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COMPLEX OSCILLATION OF DIFFERENTIAL POLYNOMIALS GENERATED BY MEROMORPHIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. We investigate the complex oscillation of some differential polynomials generated by solutions of the differential equation

$$f'' + A_1(z)f' + A_0(z)f = 0,$$

where $A_1(z), A_0(z)$ are meromorphic functions having the same finite iterated p-order.

1. Introduction and statement of results

We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions [**7, 12**]. The iterated order, an order notion for functions of fast growth, was defined by Schönhage [**14**] and Sato [**13**] (see also [**6, 8, 9**] for an extensive survey). For the definition of the iterated order of a meromorphic function, we use the same definition as in [**8**] (for iterated order of entire function see [**2**, p. 317], [**9**, p. 129]). For all $r \in \mathbb{R}$ we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.1. [**8**] Let f be a meromorphic function. Then the iterated p-order $\rho_p(f)$ of f is defined by

$$\rho_p(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p T(r,f)}{\log r} \quad (p \geqslant 1 \text{ is an integer}),$$

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where $T(r,f)$ is the Nevanlinna characteristic function of f [**7, 12**]. For $p=1$, this notation is called order and for $p=2$ hyperorder.

Definition 1.2. [**8**] The finiteness degree of the order of a meromorphic function f is defined by

$$i(f)=\begin{cases}0, & \text{if } f \text{ is rational},\\ \min\{j\in\mathbf{N}:\rho_j(f)<\infty\}, & \text{if } f \text{ is transcendental with}\\ & \rho_j(f)<\infty \text{ for some } j\in\mathbf{N},\\ \infty, & \text{if } \rho_j(f)=\infty \text{ for all } j\in\mathbf{N}.\end{cases}$$

Definition 1.3. [**4**] Let f be a meromorphic function. Then the iterated p-type of f, with iterated p-order $0<\rho_p(f)<\infty$ is defined by

$$\sigma_p(f)=\overline{\lim_{r\to+\infty}}\frac{\log_{p-1}T(r,f)}{r^{\rho_p(f)}}\quad(p\geqslant 1 \text{ is an integer}).$$

Definition 1.4. [**3, 8**] Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\bar{\lambda}_p(f)=\overline{\lim_{r\to+\infty}}\frac{\log_p\bar{N}(r,1/f)}{\log r}\quad(p\geqslant 1 \text{ is an integer}),$$

where $\bar{N}(r,1/f)$ is the counting function of distinct zeros of $f(z)$ in $\{z:|z|<r\}$. For $p=1$ this notation is called exponent of convergence of the sequence of distinct zeros and for $p=2$ hyperexponent of convergence of the sequence of distinct zeros.

Definition 1.5. [**10**] Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\bar{\tau}_p(f)=\bar{\lambda}_p(f-z)=\overline{\lim_{r\to+\infty}}\frac{\log_p\bar{N}(r,1/(F-z))}{\log r}\quad(p\geqslant 1 \text{ is an integer}).$$

For $p=1$, this notation is called exponent of convergence of the sequence of distinct fixed points and for $p=2$ hyperexponent of convergence of the sequence of distinct fixed points [**11**]. Thus $\bar{\tau}_p(f)=\bar{\lambda}_p(f-z)$ is an indication of oscillation of distinct fixed points of $f(z)$.

Definition 1.6. [**4, 8**] The finiteness degree of the iterated convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$i_\lambda(f)=\begin{cases}0, & \text{if } n(r,1/f)=O(\log r),\\ \min\{j\in\mathbb{N}:\lambda_j(f)<\infty\}, & \text{if } \lambda_j(f)<\infty \text{ for some } j\in\mathbb{N},\\ \infty, & \text{if } \lambda_j(f)=\infty \text{ for all } j\in\mathbf{N}.\end{cases}$$

Remark 1.1. Similarly, we can define the finiteness degree $i_{\bar{\lambda}}(f)$ of $\bar{\lambda}_p(f)$.

Consider the linear differential equation

$$f''+A_1(z)f'+A_0(z)f=0, \tag{1.1}$$

where $A_1(z), A_0(z)$ are meromorphic of finite iterated p-order. For almost four decades, substantial results have been obtained on the fixed points of general transcendental meromorphic functions [**16**]. However, there are few studies on the fixed points of solutions of differential equations. In [**5**] Chen firstly pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of second order linear differential equations with entire coefficients. In [**15**] Wang and Yi investigated fixed points and hyperorder of differential polynomials generated by solutions of second order linear differential equations with meromorphic coefficients. In [**10**] Laine and Rieppo gave an improvement of the results of [**15**] by considering fixed points and iterated order.

Recently, the author has studied the relation between solutions and their derivatives of the differential equation

$$f^{(k)} + A(z)f = 0, \tag{1.2}$$

where $k \geqslant 2$, $A(z)$ is a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ and have obtained the following result.

THEOREM 1.1. [**1**] *Let $k \geqslant 2$ and $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \underline{\lim}_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$. Suppose, moreover, that either:*

(i) *all poles of f are of uniformly bounded multiplicity or* (ii) $\delta(\infty, f) > 0$.

If $\varphi(z) \not\equiv 0$ is a meromorphic function with finite iterated p-order $\rho_p(\varphi) < +\infty$, then every meromorphic solution $f(z) \not\equiv 0$ of (1.2)*, satisfies*

$$\bar{\lambda}_p(f - \varphi) = \bar{\lambda}_p(f' - \varphi) = \cdots = \bar{\lambda}_p\big(f^{(k)} - \varphi\big) = \rho_p(f) = \infty,$$
$$\bar{\lambda}_{p+1}(f - \varphi) = \bar{\lambda}_{p+1}(f' - \varphi) = \cdots = \bar{\lambda}_{p+1}\big(f^{(k)} - \varphi\big) = \rho_{p+1}(f) = \rho.$$

We know that a differential equation bears a relation to all derivatives of its solutions. Hence, linear differential polynomials generated by its solutions must have special nature because of the control of differential equations.

The main purpose of this paper is to study the growth and the oscillation of some differential polynomials generated by solutions of the second order linear differential equation (1.1). We obtain some estimates of their iterated order and fixed points.

THEOREM 1.2. *Let $A_1(z), A_0(z)$ be meromorphic functions, and let $i(A_0) = p$ $(1 \leqslant p < +\infty)$. Assume that either $i_\lambda(1/A_0) < p$ or $\lambda_p(1/A_0) < \rho_p(A_0) = \rho$ $(0 < \rho < +\infty)$ and that $i(A_1) < p$. Let $d_0(z), d_1(z)$ be meromorphic functions such that at least one of $d_0(z), d_1(z)$ does not vanish identically with $\max\{\rho_p(d_j) : j = 0, 1\} < \rho_p(A_0)$. Let $\varphi(z) \not\equiv 0$ be a meromorphic function with $\rho_p(\varphi) < \rho_p(A_0)$ such that $P = (d_1' + d_0 - d_1A_1)\varphi - d_1\varphi' \not\equiv 0$. If $f \not\equiv 0$ is a meromorphic solution of* (1.1) *whose poles are of uniformly bounded multiplicity, then the differential polynomial $g_f = d_1f' + d_0f$ satisfies*

$$\bar{\lambda}_p(g_f - \varphi) = \rho_p(g_f) = \rho_p(f) = \infty,$$
$$\bar{\lambda}_{p+1}(g_f - \varphi) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho.$$

Applying Theorem 1.2 for $\varphi(z) = z$, we obtain the following result.

COROLLARY 1.1. *Suppose that* $A_1(z), A_0(z), d_0(z), d_1(z)$ *satisfy the additional hypotheses of Theorem* 1.2 *such that* $(d_1' + d_0 - d_1A_1)z - d_1 \not\equiv 0$. *If* $f \not\equiv 0$ *is a meromorphic solution of* (1.1) *whose poles are of uniformly bounded multiplicity, then the differential polynomial* $g_f = d_1f' + d_0f$ *satisfies* $\bar{\tau}_p(g_f) = \rho_p(g_f) = \rho_p(f) = \infty$ *and* $\bar{\tau}_{p+1}(g_f) = \rho_{p+1}(g_f) = \rho_p(A_0) = \rho$.

In what follows we obtain a result without the additional condition $P \not\equiv 0$.

THEOREM 1.3. *Let* $A_1(z), A_0(z)$ *be meromorphic functions, and let* $i(A_0) = p$ $(1 \leqslant p < +\infty)$. *Assume that either* $i_\lambda(1/A_0) < p$ *or* $\lambda_p(1/A_0) < \rho_p(A_0) = \rho$ $(0 < \rho < +\infty)$ *and that* $\rho_p(A_0) = \rho_p(A_1)$, $\sigma_p(A_1) < \sigma_p(A_0) = \sigma$ $(0 < \sigma < +\infty)$. *Let* $d_0(z), d_1(z)$ *be meromorphic functions such that at least one of* $d_0(z), d_1(z)$ *does not vanish identically with* $\max\{\rho_p(d_j) : j = 0, 1\} < \rho_p(A_0)$. *Let* $\varphi(z) \not\equiv 0$ *be a meromorphic function with* $\rho_p(\varphi) < \rho_p(A_0)$. *If* $f \not\equiv 0$ *is a meromorphic solution of* (1.1) *whose poles are of uniformly bounded multiplicity, then the differential polynomial* $g_f = d_1f' + d_0f$ *satisfies*

$$\bar{\lambda}_p(g_f - \varphi) = \rho_p(g_f) = \rho_p(f) = \infty,$$
$$\bar{\lambda}_{p+1}(g_f - \varphi) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho.$$

Applying Theorem 1.3 for $\varphi(z) = z$, we obtain the following result.

COROLLARY 1.2. *Suppose that* $A_1(z), A_0(z), d_0(z), d_1(z)$ *satisfy the additional hypotheses of Theorem* 1.3. *If* $f \not\equiv 0$ *is a meromorphic solution of* (1.1) *whose poles are of uniformly bounded multiplicity, then the differential polynomial* $g_f = d_1f' + d_0f$ *satisfies* $\bar{\tau}_p(g_f) = \rho_p(g_f) = \rho_p(f) = \infty$ *and* $\bar{\tau}_{p+1}(g_f) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho$.

2. Auxiliary Lemmas

We need the following lemmas in the proofs of our theorems.

LEMMA 2.1. [**8**, Remark 1.3] *If* f *is a meromorphic function with* $i(f) = p \geqslant 1$, *then* $\rho_p(f) = \rho_p(f')$.

LEMMA 2.2. [**10**] *If* f *is a meromorphic function with* $0 < \rho_p(f) < \rho$ $(p \geqslant 1)$, *then* $\rho_{p+1}(f) = 0$.

LEMMA 2.3. [**1**] *Let* $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ *be finite iterated* p*-order meromorphic functions. If* f *is a meromorphic solution with* $\rho_p(f) = \infty$ *and* $\rho_{p+1}(f) = \rho < \infty$ *of the equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F,$$

then $\bar{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = \infty$ *and* $\bar{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho$.

LEMMA 2.4. *Let* f, g *be meromorphic functions with iterated* p*-orders* $0 < \rho_p(f)$, $\rho_p(g) < \infty$ *and iterated* p*-types* $0 < \sigma_p(f), \sigma_p(g) < \infty$ $(1 \leqslant p < +\infty)$. *Then the following statements hold:*

(i)*If* $\rho_p(g) < \rho_p(f)$, *then*

$$\sigma_p(f+g) = \sigma_p(fg) = \sigma_p(f). \tag{2.1}$$

(ii) *If* $\rho_p(f) = \rho_p(g)$ *and* $\sigma_p(g) \neq \sigma_p(f)$, *then*

$$\rho_p(f+g) = \rho_p(fg) = \rho_p(f).$$

Proof. By the definition of the iterated p-type, we have

$$\sigma_p(f+g) = \varlimsup_{r\to+\infty} \frac{\log_{p-1} T(r, f+g)}{r^{\rho_p(f+g)}} \leqslant \varlimsup_{r\to+\infty} \frac{\log_{p-1}(T(r,f)+T(r,g)+O(1))}{r^{\rho_p(f+g)}}. \tag{2.2}$$

Since $\rho_p(g) < \rho_p(f)$, then $\rho_p(f+g) = \rho_p(f)$. Thus, from (2.2), we obtain

$$\sigma_p(f+g) \leqslant \varlimsup_{r\to+\infty} \frac{\log_{p-1} T(r,f)}{r^{\rho_p(f)}} + \varlimsup_{r\to+\infty} \frac{\log_{p-1} T(r,g)+O(1)}{r^{\rho_p(f)}} = \sigma_p(f). \tag{2.3}$$

On the other hand since $\rho_p(f+g) = \rho_p(f) > \rho_p(g)$, then by (2.3), we get

$$\sigma_p(f) = \sigma_p(f+g-g) \leqslant \sigma_p(f+g). \tag{2.4}$$

Hence by (2.3) and (2.4), we obtain $\sigma_p(f+g) = \sigma_p(f)$. Now, we prove $\sigma_p(fg) = \sigma_p(f)$. Since $\rho_p(g) < \rho_p(f)$, then $\rho_p(fg) = \rho_p(f)$. By the definition of the iterated p-type, we have

$$\begin{aligned} \sigma_p(fg) &= \varlimsup_{r\to+\infty} \frac{\log_{p-1} T(r,fg)}{r^{\rho_p(fg)}} = \varlimsup_{r\to+\infty} \frac{\log_{p-1} T(r,fg)}{r^{\rho_p(f)}} \\ &\leqslant \varlimsup_{r\to+\infty} \frac{\log_{p-1}(T(r,f)+T(r,g))}{r^{\rho_p(f)}} \\ &\leqslant \varlimsup_{r\to+\infty} \frac{\log_{p-1} T(r,f)}{r^{\rho_p(f)}} + \varlimsup_{r\to+\infty} \frac{\log_{p-1} T(r,g)+O(1)}{r^{\rho_p(f)}} = \sigma_p(f). \end{aligned} \tag{2.5}$$

Since $\rho_p(fg) = \rho_p(f) > \rho_p(g) = \rho_p(1/g)$, then by (2.5), we obtain

$$\sigma_p(f) = \sigma_p\Big(fg\frac{1}{g}\Big) \leqslant \sigma_p(fg). \tag{2.6}$$

Thus, by (2.5) and (2.6), we obtain $\sigma_p(fg) = \sigma_p(f)$.

(ii) Without lost of generality, we suppose that $\rho_p(f) = \rho_p(g)$ and $\sigma_p(g) < \sigma_p(f)$. Then we have $\rho_p(f+g) \leqslant \max\{\rho_p(f), \rho_p(g)\} = \rho_p(f) = \rho_p(g)$. If we suppose that $\rho_p(f+g) < \rho_p(f) = \rho_p(g)$, then by (2.1) we get

$$\sigma_p(g) = \sigma_p(f+g-f) = \sigma_p(f)$$

and this is a contradiction. Hence $\rho_p(f+g) = \rho_p(f) = \rho_p(g)$.

Now, we prove that $\rho_p(fg) = \rho_p(f) = \rho_p(g)$. Also we have

$$\rho_p(fg) \leqslant \max\{\rho_p(f), \rho_p(g)\} = \rho_p(f) = \rho_p(g).$$

If we suppose that $\rho_p(fg) < \rho_p(f) = \rho_p(g) = \rho_p(1/f)$, then by (2.1) we can write

$$\sigma_p(g) = \sigma_p\Big(fg\frac{1}{f}\Big) = \sigma_p\Big(\frac{1}{f}\Big) = \sigma_p(f)$$

and this is a contradiction. Hence $\rho_p(fg) = \rho_p(f) = \rho_p(g)$. □

Lemma 2.5. *Let $A_1(z), A_0(z)$ be meromorphic functions, and let $i(A_0) = p$ $(1 \leqslant p < +\infty)$. Assume that either $i(A_1) < p$ or $\rho_p(A_1) = \rho_p(A_0) = \rho$ $(0 < \rho < +\infty)$ and $\sigma_p(A_1) < \sigma_p(A_0) = \sigma$ $(0 < \sigma < +\infty)$. Let d_j $(j = 0, 1)$ be meromorphic functions that are not all vanish identically with* $\max\{\rho_p(d_j) : j = 0, 1\} < \rho_p(A_0)$. *Then*

$$h = d_1(d_0' - d_1A_0) - d_0(d_1' + d_0 - d_1A_1) \not\equiv 0. \tag{2.7}$$

Proof. First, we suppose that $d_1 \not\equiv 0$. If $i(A_1) < p$, then $\rho_p(A_1) = 0$ and by (2.7), we obtain $\rho_p(h) = \rho_p(A_0) = \rho > 0$. If $\rho_p(A_1) = \rho_p(A_0) = \rho$ $(0 < \rho < +\infty)$ and $\sigma_p(A_1) < \sigma_p(A_0) = \sigma$ $(0 < \sigma < +\infty)$, then by (2.7) and Lemma 2.4 we have $\rho_p(h) = \rho_p(A_0) = \rho > 0$. Thus $h \not\equiv 0$.

Now, if $d_1 \equiv 0$, $d_0 \not\equiv 0$, then $h = -d_0^2 \not\equiv 0$. □

Lemma 2.6. [**4**] *Let $A_0(z), \ldots, A_{k-1}(z)$ be meromorphic functions, and let $i(A_0) = p$ $(1 \leqslant p < +\infty)$. Assume that either $i_\lambda(1/A_0) < p$ or $\lambda_p(1/A_0) < \rho_p(A_0)$ and that either* $\max\{i(A_j) : j = 1, 2, \ldots, k-1\} < p$ *or*

$$\max\{\rho_p(A_j) : j = 1, 2, \ldots, k-1\} \leqslant \rho_p(A_0) = \rho \ \ (0 < \rho < +\infty) \ \text{ and}$$
$$\max\{\sigma_p(A_j) : \rho_p(A_j) = \rho_p(A_0)\} < \sigma_p(A_0) = \sigma \ \ (0 < \sigma < +\infty).$$

Then every meromorphic solution $f \not\equiv 0$ of the equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0$$

whose poles are of uniformly bounded multiplicity satisfies $i(f) = p+1$ and $\rho_p(f) = \infty$, $\rho_{p+1}(f) = \rho_p(A_0) = \rho$.

Lemma 2.7. *Let $A_1(z), A_0(z)$ be meromorphic functions, and let $i(A_0) = p$ $(1 \leqslant p < +\infty)$. Assume that either $i_\lambda(1/A_0) < p$ or $\lambda_p(1/A_0) < \rho_p(A_0)$ and that either $i(A_1) < p$ or $\rho_p(A_1) = \rho_p(A_0) = \rho$ $(0 < \rho < +\infty)$ and $\sigma_p(A_1) < \sigma_p(A_0) = \sigma$ $(0 < \sigma < +\infty)$. Let $d_0(z), d_1(z)$ be meromorphic functions such that at least one of $d_0(z), d_1(z)$ does not vanish identically with* $\max\{\rho_p(d_j) : j = 0, 1\} < \rho_p(A_0)$. *If $f \not\equiv 0$ is a meromorphic solution of* (1.1) *whose poles are of uniformly bounded multiplicity, then the differential polynomial*

$$g_f = d_1f' + d_0f \tag{2.8}$$

satisfies

$$\rho_p(g_f) = \rho_p(f) = \infty, \ \ \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho.$$

Proof. Suppose that $f \not\equiv 0$ is a meromorphic solution of equation (1.1) whose poles are of uniformly bounded multiplicity. Then, by Lemma 2.6, we have $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho_p(A_0) = \rho$. Differentiating both sides of equation (2.8) and replacing f'' with $f'' = -A_1f' - A_0f$, we obtain

$$g_f' = (d_1' + d_0 - d_1A_1)f' + (d_0' - d_1A_0)f. \tag{2.9}$$

Set

$$\alpha_0 = d_0' - d_1A_0, \ \ \alpha_1 = d_1' + d_0 - d_1A_1. \tag{2.10}$$

Then, by (2.8), (2.9) and (2.10), we have

$$d_1 f' + d_0 f = g_f, \quad \alpha_1 f' + \alpha_0 f = g_f'. \tag{2.11}$$

Set

$$h = d_1\alpha_0 - d_0\alpha_1 = d_1(d_0' - d_1 A_0) - d_0(d_1' + d_0 - d_1 A_1). \tag{2.12}$$

By Lemma 2.5 we have $h \not\equiv 0$. By $h \not\equiv 0$, (2.11) and (2.12), we obtain

$$f = \frac{d_1 g_f' - \alpha_1 g_f}{h}. \tag{2.13}$$

If $\rho_p(g_f) < \infty$, then by (2.13) and Lemma 2.1, we get $\rho_p(f) < \infty$ and this is a contradiction. Hence $\rho_p(g_f) = \infty$.

Now, we prove that $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$. By (2.8), Lemma 2.1 and Lemma 2.2, we get $\rho_{p+1}(g_f) \leqslant \rho_{p+1}(f)$ and by (2.13) we have $\rho_{p+1}(f) \leqslant \rho_{p+1}(g_f)$. This yields $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho$. □

3. Proof of Theorem 1.2

Suppose that $f \not\equiv 0$ is a meromorphic solution of equation (1.1) whose poles are of uniformly bounded multiplicity. Then, by Lemma 2.6, we have $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho_p(A_0) = \rho$. Set $w(z) = d_1 f' + d_0 f - \varphi$. Since $\rho_p(\varphi) < \rho_p(A_0)$, then by Lemma 2.7 we have $\rho_p(w) = \rho_p(g_f) = \rho_p(f) = \infty$ and $\rho_{p+1}(w) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho$. In order to prove $\bar{\lambda}_p(g_f - \varphi) = \infty$ and $\bar{\lambda}_{p+1}(g_f - \varphi) = \rho$, we need to prove only $\bar{\lambda}_p(w) = \infty$ and $\bar{\lambda}_{p+1}(w) = \rho$. By $g_f = w + \varphi$, we get from (2.13)

$$f = \frac{d_1 w' - \alpha_1 w}{h} + \psi, \tag{3.1}$$

where

$$\psi = \frac{d_1\varphi' - \alpha_1\varphi}{h} \tag{3.2}$$

and $\rho_p(\psi) < \infty$. Substituting (3.1) into equation (1.1), we obtain

$$\frac{d_1}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w = -(\psi'' + A_1(z)\psi' + A_0(z)\psi) = A,$$

where ϕ_j $(j = 0,1,2)$ are meromorphic functions with $\rho_p(\phi_j) \leqslant \rho_p(A_0) < \infty$ $(j = 0,1,2)$. Since $(d_1' + d_0 - d_1 A_1)\varphi - d_1\varphi' \not\equiv 0$, from (3.2) we have $\psi(z) \not\equiv 0$. Hence, by $\psi(z) \not\equiv 0$ and $\rho_p(\psi) < \infty$, it follows by Lemma 2.6 that $A \not\equiv 0$. Thus, by Lemma 2.3, we obtain $\bar{\lambda}_p(w) = \rho_p(w) = \infty$, $\bar{\lambda}_{p+1}(w) = \rho_{p+1}(w) = \rho$, i.e., $\bar{\lambda}_p(g_f - \varphi) = \rho_p(g_f) = \rho_p(f) = \infty$ and $\bar{\lambda}_{p+1}(g_f - \varphi) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho$.

Remark 3.1. From the proof of Theorem 1.2, we see that the condition $(d_1' + d_0 - d_1 A_1)\varphi - d_1\varphi' \not\equiv 0$ is necessary because if $(d_1' + d_0 - d_1 A_1)\varphi - d_1\varphi' \equiv 0$, then $\psi(z) \equiv 0$ and $A(z) \equiv 0$.

4. Proof of Theorem 1.3

Suppose that $f \not\equiv 0$ is a meromorphic solution of equation (1.1) whose poles are of uniformly bounded multiplicity. Then, by Lemma 2.6, we have $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho_p(A_0) = \rho$. Set $w(z) = d_1 f' + d_0 f - \varphi$. Since $\rho_p(\varphi) < \rho_p(A_0)$, then by Lemma 2.7 we have $\rho_p(w) = \rho_p(g_f) = \rho_p(f) = \infty$ and $\rho_{p+1}(w) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho$. In order to prove $\overline{\lambda}_p(g_f - \varphi) = \infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \rho$, we need to prove only $\overline{\lambda}_p(w) = \infty$ and $\overline{\lambda}_{p+1}(w) = \rho$. Substituting $g_f = w + \varphi$ into (2.13) and using a similar reasoning as in the proof of Theorem 1.2, we get that

$$\frac{d_1}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w = -(\psi' + A_1(z)\psi' + A_0(z)\psi) = B,$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions with $\rho_p(\phi_j) \leqslant \rho_p(A_0) < \infty$ $(j = 0, 1, 2)$ and

$$\psi = \frac{d_1\varphi' - \alpha_1\varphi}{h}. \tag{4.1}$$

Now, we prove that $\psi(z) \not\equiv 0$. Assume that $\psi(z) \equiv 0$. Then from (4.1), we obtain that

$$(d_1' + d_0 - d_1 A_1)\varphi = d_1\varphi'. \tag{4.2}$$

First, if $d_1 \equiv 0$, then by (4.2), we get $d_0 \equiv 0$ and this is a contradiction. Now if $d_1 \not\equiv 0$, since $\rho_p(\varphi) < \rho_p(A_0)$, then by (4.2), we get

$$\rho_p((d_1' + d_0 - d_1 A_1)\varphi) = \rho_p(A_1) = \rho_p(d_1\varphi') < \rho_p(A_0) = \rho_p(A_1)$$

and this is a contradiction. Hence $\psi(z) \not\equiv 0$. Since $\psi(z) \not\equiv 0$ and $\rho_p(\psi) < \infty$, it follows by Lemma 2.6 that $B \not\equiv 0$. Thus, by Lemma 2.3, we obtain $\overline{\lambda}_p(w) = \rho_p(w) = \infty$, $\overline{\lambda}_{p+1}(w) = \rho_{p+1}(w) = \rho$, i.e., $\overline{\lambda}_p(g_f - \varphi) = \rho_p(g_f) = \rho_p(f) = \infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A_0) = \rho$.

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