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THE EDGE OF THE WEDGE THEOREMS AND APPLICATIONS

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Abstract. An ultradistribution version of the edge of the wedge theorem will be given in a simpler and easier form than Martineau's theorem, but its proof is given uniquely using quite a different method from others ever known so far. As applications, we give an ultradistribution version of the reflection principle to the unit polydisk for analytic functions and to the unit ball for harmonic functions respectively.

1. Introduction

The edge of the wedge theorem was discovered by theoretical physicists in 1950's [V]. This theorem deals with a question about analytic continuation of holomorphic functions of several complex variables, which arose in physics, in connection with quantum field theory and dispersion relations.

The edge of the wedge theorem concerns the boundary values of holomorphic functions. Bogoliubov's version of this theorem can state roughly as follows:

Suppose Γ is an open convex cone (with vertex at the origin) of $\mathbb{R}^n$, V is the intersection of Γ with some bounded open ball with center at zero, E is a nonempty open set in $\mathbb{R}^n$ and $W^+ = E + iV$, $W^- = E - iV$. Suppose further that f_1 and f_2 are holomorphic functions in W^+ and W^- respectively and they have the same boundary values:

$$\lim_{\substack{y\to 0\\ y\in V}} f_1(x+iy) = \lim_{\substack{y\to 0\\ y\in(-V)}} f_2(x+iy). \tag{1.1}$$

Then f_1 and f_2 can be extended holomorphically to a complex neighborhood Ω of $W^+ \cup E \cup W^-$. (For precise statement see Theorem 3.1.)

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The limits in (1.1) were given the sense of uniform convergence at the earlier stage, but it was recognized that the theorem is still valid if one takes the limits in (1.1) in the sense of Schwartz distributions.

The purpose of this paper is to prove the case where the limits in (1.1) is given in the sense of ultradistributions and give some applications to the complex function theory. Actually, this result, so called the ultradistribution version, is a special case of Martineau's theorem [M], in which the limit in (1.1) is given in the sense of hyperfunctions of Sato. But the hyperfunctional limit is not so easy to handle with than the distributional limit or the ultradistributional limit. So we expect that the ultradistribution version will be more convenient even if the sense of the limit was somewhat weakened.

As for the proof of the ultradistribution version we use quite a different method ever known so far. In fact, we use, so called, "the parametrix method", which was motivated by Komatsu [K1–K3].

In the final section, as an applications to the complex function theory we give a reflection theorem for analytic functions in the polydisk with three equivalent conditions. Besides, a parallel result for the harmonic functions is also given here.

2. Notations and preliminaries

We introduce briefly here the Gevrey class which is an intermediate space lying between the set of infinitely differentiable functions and the set of real analytic functions. See [K1] and [K2] for more details.

DEFINITION 2.1. Let $s > 1$, $h > 0$ and K be a compact subset of $\mathbb{R}^n$. Then we denote by $\mathcal{D}_K^{s,h}$ the set of all infinitely differentiable functions ϕ in $\mathbb{R}^n$ with support K satisfying that

$$\sup_{\substack{x\in K\\ \alpha\in\mathbb{N}_0^n}} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|}\alpha!^s} < \infty \tag{2.1}$$

where $\mathbb{N}_0$ is the set of nonnegative integers and $|\alpha| = \alpha_1+\cdots+\alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$, $\partial_j = \frac{\partial}{\partial x_j}$ $(j = 1, 2, \ldots, n)$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Then $\mathcal{D}_K^{s,h}$ is a Banach space under the norm defined by

$$||\phi||_{s,K,h} = \sup_{\substack{x\in K\\ \alpha\in\mathbb{N}_0^n}} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|}\alpha!^s}. \tag{2.2}$$

and is naturally imbedded into $\mathcal{D}_K^{s,k}$ for $k > 0$.

We denote by $\mathcal{D}_K^{(s)}$ the set of all functions in $\mathcal{D}_K^{s,h}$ for every $h > 0$. For an open subset Ω of $\mathbb{R}^n$ we denote by $\mathcal{D}^{(s)}(\Omega)$ the set of all functions in $\mathcal{D}_K^{(s)}$ for some compact subset K of Ω and we call this Gevrey class. Topologies on these spaces will not be considered here. In fact, $s > 1$ make it possible to construct cut off functions and partitions of unity.

A partial differential operator $P(\partial)$ of infinite order is called a partial differential operator of Gevrey order s if

$$P(\partial) = \sum_{\alpha} a_\alpha \partial^\alpha, \quad |a_\alpha| \le CL^{|\alpha|}/\alpha!^s, \quad \alpha \in \mathbb{N}_0^n$$

for some $L > 0$ and $C > 0$. Then it is easy to see that $P(\partial)\phi \in \mathcal{D}^{(s)}(\Omega)$ for every $\phi \in \mathcal{D}^{(s)}(\Omega)$ and $P(\partial)\phi(z)$ is analytic in Ω if $\phi(z)$ is analytic there.

The following proposition is a bit variation of Lemma 2.3 in [K3], which is a key to the proof of the main theorem later on.

PROPOSITION 2.2. *For any $L > 0$, $d_1 > d_2 > 0$ and $s > 1$ there exist functions $v(x) \in \mathcal{D}^{s,2/L}(C_{d_1})$, $w(x) \in \mathcal{D}^{(s)}(C_{d_1} \backslash C_{d_2})$, and a partial differential operator $P(\partial)$ of infinite order satisfying that*

$$P(\partial)v(x) = \delta(x) + w(x), \quad x \in \mathbb{R}^n \tag{2.4}$$

$$P(\partial) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \partial^\alpha, \quad |a_\alpha| \le CM^{|\alpha|}/\alpha!^s, \quad \alpha \in \mathbb{N}_0^n, \tag{2.5}$$

for some constants $C > 0$ and M, where $C_d = \{x \in \mathbb{R}^n \mid 0 \le x_j \le d,\ j = 1, 2, \dots, n\}$ for $d > 0$ and $\delta(x)$ is the Dirac measure in $\mathbb{R}^n$.

In the above, the function $v(x)$ plays a role of parametrix of the partial differential operator $P(\partial)$ of infinite order in some sense. This fact will give an idea to the proof of the main theorem.

3. The edge of the wedge theorems

If Γ is an open cone in $\mathbb{R}^n$, let V be the intersection of Γ with some bounded open ball with center at the origin of $\mathbb{R}^n$. Let E be a nonempty open set in $\mathbb{R}^n$. Define

$$W^+ = E + iV, \quad W^- = E - iV.$$

In fact, W^+ is the set of all $z = x + iy$ in $\mathbb{C}^n$ such that $x \in E$, $y \in V$, where as $x + iy \in W^-$ if and only if $x \in E$ and $-y \in V$. Here the sets W^+ and W^- are "wedges" whose "edge" is E. (We identify E with $E + i0$, this agrees with the usual identification of a real number with a complex number whose imaginary part is 0.)

The "continuous version" of the edge of the wedge theorem can now be stated.

THEOREM 3.1. *If E, W^+, W^- are as above, then there is an open set Ω in $\mathbb{C}^n$ which contains $W^+ \cup E \cup W^-$ and which has the following property: Every continuous complex function f on $W^+ \cup E \cup W^-$ which is holomorphic in $W^+ \cup W^-$ extends to a holomorphic function F in Ω.*

The continuity assumption above on f amounts to saying that $\lim f(x + iy)$ exists, as $y \to 0$, uniformly on each compact subset of E, i.e., f has continuous boundary values on the edge E when E is approached from W^+ and W^-.

It is interesting to weaken this assumption of uniform convergence, and in fact, it turns out that the same conclusion still holds if it is only assumed that $\lim f(x+iy)$ exists as $y \to 0$ in the sense of distributions. That is to say

$$\lim_{y\to\infty} \int_E f(x+iy)\phi(x)\,dx$$

exists for every infinitely differentiable function ϕ with compact support in E (see [R]).

Here we are going to show that the same result can be obtained merely if the limit exists only for much restricted class of test functions, so called Gevrey class of some order.

Of course this result is also a special case of Martineau's generalization saying that it is true only if the hyperfunctional boundary values exist on E. But, it is not so easy to use and apply the hyperfunctional boundary values. Thus the following "ultradistribution version" will be more convenient. The proof will be done using so called the parametrix, which was constructed in the previous section.

THEOREM 3.2. *Let W^+, W^- and E be as in Theorem 3.1. Then there exists an open set Ω in $\mathbb{C}^n$ which contains $W^+ \cup W^-$ and satisfies the following:*

If f is holomorphic in $W^+ \cup W^-$ and there is a number $s > 1$ such that

$$\lim_{y\to 0} \int_E f(x+iy)\phi(x)\,dx \tag{3.1}$$

exists (as a complex number) for every $\phi \in \mathcal{D}^{(s)}(E)$, then f has a holomorphic extension F in Ω (Of course, $y \to 0$ within $V \cup (-V)$ in (3.1)).

Proof. Let Q and K be compact subsets in $\mathbb{R}^n$ with $Q \subset E$ and K so small that $Q + K \subset E$. We first choose $t > 0$ with $1 < t < s$ so that $\mathcal{D}^{t,h}(E) \subset \mathcal{D}^{(s)}(E)$ for every $h > 0$. For each $y \in V \cup (-V)$ the equation

$$\Lambda_y(\psi) = \int_E f(\xi+iy)\psi(\xi)\,d\xi \tag{3.2}$$

defines a continuous linear functional on a Banach space $\mathcal{D}^{t,h}(Q+K)$. Since $\sup_{y\in V\cup(-V)} |\Lambda_y(\psi)|$ is finite, the uniform boundedness principle implies that there exists a constant, not depending on y such that

$$|\Lambda_y(\psi)| \le C||\psi||_{t,Q+K,h}$$

For a fixed $\phi_0 \in \mathcal{D}^{t,h/2}(K) \subset \mathcal{D}^{t,h}$ we let $\tau_x\phi$ be a translation given by

$$(\tau_x\phi_0)(\xi) = \phi_0(\xi - x), \quad x \in Q.$$

Then the map $T : Q \to \mathcal{D}^{t,h}(Q+K)$ defined by $T(x) = \tau_x(\phi_0)$ is continuous since

$$\sup_{\xi,\alpha} \frac{|\partial_\xi^\alpha[\phi_0(\xi-x) - \phi_0(\xi-x_0)]|}{h^{|\alpha|}\alpha!^t} \le C|x-x_0| \sup_{\xi,\alpha} \frac{|\partial_\xi^\alpha \phi_0(\xi)|}{(h/2)^{|\alpha|}\alpha!^t}$$

for every x and $x_0 \in Q$ and for some constant $C > 0$. Since Q is compact it is clear that

$$T(Q) = \{\tau_x \phi_0 \in \mathcal{D}^{t,h}(Q+K) \mid x \in Q\} = \{\phi_0(\cdot - x) \mid x \in Q\}$$

is a compact subset of $\mathcal{D}^{t,h}(Q+K)$. Then for every $\varepsilon > 0$ there exist $x_1, x_2, \dots, x_n \in Q$ such that for every $x \in Q$

$$||\phi_0(\cdot - x) - \phi_0(\cdot - x_j)||_{t,Q+K,h} < \varepsilon/C \tag{3.4}$$

for at least one j depending on x, where C is the constant in (3.3).

On the other hand, by (3.1) the integral

$$\int_E f(x+\xi+iy)\phi_0(\xi)\, d\xi = \int_E f(\xi+iy)\phi_0(\xi - x)\, d\xi = \Lambda_y(\tau_x\phi_0)$$

converges to a complex number $\Lambda(x)$ for each $x \in Q$, as $y \to 0$.

Therefore, in view of (3.4) we have

$$\begin{aligned}
&\left|\Lambda(x) - \int_E f(x+\xi+iy)\phi_0(\xi)\, d\xi\right| \\
&\qquad \le |\Lambda(x) - \Lambda_y(\tau_{x_j}\phi_0)| + |\Lambda_y(\tau_{x_j}\phi_0) - \Lambda_y(\tau_x\phi_0)| \\
&\qquad \le |\Lambda(x) - \Lambda_y(\tau_{x_j}\phi_0)| + C||\phi_0(\cdot - x_j) - \phi_0(\cdot - x)||_{t,Q+K,h} \\
&\qquad \le |\Lambda(x) - \Lambda_y(\tau_{x_j}\phi_0)| + \varepsilon
\end{aligned}$$

This implies that

$$\max_{x\in Q}\left|\Lambda(x) - \int_E f(x+\xi+iy)\phi_0(\xi)\, d\xi\right| \to 0 \tag{3.5}$$

as $y \to 0$.

Now we choose, as in Proposition 2.2, functions $v(x) \in \mathcal{D}^{t,h/2}(C_{d_1})$, $w(x) \in \mathcal{D}^{(t)}(C_{d_1} \backslash C_{d_2})$ and a partial differential operator $P(\partial)$ of infinite order satisfying that

$$P(\partial)v(x) = \delta(x) + w(x), \quad x \in \mathbb{R}^n \tag{3.6}$$

$$P(\partial) = \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha \partial^\alpha, \quad |a_\alpha| \le CM^{|\alpha|}/\alpha!^t, \quad \alpha \in \mathbb{N}_0^n \tag{3.7}$$

for some constants $C > 0$ and $M > 0$. Here we should choose the real number d_1 so that $C_{d_1} \subset K$.

Define two functions $g(z)$ and $h(z)$ on $(Q+iV) \cup (Q-iV)$ by

$$\begin{aligned}
g(z) &= \int_E f(x+\xi+iy)v(\xi)\, d\xi \\
h(z) &= \int_E f(x+\xi+iy)w(\xi)\, d\xi
\end{aligned}$$

Then these two functions are holomorphic in $Q^0 + iV$ and $Q^0 + i(-V)$ where Q^0 denotes the interior of Q. Also, (3.5) and the continuous version imply that there exists an open set Ω_Q in $\mathbb{C}^n$ which contains Q^0, which is independent of v and w, and to which $g(z)$ and $h(z)$ extend holomorphically.

On the other hand, by (3.6) we have

$$f(z) = P(\partial)g(z) - h(z) \tag{3.8}$$

Since the partial differential operator $P(\partial)$ with (3.7) maps holomorphic functions into holomorphic functions $f(z)$ also extends holomorphically to Ω_Q. Then the open set $\Omega = \bigcup_{Q \in E} \Omega_Q$, Q compact cube, is the required one so that this completes the proof.

As a corollary of Theorem 3.2 we obtain directly the following uniqueness theorem:

COROLLARY 3.3. *Suppose $W^+ = E + iV$ is as above, $f(z)$ is holomorphic in W^+ and there exists $s > 1$ such that*

$$\int_E f(x+iy)\phi(x)\,dx \to 0 \tag{3.9}$$

as $y \to 0$ within V, for every $\phi \in \mathcal{D}^{(s)}(E)$. Then $f = 0$.

The reflection principle also can be stated in the sense of ultradistribution, in which the main hypothesis is put only on the imaginary part of f.

THEOREM 3.4. *Let E, W^+, W^- and Ω be as in Theorem 3.2. Suppose that $f(z) = u(z) + iv(z)$ is holomorphic in W^+, where u and v are real, and there is $s > 0$ such that*

$$\lim_{y \to 0} \int_E v(x+iy)\phi(x)\,dx$$

for every $\phi \in \mathcal{D}^{(s)}(E)$ (here $y \to 0$ within V). Then $f(z)$ has a holomorphic extension F in Ω by the relation $\bar{F}(z) = f(\bar{z})$ in W^-

4. Applications in the polydisk

We denote by U^n the unit polydisk in $\mathbb{C}^n$ which consists of all $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ with $|z_j| < 1$ for $j = 1, 2, \dots, n$, and by T^n the unit torus. To every f defined on U^n we associate a function $\tilde{f}(z)$ in $\mathbb{R}^n + i\mathbb{R}^n_+$ where $\mathbb{R}^n_+ = \{y \in \mathbb{R}^n \mid y_j > 0, j = 1, 2, \dots, n\}$. This can be often written in the form

$$\tilde{f}_y(x) = f_r(w) \text{ or } \tilde{f}(x+iy) = f(rw)$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, $r = (r_1, r_2, \dots, r^n) \in \mathbb{R}^n$ with $0 < r_j < 1$, $j = 1, 2, \dots, n$, $w = (w_1, w_2, \dots, w_n)$ with $|w_j| = 1$, $j = 1, 2, \dots, n$ and $rw = (r_1w_1, r_2w_2, \dots, r_nw_n)$.

It is well known that the change of variables $z \to e^{iz}$ maps all of $\mathbb{C}^n$ into $\mathbb{C}^n$ and is locally one to one. Hence any extension of f will give rise to a corresponding extension of $\tilde{f}$ and vice versa.

Let $E \subset T^n$ and $0 < \delta < 1$. Then we denote by $[E, \delta]$ the set of all rw such that $w \in E$ and $\delta < r_j < 1$, $j = 1, 2, \ldots, n$. Under the change of variables given above, $[E, \delta]$ corresponds to a set $\tilde{E} + iV$ where V is an open cube in $\mathbb{R}^n_+$.

When $E \subset T^n$ is open and f is defined in some $[E, \delta]$ we say that $f(z)$ has an (s)-ultradistribution limit on E if

$$\lim \int_E f_r(w)\phi(w)\, dm_n(w) \tag{4.1}$$

exists, for every $\phi \in \mathcal{D}^{(s)}(E)$, as $r_j \to 1$ $(1 \le j \le n)$. Here m_n denotes the Haar measure (normalized Lebesgue measure) of T^n.

Now for a $\phi \in \mathcal{D}^{(s)}(T^n)$, we consider the Fourier series expansion:

$$\phi(w) = \sum_{\gamma \in \mathbb{Z}^n} c(\gamma) w^\gamma, \quad w \in T^n \tag{4.2}$$

where $\mathbb{Z}$ is the set of all integers and

$$c(\gamma) = \int_{T^n} w^{-\gamma}\phi(w)\, dm_n(w). \tag{4.3}$$

If we denote $w_j = e^{i\theta_j}$, $\theta_j \in [0, 2\pi]$, $j = 1, 2, \ldots, n$ then for every $h > 0$ there is a constant $C > 0$ such that

$$\begin{aligned} |\gamma^\alpha c(\gamma)| &= \left| \int_{T^n} (\partial_\theta^\alpha e^{-i\gamma\theta}) \phi(e^{i\theta})\, dm_n(e^{i\theta}) \right| \\ &\le \int_{T^n} |\partial_\theta^\alpha \phi(e^{i\theta})\, dm_n(e^{i\theta})| \\ &\le C h^{|\alpha|} \alpha!^s, \quad \alpha \in \mathbb{N}_0^n,\ \gamma \in \mathbb{Z}^n \end{aligned}$$

Hence it follows that for every $z > 0$

$$|c(\gamma)| \le C_\varepsilon \exp(-\varepsilon|\gamma|^{1/s}), \quad \gamma \in \mathbb{Z}^n \tag{4.4}$$

where $|\gamma| = |\gamma_1| + |\gamma_2| + \cdots + |\gamma_n|$ for $r \in \mathbb{Z}^n$. Conversely, it is easy to see that if $C(\gamma)$ satisfies (4.4) the function given by (4.2) belongs to $\mathcal{D}^{(s)}(T^n)$.

THEOREM 4.1 *The following properties of a holomorphic function* $f(z) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha z^\alpha$ *in* U^n *are equivalent:*

(i) *There are constant* $h > 0$ *and* $C > 0$ *such that* $|a_\alpha| \le C \exp[h|\alpha|^{1/s}]$, $\alpha \in \mathbb{N}_0^n$.

(ii) f *has an* (s)*-ultradistribution limit on* T^n.

(iii) *There is some sequence of r's tending to* $(1, 1, \ldots, 1)$ *for which the integral*

$$\lim \int_E f_r(w)\phi(w)\, dm_n(w)$$

converges for every $\phi \in \mathcal{D}^{(s)}(E)$.

Proof. For a function $\phi \in \mathcal{D}^{(s)}(T^n)$ let $\hat{\phi}(\alpha)$ be Fourier coefficients of ϕ for each $\alpha \in \mathbb{Z}^n$. If (i) is true then

$$\int_{T^n} f_r(w)\phi(\bar{w})\, dm_n(w) = \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha r^\alpha \int_{T^n} w^\alpha \phi(\bar{w})\, dm_n(w) = \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha r^\alpha \hat{\phi}(\alpha)$$

Thus if (i) holds then taking $\varepsilon = 2h$ in (4.4) and $p > s(n+1)$ we have

$$\left|\int_{T^n} f_r(w)\phi(\bar{w})\, dm_n(w)\right| \le C \sum_{\alpha\in\mathbb{N}_0^n} r^{|\alpha|} \exp[-h|\alpha|^{1/s}] \le C \sum_{\alpha\in\mathbb{N}_0^n} \frac{p! r^{|\alpha|}}{h^p|\alpha|^{n+1}}$$

converges as $r_j \to 1$ $(1 \le j \le n)$. Thus, (i) implies (ii) and (ii) trivially implies (iii).

Now suppose that (i) is false. There are multi-indices $\alpha(k) \in \mathbb{N}_0^n$ with $|\alpha(k)| > k$, $k = 1, 2, \dots$, such that $|a_{\alpha(k)}| > \exp[k|\alpha(k)|^{1/s}]$, $k \in \mathbb{N}$. Taking Λ_k with $\Lambda_k a_{\alpha(k)} = |a_{\alpha(k)}|$ $(k \in \mathbb{N})$ and putting

$$\phi(w) = \sum_{k=1}^{\infty} w^{-\alpha(k)} \exp[-k|\alpha(k)|^{1/s}]$$

then $\phi(w)$ is an analytic function on T^n, so that it belongs to $\mathcal{D}^{(s)}(T^n)$. But,

$$\int_{T^n} f_r(w)\phi(w)\, dm_n(w) = \sum_{k=1}^{\infty} |a_{\alpha(k)}| \exp[-k|\alpha(k)|^{1/s}] r^{\alpha(k)} > \sum_{k=1}^{\infty} r^{\alpha(k)}$$

which tends to ∞ as $r \to (1, 1, \dots, 1)$ through any sequence. This completes the proof.

Now we are in a position to state the ultradistribution version of the Schwartz reflection principle for the polydisk.

We let W^n be the set of all $z = (z_1, z_2, \dots, z_n)$ in $\mathbb{C}^n$ such that $|z_j| > 1$ for $1 \le j \le n$.

THEOREM 4.2. *For every open $E \subset T^n$ there exists an open set Ω in $\mathbb{C}^n$ with the following properties:*

(i) *Ω contains $U^n \cup E \cup V^n$,*

(ii) *if $f(z) = u(z) + iv(z)$ is holomorphic in U^n and there exists $s > 0$ such that $v(z)$ has an (s)-ultradistribution limit 0 in E, then $f(z)$ extends holomorphically to Ω.*

Proof. Since the change of variables $z \to e^{iz}$, which is given at the beginning of this section, maps $\mathbb{C}^n$ into $\mathbb{C}^n$ and is locally one to one the proof of this theorem is immediately obtained from Theorem 3.4.

Remark. In view of Theorem 4.1 the hypothesis about $v(z)$ can be changed by one of three equivalent conditions.

We now give parallel results for the harmonic functions. Actually, if we replace U^n and T^n in the above argument by the unit ball $B(0;1) = \{x \in \mathbb{R}^n \,|\, |x| < 1\}$ and the sphere S^{n-1} we obtain the parallel results, since we have also the change of variables which maps $B(0,1)$ onto the upper half space and which is conformal. So, we state them without proof.

THEOREM 4.3. *Let E be an open subset of S^{n-1}. If $u(x)$ is harmonic in $B(0;1)$ and there exists $s > 0$ such that*

$$\lim_{|x|\to 1} \int_E u(|x|w)\phi(w)\, dw = 0,$$

for every $\phi \in \mathcal{D}^{(s)}(S^{n-1})$ then $u(x)$ extends harmonically to $B(0;1) \cup E \cup \mathbb{R}^n \setminus \bar{B}(0;1)$. Here we denote $x = |x|w$, $w \in S^{n-1}$.

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