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ON THE EIGENVALUES OF SOME CLASS OF PSEUDO-LINEAR TRANSFORMATIONS

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Abstract. We develop a connection between the eigenvalues of a class of pseudo-linear transformation over a field K and the eigenvalues of a certain linear transformation. We give a new criterion for this class to be diagonalizable over algebraically closed field.

1. Introduction

This work was inspired by [**6**] – a brief study of (σ, δ) pseudo-linear transformations together with their relations with evaluations of skew polynomial rings. It contains the necessary and sufficient conditions for the algebraic pseudo-linear transformations to be diagonalizable, as well.

If K is a division ring and V K-vector space by a (σ, δ) pseudo-linear transformation we call an additive map $T : V \to V$ such that

$$T(\alpha v) = \sigma(\alpha)T(v) + \delta(\alpha)v, \quad \alpha \in K,\, v \in V,$$

where σ is an automorphism of K and δ is a left σ-derivation i.e., δ is an additive endomorphism of K such that

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b, \quad a, b \in K.$$

Throughout this paper, we will assume that K is a field, $\delta = 0$ and σ is an automorphism of K of finite order. The reason why we switched to this case is the connection that can be developed between the eigenvalues of this class of pseudo-linear transformations and the eigenvalues of certain linear transformations. The use of linear transformations enables us to use the Cayley–Hamilton theorem which in a pseudo-linear setting does not hold.

The paper is organized as follows. In Section 2, we mention some results from [**5, 6**] in order to make the paper more self-contained. All of them are modified due to the restrictions we have made. In Section 3, the main result is presented.

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2. Preliminaries

Let K be a field and $\sigma \in \mathrm{Aut}(K)$. A skew polynomial ring (also called Ore extension), $K[t;\sigma]$ consists of polynomials $\sum_{i=0}^{n} a_i t^i$, $a_i \in K$ which are added in the usual way but are multiplied according to the following commutation rule

$$ta = \sigma(a)t, \qquad a \in K.$$

For any $c \in K^*$ element $\sigma(c)ac^{-1}$ is called the σ-conjugate of a (by c). The set $\{\sigma(c)ac^{-1} \mid c \in K^*\}$ is called the σ-conjugacy class of a.

The evaluation $f(a)$ of a polynomial $f(t) \in K[t;\sigma]$ at some element $a \in K$ is the remainder of $f(t) = \sum_{i=0}^{n} a_i t^i$ divided on the right by $t - a$. It is easy to show by induction that

$$f(a) = \sum_{i=0}^{n} a_i N_i(a)$$

where the maps N_i are defined by induction in the following way: For any $a \in K$

$$N_0(a) = 1 \text{ and } N_{i+1}(a) = \sigma(N_i(a))a,$$

which leads to

$$N_k(a) = \sigma^{k-1}(a)\sigma^{k-2}(a)\dots\sigma(a)a \qquad (k \in \mathbb{N}).$$

We define $f(A)$ for $A \in M_n(K)$ in a similar way $f(A) = \sum_{i=0}^{n} a_i N_i(A)$, where σ has been extended to $M_n(K)$ in the natural way.

Let V be a K vector space. A σ-pseudo-linear transformation of V is an additive map $T : V \to V$ such that $T(\alpha v) = \sigma(\alpha)T(v)$, $\alpha \in K$. We will use the abbreviation σ-PLT for a pseudo-linear transformation with respect to the automorphism σ. A vector $v \in V \setminus \{0\}$ is an eigenvector of σ-PLT with the corresponding eigenvalue $\lambda \in K$ iff $T(v) = \lambda v$.

If V is finite-dimensional and $e = [e_1, \dots, e_n]$ is a basis of V let us write $T(e_j) = \sum_{i=1}^{n} a_{ij} e_i$, $a_{ij} \in K$ or in the matrix notation $Te = eA$ where $A = [a_{ij}] \in M_n(K)$. The matrix A will be denoted $[T]_e$. The equality $[f(T)]_e = f([T]_e)$ holds as well [**6**, Proposition 2.13] for any polynomial $f(t) \in K[t,\sigma]$. Also if v is an eigenvector of σ-PLT T with an eigenvalue $\lambda \in K$, then $[T]_e\sigma(v_e)^T = \lambda v_e^T$, where v_e denotes the coordinates of the vector v with respect to the basis e [**3**].

A σ-PLT transformation T is algebraic if there exist $m \in \mathbb{N}, a_0, a_1, \dots, a_m \in K$, $a_m \neq 0$ such that

$$a_m T^m + \dots + a_1 T + a_0 I = 0.$$

In the case T is algebraic σ-PLT on V and $\mu_T \in K[t;\sigma]$ is its minimal polynomial, $\lambda \in K$ is an eigenvalue for T if and only if $t - \lambda$ divides on the right (left) the polynomial μ_T in $K[t;\sigma]$ [**6**, Proposition 4.5].

We will also use the notion of a Wedderburn polynomial. For $f \in K[t;\sigma]$, let

$$V(f) := \{a \in K \mid f(a) = 0\}.$$

A (monic) polynomial is said to be Wedderburn if $f = \mu_{V(f)}$ i.e., f is equal to the minimal polynomial of $V(f)$-set of its roots [**5**].

3. General results

Let K be a field, $\sigma \in \operatorname{Aut}(K)$ of an order k i.e., $\sigma \neq \mathrm{id}_K$ and k is the least nonnegative integer such that $\sigma^k = \mathrm{id}_K$. If T is σ-PLT on a vector space V over K, then T^k is a linear transformation of V since it is additive and

$$T^k(\alpha v) = \sigma^k(\alpha)T^k(v) = \alpha T^k(v), \quad \alpha \in K.$$

Therefore, if V is a finite-dimensional vector space there exist $m \in \mathbb{N}$, $a_0, \dots, a_m \in K$, $a_m \neq 0$ such that

$$a_m(T^k)^m + \cdots + a_1 T^k + a_0 I = 0,$$

which means that σ-PLT T is algebraic. We will denote its minimal polynomial with μ_T. This polynomial is invariant in $K[t;\sigma]$ and also the right factor of the polynomial $\varphi_{T^k}(t^k)$, where φ_{T^k} denotes the characteristic polynomial of T^k. What we want is to find relations between eigenvalues of linear transformation T^k and σ-PLT T.

Theorem 3.1. *Let T be σ-PLT on a finite dimensional vector space V over field K and $\sigma \in \operatorname{Aut}(K)$ of an order k. An element $\lambda \in K$ is the eigenvalue of T iff $N_k(\lambda)$ is the eigenvalue of T^k.*

Proof. Let $v \in V \setminus \{0\}$ be such that $T(v) = \lambda v$. Then

$$\begin{aligned} T^k(v) &= T^{k-1}(\lambda v) = \sigma^{k-1}(\lambda)T^{k-1}(v) = \cdots \\ &= \sigma^{k-1}(\lambda)\cdots\sigma(\lambda)\lambda v = N_k(\lambda)v. \end{aligned}$$

The polynomial $h(t) = t^k - N_k(\lambda)$ is a Wedderburn polynomial, since it is a minimal polynomial of the set $\Gamma = \{\sigma(c)\lambda c^{-1} \mid c \in K^*\}$. For any $c \in K^*$, we have

$$N_k(\sigma(c)\lambda c^{-1}) = \sigma^k(c)N_k(\lambda)c^{-1} = N_k(\lambda).$$

This shows that h vanishes on Γ. Let $f(t) = \sum_{i=1}^m a_i t^i$ be the monic minimal polynomial of Γ. Then $m = \deg f \leqslant k$, and $a_0 \neq 0$. Let $d \in K^*$. For any $e \in \Gamma$, we have $0 = \sum_{i=0}^m a_i \sigma^i(d) N_i(e) d^{-1}$. Thus, Γ satisfies the polynomial $\sum_{i=0}^m a_i \sigma^i(d) t^i$. By the uniqueness of the minimal polynomial, we must have $\sigma^m(d)a_i = a_i\sigma^i(d)$ for every i. Since $a_0 \neq 0$, this implies that $\sigma^m = \mathrm{id}_K$. Therefore, we have $m = k$ and $f(t) = t^k - N_k(\lambda)$.

We can write $t^k - N_k(\lambda) = (t-\lambda_k)(t-\lambda_{k-1})\cdots(t-\lambda_1)$, where $\lambda_1, \dots, \lambda_k$ are σ-conjugated to λ [**5**, Theorem 5.1], [**3**, Lemma 5]. This gives us

$$T^k - N_k(\lambda)\,\mathrm{id}_K = (T - \lambda_k\,\mathrm{id}_K)(T - \lambda_{k-1}\,\mathrm{id}_K)\cdots(T - \lambda_1\,\mathrm{id}_K).$$

We can conclude that if there exists $0 \neq v \in V$ such that $(T^k - N_k(\lambda)\,\mathrm{id}_K)(v) = 0$, then there exists $l \in \{1, \dots, k\}$ and $0 \neq u \in V$ such that $(T - \lambda_l\,\mathrm{id}_K)(u) = 0$. Since that λ_l is σ-conjugated to λ, there exists $a \in K^*$ such that $\lambda_l = \sigma(a)\lambda a^{-1}$. Then for $u_0 = a^{-1}u$ we obtain

$$T(u_0) = T(a^{-1}u) = \sigma(a^{-1})T(u) = \sigma(a^{-1})\sigma(a)\lambda a^{-1}u = \lambda u_0$$

i.e., λ is an eigenvalue for T, as desired. □

According to [**6**, Theorem 4.6] the minimal polynomial of an algebraic σ-PLT T has roots in at most $n = \dim V$ σ-conjugacy classes. Moreover [**6**, Proposition 4.3] shows that the set

$$\Gamma_T = \{\alpha \in K \mid T(v) = \alpha v\}$$

is closed by σ-conjugations. We can write $\Gamma_T = \Gamma_1 \cup \cdots \cup \Gamma_r$, where $\Gamma_i = \{\sigma(c)\lambda_i c^{-1} \mid c \in K^*\}$ and $r \leqslant n$ i.e., we can see Γ_T as a disjoint union of different σ-conjugacy classes of eigenvalues of T.

THEOREM 3.2. *A σ-PLT T on a finite dimensional vector space V over algebraically closed field K, $\sigma \in \mathrm{Aut}(K)$ of an order k, is diagonalizable iff $\mu_{T^k}(t) = \prod_{i=1,\lambda_i \neq \lambda_j}^{r}(t - N_k(\lambda_i))$, where r is the number of σ-conjugacy classes containing eiganvalues of T.*

PROOF. [**6**, Theorem 4.9] says that algebraic σ-PLT T is diagonalizable iff $\mu_T = \prod_{i=1}^{r} \mu_{\Gamma_i}$ where μ_{Γ_i} stands for the minimal polynomial of the set $\Gamma_i = \{\sigma(c)\lambda_i c^{-1} \mid c \in K^*\}$ of the eigenvalues of T σ-conjugated to λ_i i.e., $\mu_T(t) = \prod_{i=1}^{r}(t^k - N_k(\lambda_i))$, since that $\mu_{\Gamma_i}(t) = t^k - N_k(\lambda_i)$. The polynomial $\prod_{i=1}^{r}(t - N_k(\lambda_i))$ vanishes at T^k and is its minimal polynomial, as well. Otherwise, we would get the polynomial of degree less than $\deg \mu_T$ vanishing at T. Thus, $\mu_{T^k}(t) = \prod_{i=1}^{r}(t - N_k(\lambda_i))$ as desired. □

EXAMPLE 3.1. Let

$$A = \begin{bmatrix} i+1 & 1 \\ -1 & -i \end{bmatrix} \in M_2(\mathbb{C}),$$

$\sigma \in \mathrm{Aut}(\mathbb{C})$, $\tau(x) = \bar{x}$, be complex conjugation and $T(X) = \bar{X}A - A\bar{X}$. In this case, σ is the automorphism of $\mathbb{C}$ of order $k = 2$.

First, we determine the matrix $[T]_e$, where e is the canonical base of $M_2(\mathbb{C})$,

$$P = [T]_e = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & -2i-1 & 0 & -1 \\ 1 & 0 & 2i+1 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

then the matrix $N_2(P)$ becomes

$$N_2(P) = \bar{P}P = \begin{bmatrix} -2 & 2i+1 & -2i-1 & 2 \\ 2i-1 & 3 & -2 & -2i+1 \\ -2i+1 & -2 & 3 & 2i-1 \\ 2 & -2i-1 & 2i+1 & -2 \end{bmatrix}.$$

Next, we calculate the characteristic polynomial $\varphi_{N_2(P)}$. In this case we have

$$\varphi_{T^2}(t) = t^2(t-1)^2$$

and $\mu_{T^2}(t) = t(t-1)$. According to Theorem 2 σ-PLT is diagonalizable.

All the eigenvectors of T for the eigenvalue $\lambda = 1$ belong to the set $U = \ker(T^2 - I)$ which has the basis $[C, D]$, where

$$C = \begin{bmatrix} -1 & 0 \\ 1-2i & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The vectors $C + T(C)$, $D + T(D)$ are the eigenvectors of T for $\lambda = 1$, and $c^{-1}(C + T(C))$, $c^{-1}(D + T(D))$ are the eigenvectors of T for $\lambda = i$, where c is any complex number which satisfies $1 = \bar{c}ic^{-1}$. Similarly, we get the basis $[E, F]$ of $W = \ker T^2$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad F = \begin{bmatrix} -2i - 1 & -1 \\ 1 & 0 \end{bmatrix}$$

and one basis of $\ker T$ $[E, T(F)]$.

Finally, we get that with respect to the basis $[C + T(C), D + T(D), E, T(F)]$ σ-PLT has the matrix $\operatorname{diag}(1, 1, 0, 0)$. We can also get $[T]_f = \operatorname{diag}(i, i, 0, 0)$ with respect to the basis $f = [c^{-1}(C + T(C)), c^{-1}(D + T(D)), E, T(F)]$, for $c = 1 + i$.

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