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EXISTENCE OF REGULARLY AND RAPIDLY VARYING SOLUTIONS FOR A CLASS OF THIRD ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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To the memory of Professor Michal Greguš

ABSTRACT. The existence of solutions belonging to the Karamata class of functions of a class of third order nonlinear differential equations is proved via some more general results on asymptotic equivalence.

1. Introduction

The theory of Karamata regularly varying functions has proved to be a powerful tool for the study of asymptotic behavior of nonoscillatory solutions of linear and nonlinear differential equations. For a variety of results produced in the framework of regular variation the reader is referred to the monograph [9] and the papers [4–8].

Most of the results found in the literature are concerned exclusively with second order differential equations, and so it is natural to raise the question as to whether higher order differential equations could be studied in the framework of regularly and rapidly varying functions.

The purpose of this paper is to provide a partial affirmative answer to this question by demonstrating the existence of regularly varying and rapidly varying solutions for third order differential equations of the type

$$x''' + 2P(t)x' + P'(t)x = F(t, x) \tag{A}$$

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where P is continuously differentiable on $[a,\infty)$, for some $a>0$, F is continuous on $[a,\infty)\times R$ and

$$|F(t,x)| \leqslant G(t,x) \tag{1.1}$$

where $G:[a,\infty)\times R \to R_+$, $R_+=[0,\infty)$ is a continuous function which is nondecreasing in the second variable for all $t \geqslant a$.

More specifically, in Section 2 we indicate a situation in which the self-adjoint linear equation

$$y''' + 2P(t)y' + P'(t)y = 0 \tag{A$_0$}$$

has a fundamental set of solutions consisting of regularly and rapidly varying functions.

In Section 3 we establish conditions under which equation (A) possesses a fundamental set of solutions which are asymptotic as $t\to\infty$ to the indicated regularly and rapidly varying solutions of equation (A$_0$).

For that purpose a more general result on asymptotic equivalence between solutions of (A) and (A$_0$) is proved (Theorems 3.1–3.3).

To reader's benefit we summarize below the definitions and basic properties of Karamata regularly varying and rapidly varying functions which are needed in developing the main existence theorems (cf. [1]).

Definition. A measurable positive function f(t) defined in some neighborhood of infinity is said to be regularly varying (at infinity) of index ρ if

$$\lim_{t\to\infty}\frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for any} \quad \lambda>0.$$

The set of all regularly varying functions of index ρ is denoted by RV(ρ). If in particular $\rho=0$, the symbol SV is often used for RV(0) and a member of SV is called a slowly varying function.

From the definition it follows that $f(t)\in \mathrm{RV}(\rho)$ is expressed as $f(t)=t^\rho L(t)$ for some $L(t)\in \mathrm{SV}$. So the class SV plays a central role in the theory of regular variation. Of fundamental importance are the following facts holding for any $L(t)\in$ SV:

The function $f(t)\in\mathrm{RV}(0)$ if and only if it can be represented in the form

$$f(t) = c(t)\exp\left\{\int_{t_0}^t \frac{\delta(s)}{s}ds\right\}, \quad t\geqslant t_0, \tag{1.2}$$

for some $t_0>0$ and some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty}\delta(t) = 0.$$

There holds

$$\lim_{t\to\infty} t^{\varepsilon} L(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} t^{-\varepsilon} L(t) = 0 \quad \text{for any} \quad \varepsilon > 0. \tag{1.3}$$

It is obvious that $L_i(t) \in \mathrm{SV}$, $f_i(t) \in \mathrm{RV}(\rho_i)$, $i = 1, 2$, implies

$$L_1(t)L_2(t) \in \mathrm{SV}, \quad f_1(t)f_2(t) \in \mathrm{RV}(\rho_1 + \rho_2). \tag{1.4}$$

All functions tending to positive constants or the function $\Pi_1^n (\log_\nu t)^{\xi_\nu}$, ξ_ν real and $\log_\nu$ the ν-th iteration of the logarithm, are simple examples of slowly varying functions.

Definition. A measurable function $f : [a, \infty) \to (0, \infty)$ is called rapidly varying (at infinity) of index ∞ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \infty \quad \text{for} \quad \lambda > 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = 0 \quad \text{for} \quad 0 < \lambda < 1$$

and is called rapidly varying (at infinity) of index $-\infty$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = 0 \quad \text{for} \quad \lambda > 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \infty \quad \text{for} \quad 0 < \lambda < 1.$$

The set of all rapidly varying functions of index ∞ (or $-\infty$) is denoted by $\mathrm{RV}(\infty)$ (or $\mathrm{RV}(-\infty)$).

The function $\exp(at^{\alpha})$, a, α real and $a \neq 0$, $\alpha > 0$, is a simple example of rapidly varying function.

2. The self-adjoint equation $(\mathbf{A}_0)$

2.1. Preliminaries. Consider the second order linear differential equation

$$z'' + \frac{1}{2} P(t) z = 0 \tag{2.1}$$

and suppose it to be nonoscillatory. The following is known:

Lemma 2.1. [3] *There exist two linearly independent solutions $u(t)$ and $v(t)$ of* (2.1) *such that*

$$\begin{gathered} u(t)/v(t) \to 0 \quad \text{as} \quad t \to \infty, \\ \int_a^\infty \frac{dt}{u(t)^2} = \infty, \quad \int_a^\infty \frac{dt}{v(t)^2} < \infty, \\ v(t) = u(t) \int_{t_0}^t \frac{ds}{u(s)^2}, \quad u(t) = v(t) \int_t^\infty \frac{ds}{v(s)^2}. \end{gathered} \tag{2.2}$$

The solution $u(t)$ satisfying (2.2) is called *principal solution* and $v(t)$ is called *nonprincipal* one.

There is no loss of generality in assuming that both $u(t)$ and $v(t)$ are eventually positive.

LEMMA 2.2. [2], [10] *Let $\{u(t), v(t)\}$ be a fundamental set of solutions on a certain interval I of equation* (2.1). *Then the functions $y_1(t) = u(t)^2$, $y_2(t) = u(t)v(t)$ and $y_3(t) = v(t)^2$ form a fundamental set of solutions on I of equation* (A_0).

LEMMA 2.3. [12] *Let u(t) be a principal solution of equation* (2.10). *Then equation* (A) *can be written in the canonical factorized form:*

$$L_3 x \equiv \frac{2}{u(t)^2}\Big(\frac{u(t)^2}{2}\Big(u(t)^2\Big(\frac{1}{u(t)^2}x\Big)'\Big)'\Big)' = F(t,x). \tag{A$'$}$$

To construct a fundamental set of solutions of (A) choose $u(t)$ and $v(t)$ as positive principal and nonprincipal solutions of (2.1). Then by Lemma 2.2 and (2.2), the set of functions

$$\begin{aligned} &y_1(t) = u(t)^2, \\ &y_2(t) = u(t)v(t) = u(t)^2 \int_a^t \frac{ds}{u(s)^2}, \\ &y_3(t) = v(t)^2 = u(t)^2 \int_a^t \frac{1}{u(s)^2} \int_a^s \frac{2}{u(r)^2}\, dr\, ds \end{aligned} \tag{2.3}$$

forms a fundamental set of solutions for the equation $L_3 x = 0$ (and thus for (A_0)).

Notice also that

$$\int_a^t \frac{1}{u(s)^2} \int_a^s \frac{1}{u(r)^2}\, dr\, ds = \frac{1}{2}\frac{v(t)^2}{u(t)^2}. \tag{2.4}$$

These solutions are asymptotically ordered in the sense that

$$\lim_{t\to\infty} \frac{y_i(t)}{y_j(t)} = 0 \quad \text{and} \quad 1 \leqslant i < j \leqslant 3.$$

2.2. Results. We present here some sufficient conditions which guarantee the existence of regularly and rapidly varying solutions of equation (A_0).

THEOREM 2.1. *Let c be a constant such that $c < 1/4$ and let ρ, σ ($\rho < \sigma$) denote the real roots of the equation*

$$\lambda^2 - \lambda + c = 0. \tag{2.5}$$

If P(t) is conditionally integrable on $[a, \infty)$ and satisfies

$$\lim_{t\to\infty} t \int_t^\infty P(s)\, ds = 2c, \tag{2.6}$$

then equation (A_0) *has a fundamental set of regularly varying (hence nonoscillatory) solutions* $y_i(t)$, $i = 1, 2, 3$, *of the form*

$$y_1(t) = t^{2\rho} L_1(t), \quad y_2(t) = t L_2(t), \quad y_3(t) = t^{2\sigma} L_3(t), \tag{2.7}$$

where $L_1(t)$ *is some slowly varying function,* $L_2(t) \to (1-2\rho)^{-1}$ *and* $L_3(t) \sim \{(1-2\rho)^2 L_1(t)\}^{-1}(t)$ *as* $t \to \infty$.

Proof. By [9, Theorem 1.11], condition (2.6) is necessary and sufficient for equation (2.1) to possess two linearly independent regularly varying (hence nonoscillatory) solutions of the form $u(t) = t^{\rho} L_1(t)$ and $v(t) = t^{\sigma} L_2(t)$ where $L_1(t)$ is some slowly varying function and $L_2(t) \sim \{(1-2\rho)L_1(t)\}^{-1}$, as $t \to \infty$. An application of Lemma 2.2 and the use of property (1.4) of regularly varying functions give the result. $\square$

If $c = 1/4$ which is the borderline case between nonoscillation and oscillation of equation (2.1), there holds

Theorem 2.2. *Suppose that*

$$\lim_{t\to\infty} t \int_t^{\infty} P(s)\, ds = \frac{1}{2}.$$

Suppose furthermore that the function

$$\phi(t) = t \int_t^{\infty} P(s)\, ds - \frac{1}{2}$$

satisfies

$$\int^{\infty} \frac{|\phi(t)|}{t}\, dt < \infty$$

and

$$\int^{\infty} \frac{|\psi(t)|}{t}\, dt < \infty, \quad \text{where} \quad \psi(t) = \int_t^{\infty} \frac{|\phi(s)|}{t}\, ds.$$

Then, equation (A_0) *has a fundamental set of regularly varying (hence nonoscillatory) solutions* $y_i(t)$, $i = 1, 2, 3$, *of index one and of the form*

$$y_1(t) = tL_1(t), \quad y_2(t) = t \log t\, L_2(t), \quad y_3(t) = t \log^2 t\, L_3(t)$$

where $L_1(t) \to \alpha$, $\alpha \in (0, \infty)$, $L_2(t) \to 1$, $L_3(t) \to 1/\alpha^2$ *as* $t \to \infty$.

Proof. One proceeds exactly as in the proof of Theorem 2.1, using this time [9, Theorem 1.12]. $\square$

Theorem 2.3. *Let $P(t) < 0$ for $t \geqslant a$. If for each $\lambda > 1$*

$$\lim_{t\to\infty}\left(-t\int_t^{\lambda t} P(s)\,ds\right) = \infty, \tag{2.8}$$

then equation (A_0) *has at least two rapidly varying solutions such that the first of these solutions decreases and is of the class* $\mathrm{RV}(-\infty)$ *whereas the second one increases and is of the class* $\mathrm{RV}(\infty)$.

Proof. It is known (e.g., [9, Lemma 1]) that (2.1) has a (positive) decreasing solution $u(t)$. Let $v(t)$ be the corresponding linearly independent solution defined by (2.2). Then, by [9, Theorem 1.3], one has $u(t) \in \mathrm{RV}(-\infty)$ and $v(t) \in \mathrm{RV}(\infty)$ if and only if condition (2.8) holds. It is clear from the definition of rapidly varying functions that $y_1(t) = u(t)^2 \in \mathrm{RV}(-\infty)$ and $y_3(t) = v(t)^2 \in \mathrm{RV}(\infty)$ which completes the proof. □

However the third linearly independent solution $y_2(t) = u(t)v(t)$ need not to be rapidly varying at all. This is shown by the example:

$$u(t) = e^{-t}, \quad v(t) = e^{t}, \quad \text{so that} \quad y_2(t) = 1.$$

3. Regularly and rapidly varying solutions of perturbed equation (A)

The objective of this section is twofold. First, we establish sufficient conditions for equation (A) to have solutions $x_1(t)$, $x_2(t)$ and $x_3(t)$ with the same asymptotic behavior as solutions $y_1(t) = u(t)^2$, $y_2(t) = u(t)v(t)$ and $y_3(t) = v(t)^2$ of equation (A_0), respectively. Secondly, we apply these results to construct regularly and rapidly varying solutions of (A).

3.1. Asymptotic equivalence between $x_i(t)$ and $y_i(t)$, $i = 1, 2, 3$. We prove

Theorem 3.1. *If for some $\alpha > 0$*

$$\int_a^\infty v(t)^2 G\big(t, \alpha u(t)^2\big)\,dt < \infty, \tag{3.1}$$

then there exists an eventually positive solution $x_1(t)$ of equation (A) *such that for $t \to \infty$, $x_1(t) \sim \frac{\alpha}{2} u(t)^2$.*

Proof. Choose $T \geqslant a$ such that

$$\int_T^\infty v(t)^2 G\big(t, \alpha u(t)^2\big) dt \leqslant \frac{\alpha}{2}, \tag{3.2}$$

which is possible by (3.1). Further, define the set X_1 by

$$X_1 = \big\{x \in C[T,\infty) : 0 \leqslant x(t) \leqslant \alpha u(t)^2,\ t \geqslant T\big\}$$

and the integral operator $\mathcal{F}_1$ by

$$\mathcal{F}_1 x(t) = \frac{\alpha}{2}\, u(t)^2 + u(t)^2 \int_t^\infty \left[\int_t^s \frac{1}{u(s_2)^2} \int_t^{s_2} \frac{1}{u(s_1)^2}\, ds_1\, ds_2 \right] u(s)^2 F\big(s, x(s)\big)\, ds, \quad t \geqslant T.$$

The set X_1 is a convex closed subset of a locally convex space $C[T,\infty)$ equipped with the usual metric topology of uniform convergence on compact subintervals of $[T,\infty)$. We will apply the Schauder–Tychonoff fixed point theorem to prove that $\mathcal{F}_1$ has a fixed element in X_1. To this end it suffices to verify that $\mathcal{F}_1$ maps X_1 into itself, that $\mathcal{F}_1$ is a continuous mapping, and that $\mathcal{F}_1(X_1)$ is relatively compact in $C[T,\infty)$.

(i) Using (3.2), (2.2) and (2.4) one easily checks that $x \in X_1$ implies $\mathcal{F}_1 x \in X_1$, that is, $\mathcal{F}_1$ is a self-map on X_1.

(ii) Let $\{x_\nu\}$ be a sequence of elements of X_1 converging to $x_0 \in X_1$ in the topology of $C[T,\infty)$. Because of (3.1), the Lebesgue dominated convergence theorem can be applied to prove without difficulty that the sequence of functions $\{\mathcal{F}_1 x_\nu(t)\}$ converges to $\mathcal{F}_1 x_0(t)$ uniformly on any compact subinterval of $[T,\infty)$. This means that the $\mathcal{F}_1$ is a continuous mapping on X_1.

(iii) Since the set $\{\mathcal{F}_1 x(t)\}$ is locally uniformly bounded and locally equicontinuous on $[T,\infty)$, it follows from the Arzela–Ascoli lemma that $\mathcal{F}_1(X_1)$ is relatively compact in $C[T,\infty)$.

Therefore the Schauder–Tychonoff fixed point theorem ensures the existence of an element $x_1 \in X_1$ such that $x_1 = \mathcal{F}_1 x_1$, which is equivalent to the integral equation

$$x_1(t) = \frac{\alpha}{2}\, u(t)^2 + u(t)^2 \int_t^\infty \left[\int_t^s \frac{1}{u(s_2)^2} \int_t^{s_2} \frac{1}{u(s_1)^2}\, ds_1\, ds_2 \right] \times u(s)^2 F\big(s, x_1(s)\big)\, ds, \quad t \geqslant T. \tag{3.3}$$

It is a matter of simple calculation to verify via differentiation of (3.3) that the function $x_1(t)$ is a solution of the equation (A') and hence of (A) on $[T,\infty)$. Also the asymptotics of $x_1(t)$ is an immediate consequence of (3.3). This completes the proof. □

Theorem 3.2. *If for some $\beta > 0$*

$$\int_a^\infty u(t)\, v(t)\, G\big(t, \beta\, u(t)\, v(t)\big)\, dt < \infty, \tag{3.4}$$

then there exists an eventually positive solution $x_2(t)$ of equation (A) *such that for $t \to \infty$, $x_2(t) \sim \frac{\beta}{2} u(t)v(t)$.*

Proof. Such solution is obtained as before via the Schauder–Tychonoff theorem, as a solution of the integral equation

$$x_2(t) = \frac{\beta}{2}\, u(t)\, v(t) + u(t)^2 \int_T^t \frac{1}{u(s)^2} \int_s^\infty \left[\int_s^r \frac{d\sigma}{u(\sigma)^2} \right] u(r)^2 F\big(r, x_2(r)\big)\, dr\, ds \tag{3.5}$$

for some $T \geqslant a$.

Choose $T \geqslant a$ such that

$$\int_T^\infty u(t)\, v(t)\, G\big(t, \beta\, u(t)\, v(t)\big)\, dt \leqslant \frac{\beta}{2}. \tag{3.6}$$

Further, consider the set $X_2 \subset C[T,\infty)$ and the integral operator $\mathcal{F}_2 : X_2 \to C[T,\infty)$ defined by

$$X_2 = \big\{x \in C[T,\infty) : 0 \leqslant x(t) \leqslant \beta\, u(t)\, v(t),\ t \geqslant T\big\}$$

and

$$\mathcal{F}_2 x(t) = \frac{\beta}{2}\, u(t)\, v(t) + u(t)^2 \int_T^t \frac{1}{u(s)^2} \int_s^\infty \left[\int_s^r \frac{d\sigma}{u(\sigma)^2}\right] u(r)^2 F\big(r, x(r)\big)\, dr\, ds.$$

Using (3.4) and proceeding exactly as in the proof of Theorem 3.1, $\mathcal{F}_2$ is shown to be a continuous mapping which sends X_2 into a relatively compact subset of $C[T,\infty)$, and consequently there exists an element $x_2 \in X_2$ such that $x_2 = \mathcal{F}_2 x_2$, which is nothing else but (3.5). That $x_2(t)$ is a solution of (A) on $[T,\infty)$ with the desired asymptotic property, one obtains as before. This completes the proof. □

Theorem 3.3. *If for some* $\gamma > 0$

$$\int_a^\infty u(t)^2 G\big(t, \gamma v(t)^2\big)\, dt < \infty, \tag{3.7}$$

then there exists an eventually positive solution $x_3(t)$ *of* (A) *such that* $x_3(t) \sim \frac{\gamma}{2} v(t)^2$ *as* $t \to \infty$.

Proof. Again take $T \geqslant a$ such that

$$\int_T^\infty u(t)^2 G\big(t, \gamma v(t)^2\big) dt \leqslant \frac{\gamma}{2}. \tag{3.8}$$

The integral equation which generates a sought solution of (A) is this time

$$x_3(t) = \frac{\gamma}{2} v(t)^2 + u(t)^2 \int_T^t \frac{1}{u(s)^2} \int_s^t \frac{d\sigma}{u(\sigma)^2} \int_s^\infty u(r)^2 F\big(r, x_3(r)\big)\, dr\, ds. \tag{3.9}$$

Let X_3 denote the set

$$X_3 = \big\{x \in C[T,\infty) : 0 \leqslant x(t) \leqslant \gamma v(t)^2,\ t \geqslant T\big\}.$$

It is easy to see that the operator $\mathcal{F}_3$ defined by

$$\mathcal{F}_3 x(t) = \frac{\gamma}{2} v(t)^2 + u(t)^2 \int_T^t \frac{1}{u(s)^2} \int_s^t \frac{d\sigma}{u(\sigma)^2} \int_s^\infty u(r)^2 F\big(r, x(r)\big)\, dr\, ds$$

maps X_3 into X_3. Since the continuity of $\mathcal{F}_3$ and the relative compactness of $\mathcal{F}_3(X_3)$ are verified routinely, there exists a fixed element x_3 of $\mathcal{F}_3$ in X_3 which satisfies (3.9) and hence solves the equation (A). The asymptotic property of $x_3(t)$ follows from (3.9). This completes the proof. $\square$

In the case of the specialization

$$x''' + 2P(t)x' + P'(t)x = Q(t)|x|^m \operatorname{sgn} x, \tag{3.10}$$

where $m > 0$ is a constant and $Q : [a, \infty) \to R$ is continuous, the function G in (1.1) can be taken to be

$$G(t, x) = |Q(t)||x|^m, \quad t \geqslant a, \quad x \in R,$$

and conditions (3.1), (3.4) and (3.7) reduce to

$$\int_a^\infty v(t)^2 u(t)^{2m} |Q(t)| \, dt < \infty, \tag{3.11}$$

$$\int_a^\infty [u(t)v(t)]^{m+1} |Q(t)| \, dt < \infty, \tag{3.12}$$

$$\int_a^\infty u(t)^2 v(t)^{2m} |Q(t)| \, dt < \infty, \tag{3.13}$$

respectively.

Corollary 3.1. *Condition* (3.11) *is sufficient for equation* (3.10) *to have for any $\alpha > 0$ an eventually positive solution $x_1(t)$ such that $x_1(t) \sim \alpha u(t)^2$ as $t \to \infty$.*

Corollary 3.2. *If* (3.12) *holds, then for any $\beta > 0$ the equation* (3.10) *has an eventually positive solution $x_2(t)$ such that $x_2(t) \sim \beta u(t)v(t)$ as $t \to \infty$.*

Corollary 3.3. *Condition* (3.13) *is sufficient for equation* (3.10) *to have for any $\gamma > 0$ an eventually positive solution $x_3(t)$ such that $x_3(t) \sim \gamma v(t)^2$ as $t \to \infty$.*

3.2. Regularly and rapidly varying solutions. If $P(t)$ satisfies condition (2.6), then Theorem 2.1 guarantees the existence of a fundamental set of regularly varying solutions of equation (A_0) as given by (2.7). Then, if conditions (3.1), (3.4) and (3.7) hold, Theorems 3.1, 3.2 and 3.3, respectively, guarantee the existence of a fundamental set of solutions $x_i(t)$, $i = 1, 2, 3$, of equation (A) such that for $t \to \infty$, $x_i(t) \sim \alpha_i y_i(t)$, $\alpha_i > 0$. But this also means that $x_i(t)$ are regularly varying since $\phi(t) \sim L(t)$ as $t \to \infty$ implies $\phi(t) = L^*(t)$ where $L^*(t)$ is some slowly varying function such that $L^*(t) \sim L(t)$ as $t \to \infty$ [9, Proposition 7].

As an illustration, take in (A)

$$F(t, x) = t^q M(t)|x|^m \operatorname{sgn} x, \quad q \text{ real}, \quad m > 0, \quad M(t) \in \mathrm{SV}, \tag{3.14}$$

which case might occur in some applications. Here all three conditions (3.11), (3.12) and (3.13) reduce to a single one: for $m > 1$ (superlinear case), to $q <$

$-2\sigma(m-1)-3$, for $m<1$ (sublinear one), to $q<-2\rho(m-1)-3$ and for the linear case $m=1$ all three conditions would reduce to $q<-3$. This is obtained by using (1.3), (1.4) and since $\rho+\sigma=1$. Thus one obtains in the special case (3.14) of (A) that the fundamental set $x_i(t)$, $i=1,2,3$, exists and that for $t\to\infty$

$$x_1(t)\sim\alpha t^{2\rho}L(t),\quad x_2(t)\sim\beta t,\quad x_3(t)\sim\gamma t^{2\sigma}/L(t)^2$$

for any positive constants α,β,γ and for some slowly varying $L(t)$.

To obtain the existence and the asymptotic behaviour of rapidly varying solutions of the perturbed equation (A) with $P(t)<0$, one proceeds in the same way as above in the regularly varying case. That is: assume condition (2.7) and apply first Theorem 2.3 to obtain two rapidly varying solutions $y_1(t)$ and $y_2(t)$. Then assuming conditions (3.11) and (3.13), and applying Theorem 3.1 and Theorem 3.3 one concludes the existence of two rapidly varying solutions $x_1(t)$ and $x_2(t)$ such that for $t\to\infty$

$$x_i(t)\sim\alpha_i y_i(t),\quad \alpha_i>0,\quad i=1,2.$$

4. Examples

Example 1. Consider the equation

$$x'''+2\sin(e^t)x'+e^t\cos(e^t)x=t^qM(t)|x|^m\operatorname{sgn}x,\quad q \text{ real},\quad M(t)\in\mathrm{SV}, \tag{4.1}$$

which is a special case of (A) with $P(t)=\sin(e^t)$. Since this $P(t)$ satisfies (2.6) with $c=0$, the second order equation

$$z''+\sin(e^t)z=0$$

has linearly independent solutions $u(t)\in\mathrm{RV}(0)=\mathrm{SV}$ and $v(t)\in\mathrm{RV}(1)$. A further analysis shows that

$$u(t)\sim A_0\quad\text{and}\quad v(t)\sim\frac{t}{A_0}\quad\text{for some constant } A_0$$

[9, Example 2.1]. From the above observation, it follows that (4.1) by Section 3.2 has solutions $x_i(t)$, $i=1,2,3$, such that

$$x_1(t)\sim\alpha,\quad x_2(t)\sim\beta t,\quad x_3(t)\sim\gamma t^2,$$

for any positive α,β and γ, provided that

$$q<-3,\quad q<-m-2,\quad q<-2m-1,$$

respectively. Clearly, if $m\geqslant 1$ (resp. $m\leqslant 1$), then $q<-2m-1$ (resp. $q<-3$) implies the satisfaction of other two conditions.

We next consider the case where the function $P(t)$ satisfies the conditions of Theorem 2.2. The equation (2.1) with this $P(t)$ has two linearly independent solutions $u(t)$ and $v(t)$ such that $u(t) \sim At^{\frac{1}{2}}$ and $v(t) \sim t^{\frac{1}{2}}(\log t)/A$ for some $A > 0$, and so the equation (A_0) has a fundamental set of solutions $\{u(t)^2, u(t)v(t), v(t)^2\}$ satisfying

$$u(t)^2 \sim A^2 t, \quad u(t)v(t) \sim t\log t, \quad v(t)^2 \sim \frac{t(\log t)^2}{A^2}.$$

It can be shown easily that the conditions (3.1), (3.4) and (3.7) formulated for this case are fulfilled if, for example, $q < -m-2$. If this inequality holds, our Theorems 3.1–3.3 guarantee the existence of three types of solutions $x_i(t)$, $i = 1, 2, 3$, having the properties

$$x_1(t) \sim \alpha t, \quad x_2(t) \sim \beta t\log t, \quad x_3(t) \sim \gamma t(\log t)^2$$

for any given positive constants α, β and γ.

Notice that this is in particular true if in (A) $P(t) = 1/2t^2$, in which case (2.1) is an Euler equation with solutions $u(t) = t^{1/2}$ and $v(t) = t^{1/2}\log t$.

Example 2. Consider the equation

$$x''' - 4k^2t^{2r}x' - 4k^2rt^{2r-1}x = Q(t)|x|^m \operatorname{sgn} x, \quad t \geqslant 1, \tag{4.2}$$

where $k > 0$ is a constant and $2r$ is a positive integer. The associated second order linear equation is the generalized Airy's differential equation

$$z'' - k^2t^{2r}z = 0. \tag{4.3}$$

Since the condition (2.8) is clearly satisfied, the equation (4.3) has a fundamental set $\{u(t), v(t)\}$ of eventually positive solutions, of which $u(t)$ is a (principal) rapidly varying solution of index $-\infty$ and $v(t)$ is a (non-principal) rapidly varying solution of index ∞ with the asymptotic behavior

$$u(t) \sim t^{-r/2}\exp\Big(-\frac{kt^{r+1}}{r+1}\Big), \quad v(t) \sim t^{-r/2}\exp\Big(\frac{kt^{r+1}}{r+1}\Big)$$

as $t \to \infty$ (see [11, p. 285]). It follows that the third order linear differential equation

$$y''' - 4k^2t^{2r}y' - 4k^2rt^{2r-1}y = 0, \quad t \geqslant 1, \tag{4.4}$$

has a fundamental set of eventually positive, asymptotically ordered solutions $y_1(t)$, $y_2(t)$ and $y_3(t)$ such that

$$\begin{aligned} y_1(t) &= u(t)^2 \sim t^{-r}\exp\Big(-\frac{2kt^{r+1}}{r+1}\Big), \\ y_2(t) &= u(t)v(t) \sim t^{-r}, \\ y_3(t) &= v(t)^2 \sim t^{-r}\exp\Big(\frac{2kt^{r+1}}{r+1}\Big). \end{aligned}$$

The conditions (3.11), (3.12) and (3.13) reduce to

$$\int_1^\infty t^{-r(m+1)} \exp\left\{\frac{2k(1-m)t^{r+1}}{r+1}\right\}|Q(t)|\,dt < \infty, \tag{4.5}$$

$$\int_1^\infty t^{-r(m+1)}|Q(t)|\,dt < \infty \tag{4.6}$$

$$\int_1^\infty t^{-r(m+1)} \exp\left\{\frac{2k(m-1)t^{r+1}}{r+1}\right\}|Q(t)|\,dt < \infty, \tag{4.7}$$

respectively.

It is a matter of elementary calculation to check that, in case the function $Q(t)$ satisfies

$$|Q(t)| \leqslant Mt^\rho \exp(At^{r+1}), \quad t \geqslant 1, \tag{4.8}$$

for some constants A, M and ρ,

(i) the condition (4.5) guaranteeing the existence of a solution $x_1(t) \sim \alpha u(t)^2$ is fulfilled if either

$$A < \frac{2k(m-1)}{r+1} \quad \text{or} \quad A = \frac{2k(m-1)}{r+1} \quad \text{and} \quad \rho < r(m+1)-1;$$

(ii) the condition (4.6) guaranteeing the existence of a solution $x_2(t) \sim \beta u(t)v(t)$ is satisfied if either

$$A < 0 \quad \text{or} \quad A = 0 \quad \text{and} \quad \rho < r(m+1)-1;$$

(iii) the condition (4.7) guaranteeing the existence of a solution $x_3(t) \sim v(t)^2$ is fulfilled if either

$$A < \frac{2k(1-m)}{r+1} \quad \text{or} \quad A = \frac{2k(1-m)}{r+1} \quad \text{and} \quad \rho < r(m+1)-1.$$

If $m = 1$, that is, if the equation (4.2) is linear, then clearly $A = 0$ and the above conditions (i), (ii) and (iii) reduce to the single condition $\rho < 2r - 1$ which implies the existence of positive solutions $x_1(t), x_2(t)$ and $x_3(t)$ of all three asymptotic types.

In the inequality (4.8) instead of $M = \text{const}$ one may take some slowly varying function $M(t)$. Then the requested convergence may hold even for $\rho = r(m+1)-1$ when the exp factor disappears, but that will depend on $M(t)$. Namely, all three conditions (4.5), (4.6) and (4.7) will reduce to

$$\int_1^\infty \frac{M(t)}{t}dt < \infty.$$

Observe that in this example one can get a more precise asymptotic formula for $x_i(t)$ in the case when $r = 1/2$ and $k = 1$ (i.e., when (4.3) is the Airy equation). Then

$$
\begin{aligned}
u(t) &= t^{-1/4} \exp\Big(-\frac{2}{3}t^{3/2}\Big)\big(1 + O(t^{-3/2})\big) \\
v(t) &= t^{-1/4} \exp\Big(\frac{2}{3}t^{3/2}\Big)\big(1 + O(t^{-3/2})\big)
\end{aligned}
\tag{4.9}
$$

using (4.8) one can estimate the integral in (3.3) by

$$O(t^{-\frac{1}{2}(m+1)+\rho+1} \exp\big(-Bt^{3/2}\big)), \quad \text{where } B = \frac{2}{3}(m-1) - A.$$

Hence, from (3.3), due to (4.9) one obtains for $t \to \infty$:

a) For $B > 0$

$$y_1(t) = t^{-1/2} \exp\Big(-\frac{4}{3}t^{3/2}\Big)(1 + O(t^{-3/2})).$$

b) For $B = 0$ the same formula holds when $\frac{1}{2}m - \rho \leqslant 2$ and

$$y_1(t) = t^{-1/2} \exp\Big(-\frac{4}{3}t^{3/2}\Big)\Big(1 + O\big(t^{-\frac{1}{2}(m+1)+\rho+1}\big)\Big)$$

when $\frac{1}{2}m - \rho > 2$.

By using the same reasoning one obtains analogous formulae for $y_3(t)$ and $y_2(t)$. The cases to be distinguished in the later case are $A < 0$ and $A = 0$, because of (4.6).

Remark. The importance of the equation of the form (A) is apparent in the study of a ladder-like electrical network with both nonlinear inductor and capacitor (see [13]). In particular, if the line possesses nonhomogeneous features and possible losses, one arrives to the equation

$$u_t + au^p u_x + bu_{xxx} + g(x)u_x + du = 0.$$

Here x and t are space and time variables, respectively, u is dimesionless voltage, $g(x)$ is a function reflecting nonhomogeneity, p, a, b and d are real constants.

If the nonlinearity of the line is absent (i.e., $p = 0$), $g(x)$ is linear (as often happens in applications), e.g., $g(x) = 2dx$, and the voltage is of travelling wave form $u = u(x - mt)$, the above equation is reduced to the form (A_0) with $P(x) = \frac{1}{b}(dx + a - m)$. If in addition an external force $F(u, x)$ is present satisfying the conditions of this paper, then our results are applicable.

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