathrm{Lat}\,\mathcal{A}$. Let F be the spectral measure of P. Then

$$F(2^{-k}, 1] = E_k \in \mathrm{Lat}\,\mathcal{A}$$

which is contained in both $\mathrm{Lat}\,T$ and $\mathrm{Lat}\,T^*$. Thus the range of $F(2^{-k}, 1]$ is invariant under T and T^*, for $k = 1, 2, 3, \ldots$. By [**7**, Theorem B], the range R of P is invariant under both T and T^*. This implies that

$$\mathrm{Lat}_{1/2}\,\mathcal{A} \subseteq \mathrm{Lat}_{1/2}\,T \quad \text{and} \quad \mathrm{Lat}_{1/2}\,\mathcal{A} \subseteq \mathrm{Lat}_{1/2}\,T^*.$$

This gives

$$T \in (\mathrm{Alg}\,\mathrm{Lat}_{1/2}\,\mathcal{A}) \cap (\mathrm{Alg}\,\mathrm{Lat}_{1/2}\,\mathcal{A})^* = \mathcal{A} \cap \mathcal{A}^*.$$

Hence $\mathcal{A}$ is prereflexive.

COROLLARY 11. *Let $\mathcal{A}$ be a weakly closed algebra containing a maximal abelian selfadjoint algebra. Then $\mathcal{A}$ is prereflexive if and only if it is pre-parareflexive.*

An algebra $\mathcal{A}$ is said to be strictly cyclic [**5**] on a Hilbert space H if there exists a vector x_0 in H such that $\mathcal{A}x_0 = H$. In the following we prove that abelian strictly cyclic prealgebraically reflexive algebras are pre-parareflexive.

THEOREM 12. *Let $\mathcal{A}$ be a strictly cyclic abelian algebra of operators. Then $\mathcal{A}$ is pre-parareflexive if and only if $\mathcal{A}$ is prealgebraically reflexive.*

Proof. For any $x \in H$, $\mathcal{A}x$ is an operator range invariant under $\mathcal{A}$ and any invariant linear manifold of $\mathcal{A}$ is the sum of spaces of the form $\mathcal{A}x$. Thus $\text{Lat}_0 \mathcal{A} \subseteq \text{Lat}_{1/2} \mathcal{A}$ and therefore

$$(\text{Alg}\,\text{Lat}_{1/2}\,\mathcal{A}) \cap (\text{Alg}\,\text{Lat}_{1/2}\,\mathcal{A})^* = (\text{Alg}\,\text{Lat}_0\,\mathcal{A}) \cap (\text{Alg}\,\text{Lat}_0\,\mathcal{A})^*.$$

Definition 13. For an algebra $\mathcal{A}$, $\text{Lat}_{1/2}\,\mathcal{A}$ is said to be commutative if there exists a commuting set of positive operators such that every element of $\text{Lat}_{1/2}\,\mathcal{A}$ is the range of some operator in this set.

THEOREM 14. *Let $\mathcal{A}$ be pre-parareflexive algebra. The following are equivalent.*

(i) *$\mathcal{A}$ contains a maximal abelian selfadjoint algebra.*

(ii) *$\text{Lat}_{1/2}\,\mathcal{A}$ is commutative.*

Proof. Let $\text{Lat}_{1/2}\,\mathcal{A}$ be commutative. Let $S \subseteq B(H)$ be a commutative set of positive operators such that every element of $\text{Lat}_{1/2}\,\mathcal{A}$ is the range of some operator in S. Then S', the commutant of S, and $\mathcal{A}(S)$, the weakly closed algebra generated by S, leave invariant all the elements of $\text{Lat}_{1/2}\,\mathcal{A}$ and thus are contained in $\text{Alg}\,\text{Lat}_{1/2}\,\mathcal{A}$. Between $\mathcal{A}(S)$ and S', there is a maximal abelian selfadjoint algebra, say M. As M is selfadjoint, $M \subseteq (\text{Alg}\,\text{Lat}_{1/2}\,\mathcal{A})^*$. This implies that

$$M \subseteq (\text{Alg}\,\text{Lat}_{1/2}\,\mathcal{A}) \cap (\text{Alg}\,\text{Lat}_{1/2}\,\mathcal{A})^* = \mathcal{A} \cap \mathcal{A}^*.$$

If $\mathcal{A}$ contains a maximal abelian selfadjoint algebra, then every element of $\text{Lat}_{1/2}\,\mathcal{A}$ is the range of some operator in the maximal abelian selfadjoint algebra [3]. Thus $\text{Lat}_{1/2}\,\mathcal{A}$ is commutative.

Corollary 11 together with Theorem 15 imply the following.

COROLLARY 15. *If $\mathcal{A}$ is a weakly closed algebra with commutative $\text{Lat}_{1/2}\,\mathcal{A}$, then $\mathcal{A}$ is pre-parareflexive if and only if $\mathcal{A}$ is prereflexive.*

Definition 16. An operator T on H is called a parareflexive operator if every operator S on H leaving invariant all invariant operator ranges of T, is an entire function of T. Equivalently, if $S \in \text{Alg}\,\text{Lat}_{1/2}\,T$, then S is an entire function of T.

THEOREM 17. *Let $\mathcal{A}$ be a commutative parareflexive algebra on a finite-dimensional space. Then each element of $\mathcal{A}$ is parareflexive.*

Proof. Let $T \in \mathcal{A}$. Let $B \in \text{Alg}\,\text{Lat}_{1/2}\,T$. Then

$$\text{Lat}_{1/2}\,\mathcal{A} \subseteq \text{Lat}_{1/2}\,T \subseteq \text{Lat}_{1/2}\,B.$$

This implies that $B \in \text{Alg}\,\text{Lat}_{1/2}\,\mathcal{A} = \mathcal{A}$. As $\mathcal{A}$ is commutative, $BT = TB$. Also $B \in \text{Alg}\,\text{Lat}\,T$ and acts on a finite-dimensional space. By [2, Theorem 10], B is a polynomial in T. Thus T is a parareflexive operator.

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