

ON A THEOREM OF M. VUILLEUMIER

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ABSTRACT. We give an improvement of a well-known theorem on matrix transforms of slowly varying sequences in the sense of Karamata.

1. Introduction

A sequence of positive numbers (ℓ_n) is said to be slowly varying in the sense of Karamata if

$$\lim_n \left(\frac{\ell_{[cn]}}{\ell_n} \right) = 1, \quad \forall c > 0.$$

The essential properties of these sequences were studied by Karamata [5], [6], Bojanic and Seneta [2] and many others.

Some examples of ℓ_n are:

$$1, \quad \log^a 2n, \quad \log^b(\log 3n), \quad \exp(\log^c 2n); \quad a, b \in \mathbb{R}; \quad 0 < c < 1.$$

The main tool in dealing with matrix transforms of slowly varying sequences is a theorem of Vuilleumier [4]. Her result specialized to triangular real-valued matrices (A_{nk}) , $1 \leq k \leq n$ can be stated as follows:

THEOREM A. *In order that*

$$\sum_{k \leq n} A_{nk} \ell_k \sim \ell_n \quad (n \rightarrow \infty),$$

for each slowly varying sequence (ℓ_n) , it is necessary and sufficient that

$$(1) \quad I. \sum_{k \leq n} A_{nk} \rightarrow 1 \quad (n \rightarrow \infty); \quad II. \sum_{k \leq n} |A_{nk}| k^{-\eta} = O(n^{-\eta}) \quad (n \rightarrow \infty),$$

for some $\eta > 0$.

This theorem plays a fundamental role in the theory of R-regular or R-mercian matrices [3], [10]. But, although it is self-sufficient, there are some inner limitations as we are going to show.

Consider a real-valued sequence (a_n) , $\forall M \in \mathbb{N} : \sum_{n \leq M} a_n \neq 0$, and let $A_{nk} := a_k / \sum_{i \leq n} a_i$.

Then, the condition I of the Theorem A is trivially satisfied and for II, using an inequality for convex means (Lemma 3, below), we obtain

$$\begin{aligned}
 \sum_{k \leq n} |A_{nk}| k^{-\eta} &= \frac{\sum_{k \leq n} |a_k|}{|\sum_{k \leq n} a_k|} \frac{\sum_{k \leq n} |a_k| k^{-\eta}}{\sum_{k \leq n} |a_k|} \\
 (2) \quad &\geq \frac{\sum_{k \leq n} |a_k|}{|\sum_{k \leq n} a_k|} \left(\frac{\sum_{k \leq n} k |a_k|}{\sum_{k \leq n} |a_k|} \right)^{-\eta} \\
 &= n^{-\eta} \left(\frac{|\sum_{k \leq n} a_k|}{\sum_{k \leq n} |a_k|} \right)^{-1} \left(\frac{\sum_{k \leq n} k |a_k|}{n \sum_{k \leq n} |a_k|} \right)^{-\eta}.
 \end{aligned}$$

Since both expressions in parenthesis are positive and not greater than one, we see from (2) that if

$$\liminf_n \frac{|\sum_{k \leq n} a_k|}{\sum_{k \leq n} |a_k|} = 0 \quad \text{or} \quad \liminf_n \frac{\sum_{k \leq n} k |a_k|}{n \sum_{k \leq n} |a_k|} = 0,$$

the condition II is not satisfied so that Theorem A is not applicable.

We will remove such obstacles and thus extend the field of applications.

2. Results

In order to produce a proof of Theorem B below, we need some well-known properties of slowly varying sequences and some elementary inequalities.

LEMMA 1. For each $c > 0$, a slowly varying sequence ℓ_n satisfies (cf. [1, p. 52])

$$\ell_{[x]} \sim \ell_{[cx]} \sim \ell_{[c[x]]} \quad (x \rightarrow \infty).$$

LEMMA 2. For $\eta > 0$, the following relations hold

$$\sup_{k \leq y} k^\eta \ell_k \sim y^\eta \ell_{[y]}; \quad \sup_{k \geq y} k^{-\eta} \ell_k \sim y^{-\eta} \ell_{[y]} \quad (y \rightarrow \infty).$$

A variant of the convex means inequality (cf. [7, p. 76]) is

LEMMA 3. For a sequence of non-negative numbers α_k and $\mu \leq 0$ or $\mu \geq 1$,

$$\frac{\sum k^\mu \alpha_k}{\sum \alpha_k} \geq \left(\frac{\sum k \alpha_k}{\sum \alpha_k} \right)^\mu,$$

and the converse inequality holds for $0 < \mu < 1$.

For an application we need the following

LEMMA 4. For each $a, \eta \in R$, we have

$$\sum_{1 \leq k \leq n} \binom{n+a}{n-k} \frac{k^\eta}{k!} \sim n^{\eta/2} L_n^{(a)}(-1) \quad (n \rightarrow \infty),$$

where $L_n^{(a)}(z)$ is the Laguerre polynomial [8].

Now, we can prove our main result. For a complex-valued triangular matrix (a_{nk}) , $1 \leq k \leq n$, define

$$\sigma_n := \frac{|\sum_{k \leq n} a_{nk}|}{\sum_{k \leq n} |a_{nk}|}; \quad t_n := \frac{\sum_{k \leq n} k |a_{nk}|}{\sum_{k \leq n} |a_{nk}|}.$$

We can prove the following

THEOREM B. If the matrix (a_{nk}) satisfies for $n \rightarrow \infty$

$$(i) \ t_n \rightarrow \infty, \ t_n = o(n); \quad (ii) \ \liminf_n \sigma_n > 0; \quad (iii) \ \frac{\sum_{k \leq n} k^{-\eta} |a_{nk}|}{\sum_{k \leq n} |a_{nk}|} = O(t_n^{-\eta})$$

for some $\eta > 0$, then

$$\frac{\sum_{k \leq n} \ell_k a_{nk}}{\ell_{[t_n]} \sum_{k \leq n} a_{nk}} \rightarrow 1 \quad (n \rightarrow \infty),$$

for all slowly varying sequences (ℓ_n) .

Proof. The condition (ii) guarantees that, for sufficiently large n , we have $\sum_{k \leq n} a_{nk} \neq 0$ and $1/\sigma_n = O(1)$. Therefore, for such n and all fixed c , $0 < c < 1$, we obtain

$$\begin{aligned} \sigma_n \left| \frac{\sum_{k \leq n} \ell_k a_{nk}}{\ell_{[t_n]} \sum_{k \leq n} a_{nk}} - 1 \right| &= \frac{|\sum_{k \leq n} a_{nk} (\ell_k / \ell_{[t_n]} - 1)|}{\sum_{k \leq n} |a_{nk}|} \leq \\ &= \frac{1}{\sum_{k \leq n} |a_{nk}|} \left(\left| \sum_{k \leq ct_n} \right| + \left| \sum_{ct_n < k < t_n/c} \right| + \left| \sum_{k \geq t_n/c} \right| \right) = S_1 + S_2 + S_3. \end{aligned} \quad (B_1)$$

Applying Lemmas 1 and 2, by (iii), we get

$$\begin{aligned} S_1 &\leq \frac{\sum_{k \leq ct_n} |a_{nk}| |\ell_k / \ell_{[t_n]} - 1|}{\sum_{k \leq n} |a_{nk}|} = \frac{\sum_{k \leq ct_n} k^{-\eta} |a_{nk}| |k^\eta \ell_k / \ell_{[t_n]} - k^\eta|}{\sum_{k \leq n} |a_{nk}|} \leq \\ &\sup_{k \leq ct_n} (k^\eta \ell_k / \ell_{[t_n]} + k^\eta) \frac{\sum_{k \leq n} k^{-\eta} |a_{nk}|}{\sum_{k \leq n} |a_{nk}|} \sim 2(ct_n)^\eta \cdot O(t_n^{-\eta}) \ll c^\eta. \end{aligned} \quad (B_2)$$

Similarly, using Lemmas 1, 2 and 3 with $0 < \mu < 1$, we obtain

$$S_3 \leq \sup_{k \geq \frac{1}{c} t_n} (k^{-\mu} \ell_k / \ell_{[t_n]} + k^{-\mu}) \frac{\sum_{k \leq n} k^\mu |a_{nk}|}{\sum_{k \leq n} |a_{nk}|} \sim 2\left(\frac{1}{c} t_n\right)^{-\mu} \cdot O(t_n)^\mu \ll c^\mu. \quad (B_3)$$

Finally, arguing as before and using (i), one has

$$S_2 \leq \frac{\sum_{ct_n < k < 1/ct_n} |a_{nk}| |\ell_k / \ell_{[t_n]} - 1|}{\sum_{k \leq n} |a_{nk}|} \leq \sup_{ct_n < k < 1/ct_n} |\ell_k / \ell_{[t_n]} - 1| = o(1) \quad (t_n \rightarrow \infty),$$

by the uniform convergence (see [1, pp. 6–11]).

Since c can be taken arbitrarily small, from $(B_1), (B_2), (B_3)$ we deduce

$$\left| \frac{\sum_{k \leq n} \ell_k a_{nk}}{\ell_{[t_n]} \sum_{k \leq n} a_{nk}} - 1 \right| = \frac{1}{\sigma_n} (S_1 + S_2 + S_3) = O(1)o(1) = o(1) \quad (n \rightarrow \infty),$$

i.e., Theorem B is proved.

Remark. Comparing theorems A and B, two advantages of the second one become clear. Firstly, Theorem A is not applicable when $t_n = o(n)$ (see Introduction), while in Theorem B it is enough that $t_n \rightarrow \infty$ ($n \rightarrow \infty$). Secondly, Theorem B is valid for complex-valued matrices. Closer connection between theorems A and B can be established if we replace the condition (i) by: $t_n \rightarrow \infty$ and $\lim_n t_n/n$ exist. The proof is carried out as before.

To illustrate the power of the assertion from Theorem B, we give a nontrivial example.

Consider the class of Laguerre polynomials $L_n^{(a)}(z)$ defined by

$$L_n^{(a)}(z) := \sum_{k \leq n} \binom{n+a}{n-k} \frac{(-z)^k}{k!}$$

and take $a_{nk} := \binom{n+a}{n-k} \frac{\exp(ibkn^{-1/4})}{k!}$, $b \in R$. Then

$$\sum_{k \leq n} a_{nk} = L_n^{(a)}(-\exp(ibn^{-1/4})); \quad \sum_{k \leq n} |a_{nk}| = L_n^{(a)}(-1); \quad t_n = \frac{L_n^{(a)}(-1)}{L_n^{(a)}(-1)}.$$

Perron's formula for the asymptotic behavior of Laguerre polynomials in the complex plane cut along the positive part of the real axis says that (cf. [9, p. 197])

$$L_n^{(a)}(z) = 1/2\pi^{-1/2} e^{z/2} (-z)^{-a/2-1/4} n^{a/2-1/4} \exp(2(-nz)^{1/2}) (1 + O(n^{-1/2}))$$

when $n \rightarrow \infty$. Using this formula and the properties of Laguerre polynomials we get $t_n \sim \sqrt{n}$, ($n \rightarrow \infty$) i.e., Theorem A is not applicable in this case.

On the other hand, taking into account Lemma 4, we see that the condition (iii) is satisfied and

$$\sigma_n \sim \frac{|\exp(2\sqrt{n}e^{ib/2n^{-1/4}})|}{\exp(2\sqrt{n})} = \exp(2\sqrt{n}(\cos(b/2n^{-1/4}) - 1)) \rightarrow e^{-b^2/4} \quad (n \rightarrow \infty),$$

i.e., (ii) is also valid; hence, applying Theorem B, we obtain for $n \rightarrow \infty$

$$\sum_{k \leq n} \binom{n+a}{n-k} \frac{\exp(ikn^{-1/4})}{k!} \ell_k = L_n^{(a)}(-\exp(ibn^{-1/4})) \ell_{[\sqrt{n}]} (1 + o(1)).$$

In addition, by separating the real and imaginary parts on the left and applying Perron's formula on the right side of the last expression, we obtain the following two asymptotic relations for $n \rightarrow \infty$

$$\sum_{k \leq n} \binom{n+a}{n-k} \frac{\cos(bkn^{-1/4})}{k!} \ell_k \sim \frac{1}{2\sqrt{\pi e}} n^{a/2-1/4} \ell_{[\sqrt{n}]} e^{2\sqrt{n}-b^2/4} \cos(bn^{1/4});$$

$$\sum_{k \leq n} \binom{n+a}{n-k} \frac{\sin(bkn^{-1/4})}{k!} \ell_k \sim \frac{1}{2\sqrt{\pi e}} n^{a/2-1/4} \ell_{[\sqrt{n}]} e^{2\sqrt{n}-b^2/4} \sin(bn^{1/4}).$$

References

- [1] N.H. Bingham et al., *Regular Variation*, Cambridge University Press, 1989.
- [2] R. Bojanić, E. Seneta, *A unified theory of regularly varying sequences*, Math. Zeitsch. 134 (1973), 91-106
- [3] V. Vučković, *Mercerian theorems for Beekmann matrices*, Publ. Inst. Math. (Beograd) 41(55) (1987), 83-89
- [4] M. Vuilleumier, *Sur le comportement asymptotique des transformations linéaires des suites*, Math. Zeitsch. 98 (1967), 126-139
- [5] J. Karamata, *Sur un mode de croissance régulière; théorèmes fondamentaux*, Bull. Soc. Math. France 61 (1933),
- [6] J. Karamata, *Sur un mode de croissance régulière des fonctions*, Mathematica (Cluj) 4 (1930), 38-53
- [7] G. Pólya, G. Szegő, *Aufgaben und Lehrsätze aus der Analysis, Vol. I*, Springer-Verlag, 1925.
- [8] S. Simić, *Orthogonal polynomials and regularly varying sequences*, Publ. Inst. Math. (Beograd), to appear.
- [9] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc., Providence, R.I., 1959.
- [10] S. Zimering, *Some Mercerian theorems for regularly varying sequences*, Publ. Inst. Math. (Beograd) 15(29) (1973), 171-177

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